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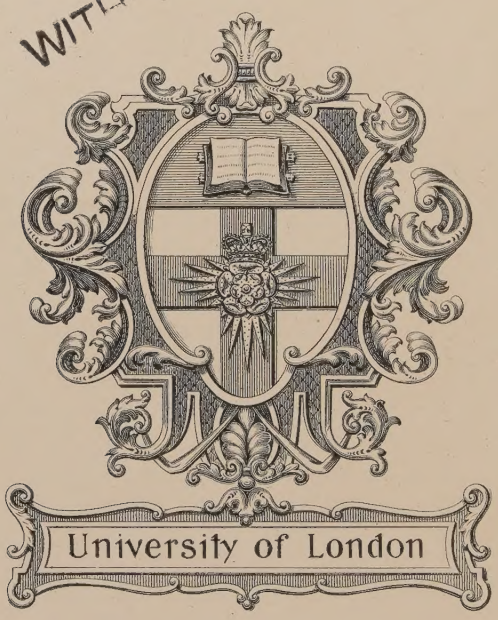

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This theorem is stated for the first time in general form by Rogosinski.* It is more or less equivalent to (though superficially very unlike) a well known general principle due to Lindelöf,† and particular cases of it have been considered by other writers.

2. We write

$$M_\lambda(\rho) = M_\lambda(\rho, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^\lambda \, d\theta\right\}^{1/\lambda} \quad (\lambda > 0).$$

For $\lambda \to 0$ and $\lambda \to \infty$ we obtain the limiting forms

$$M_\infty(\rho) = M(\rho) = \operatorname*{Max}_{|z|=\rho} |f(z)|, \tag{2.1}$$

$$M_0(\rho) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})| \, d\theta\right\}. \tag{2.2}$$

THEOREM 2.—*If f is subordinate to f in $|z| < 1$, then*

$$M_\lambda(\rho, f) \leqslant M_\lambda(\rho, \bar{f}) \quad (\rho < 1, \ 0 \leqslant \lambda \leqslant \infty).$$

This is the principal theorem of the paper.

3. In describing applications of these general theorems it will be convenient to use a systematic notation for functions belonging to various classes. All our functions are regular for $|z| < 1$.

A ("Picard") function which never takes the values 0 or 1 we denote by $p(z) = \sum_{n=0}^{\infty} p_n z^n$. Similarly $q(z) = \Sigma q_n z^n$ is to have the property $q(z) \neq 2n\pi i$ $(n = \ldots, -1, 0, 1, 2, \ldots)$; $r(z) = \Sigma r_n z^n$ the property

$$r(z) \neq -4\pi^2 m^2 \quad (m = 0, 1, 2, \ldots).$$

Evidently a $\log p(z)$ is a $q(z)$ and conversely; a $q^2(z)$ is an $r(z)$ and conversely. In conformity with the notation of §§ 1, 2 the simultaneous appearance of a pair of functions f and $\bar{f}$, p and $\bar{p}$, ... denotes that f is subordinate to $\bar{f}$, p to $\bar{p}$, ...‡ We use $\bar{s}(z)$ [but *not* $s(z)$] to denote a function "*schlicht* in $|z| < 1$," that is, a function for which, for every value of c, the equation $\bar{s}(z) = c$ has at most one (simple) root in $|z| < 1$. For convenience we lay down the further convention $s_0 (= \bar{s}_0) = 0$.

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* W. Rogosinski, "Über Bildschranken . . . ", *Math. Zeitschrift*, 17 (1923), 260.

† *Loc. cit.*

‡ $\bar{p}$ is of course a p, $\bar{q}$ a q, and so on.

Finally, $\omega(z)$ denotes a function with the property $|\omega| \leqslant 1$, $\omega(0) = 0$ (which relations are equivalent to $|\omega| \leqslant |z|$).

A symbol $A(\alpha, \beta, \ldots)$ denotes a positive constant, not necessarily the same at every occurrence, which depends only on the arguments explicitly shown. Thus A denotes a positive absolute constant.

4. *The functions* $p(z)$, $q(z)$, $r(z)$.

It may be shown that p is subordinate to the function defined in terms of the elliptic modular function by

$$\bar{p}(z) = k^2(\tau), \qquad z = \frac{\tau - \tau_0}{\tau - \bar{\tau}_0},$$

where $k^2(\tau_0) = p_0$, the determination of τ_0 is fixed by $-1 < \Re\tau_0 \leqslant 1$, $|\tau_0 - \frac{1}{2}| \geqslant \frac{1}{2}$, $|\tau_0 + \frac{1}{2}| \geqslant \frac{1}{2}$, and $\bar{\tau}_0$ is the conjugate of τ_0. It follows that $q(z)$ is subordinate to $\bar{q}(z) = \log k^2(\tau)$ $\{\log k^2(\tau_0) = q_0\}$.* A detailed study of the function $\bar{q}(z)$, combined with Theorem 2, leads to

THEOREM 3.—

$$M(\rho, q) < A(q_0)(1-\rho)^{-1} \dagger \tag{4.1}$$

$$\sqrt{(\Sigma |q_n|^2 \rho^{2n})} = M_2(\rho, q) < A(q_0)(1-\rho)^{-\frac{1}{2}} \tag{4.2}$$

$$M_1(\rho, q) = \frac{1}{2\pi} \int_0^{2\pi} |q(\rho e^{i\theta})|\, d\theta < A(q_0, \epsilon)(1-\rho)^{-\epsilon} \tag{4.3}$$

$$|q_n| < A(q_0, \epsilon) n^{\epsilon}. \tag{4.4}$$

The result (4.3) (or rather the particular case $q = \bar{q}$) lies very deep in the theory of modular functions; (4.4) is an immediate corollary. It is possible that the right-hand sides of (4.3) and (4.4) should be $A(q_0) \log\{1/(1-\rho)\}$ and $A(q_0)$. More than this *cannot* be true, as is shown by the example $q(z) = (1-z)^{-1}$.

A single result about $r(z)$ may be noted:

$$M(\rho, r) < A(r_0)(1-\rho)^{-2}. \tag{4.5}$$

5. *Functions* $s(z)$.

To be an s is to be subordinate to an $\bar{s}$. Hence it may be shown that to be an s is equivalent to being a function $s_1 z + s_2 z^2 + \ldots$ which takes no value belonging to some connected continuum C extending to ∞. To make

* See P. Lévy, "Remarques sur le théorème de M. Picard", *Bull. de la Soc. Math. de France*, 40 (1912), 25–39.

† This is a known result. See Lévy, *loc. cit.*

inferences from Theorem 2 about functions s we require information about

$$M_\lambda(\rho, \bar{s}).$$

THEOREM 4.—

$$M(\rho, \bar{s}) \leqslant |\bar{s}_1| \rho(1-\rho)^{-2} \tag{5.1}$$

$$M_1(\rho, \bar{s}) \leqslant A|\bar{s}_1| \rho(1-\rho)^{-1} \tag{5.2}$$

$$|\bar{s}_n| \leqslant A|\bar{s}_1| n. \tag{5.3}$$

(5.1) is a known theorem due to G. Pick and L. Bieberbach.* (5.2) and (5.3) are new, the best previous result about $\bar{s}_n$ being

$$|\bar{s}_n| \leqslant A|\bar{s}_1| n^2.$$

The inferences about functions s are given by

THEOREM 5.—

$$M(\rho, s) \leqslant 4d\rho(1-\rho)^{-2} \tag{5.4}$$

$$M_1(\rho, s) \leqslant Ad\rho(1-\rho)^{-1} \tag{5.5}$$

$$|s_n| \leqslant Adn, \tag{5.6}$$

where d is the distance of the continuum C from $w = 0$.

(5.4) is a best possible result, and (5.5), (5.6) are best possible except for the values of the constants A. This is shown by the example $s(z) = \bar{s}(z) = z(1-z)^{-2}$. It is odd that we can prove the best possible result

$$|s_2| \leqslant 2|\bar{s}_1|. \tag{5.7}$$

A curious fact emerges from a comparison of Theorems 3 and 5. The effect of a set of missing values $2n\pi i$ is to restrict the order of $M(\rho)$ to that of $(1-\rho)^{-1}$ at most, that of $M_1(\rho)$ to $(1-\rho)^{-\epsilon}$ at most and that of the nth coefficient to n^ϵ at most. But no substantially greater restriction is effected by a set of missing values filling a straight line, as is shown by the function $(1-z)^{-1}$, whose real part is bounded below. Similar results hold for $r(z)$ as compared with $z(1-z)^{-2}$. On the other hand, even a *domain* of missing values, *provided it is bounded*, need not restrict the order of $M(\rho)$ below $\exp\{A/(1-\rho)\}$, as is shown by the

* See L. Bieberbach, "Analytische Funktionen in der Ungebung wesentlich singulärer Stellen", *Math. Zeitschrift*, 2 (1918), 158–170 (foot-note p. 161).

example $f = \exp\{1/(1-z)\}$, for which $|f| \geqslant \frac{1}{2}$. And substantially *this* restriction is effected by only a pair of missing values.

Finally the following theorem shows (among other things) that even a very sparse distribution of missing values involves a heavy restriction on the order of $M(\rho)$.

THEOREM 6.—*Suppose we are given a positive integral k, a constant $c > 1$, and an infinite sequence $w_1, w_2, \ldots$, where*

$$|w_n| = r_n, \quad r_1 > 0, \quad r_n < r_{n+1} < cr_n.$$

Then for every function $f(z) = a_0 + a_1 z + \ldots$ which is regular in $|z| < 1$ and takes no value w_n more than k times, we have

$$|f(\rho e^{i\theta})| < A(k)\mu(1-\rho)^{-h},$$

where h depends only on k, c, and r_1, and

$$\mu = \text{Max}(1, |a_0|, |a_1|, \ldots, |a_k|).$$

6. We have not so far made any application of the principle

$$M_0(\rho, f) \leqslant M_0(\rho, \bar{f}),$$

which may be rewritten as

$$\int_0^{2\pi} \log|f(\rho e^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} \log|\bar{f}(\rho e^{i\theta})|\, d\theta. \tag{6.1}$$

Since $f-c$ is subordinate to $\bar{f}-c$ it is evident that (6.1), applied to $f-c$ and $\bar{f}-c$, and translated by means of Jensen's theorem, gives important connexions between $\Pi\rho_n(c)$ and $\Pi\bar{\rho}_n(c)$, where the $\rho_n(c)$ are the moduli of the roots of the equation $f(z)-c=0$. It is enough to note a single result. *For functions $\omega(z)$ we have, for $0 < |c| < 1$,*

$$\Pi\rho_n(c) \geqslant |\omega_\nu c|,$$

where ω_ν is the first coefficient of $\omega(z)$ that does not vanish.

On the Factorization of Certain Functions Rational and Integral in one of Several Variables

S. BEATTY.

It is known that certain functions which are rational and integral in one variable and have coefficients expansible in series of powers related to the remaining variables are products of functions of the same general

type but linear in that one variable. It is the custom to demonstrate the existence of one such linear factor of a function of this kind and to infer the existence of a representation of the function as a product of linear factors. In the present paper the existence of the product representation is established directly, which not only avoids the necessity of creating criteria for distinguishing the linear factors but is applicable should some of them be repeated. The procedure for effecting the formal factorization in the case of $N+1$ additional variables is based by analogy on that for N, and the result in the latter case is employed in the former. Certain conditions which insure that the factorization is numerical are discussed.

On Cubic Fields with a Given Discriminant

W. E. H. BERWICK.

The question proposed in this paper is the determination of all the cubic fields belonging to an assigned discriminant D. It appears that the solution of this problem is connected with the arrangement of ideals in the quadratic field $[\sqrt{-3D}]$. The class-symbols of the ideals in a quadratic field define an Abelian group, but it has not yet been determined whether every Abelian group can be so defined. In this paper the outstanding difficulty is due to uncertainty whether there exists a quadratic field whose principal genus contains an assigned number of exponents of irregularity, each divisible by 3. The investigation is complete except in deciding whether every possible type of discriminant actually exists. Another difficulty is caused by the peculiarities of the prime number 3 as a factor of the discriminant.

On the Radiation of Electric Dipoles

H. BATEMAN.

In a former paper (*Proceedings*, Ser. 2, Vol. 18, p. 115) expressions were given for the potentials and field vectors in an electromagnetic field in which electric dipoles are projected from a moving singularity O, and travel along straight lines with the velocity of light. The moments of these dipoles do not, however, remain constant in time, and the surmise that a cancelling of singularities can be obtained by superposing two such fields proves to be incorrect. It is possible, however, to write down

expressions for the field vectors of a field in which electric dipoles are projected from a moving point and move without change of moment, and then a cancelling of singularities with superposed fields can be obtained. In the case when O is stationary, a field of this type may be obtained by differentiating with respect to x the components

$$E_x = \frac{x}{r^3} f(\tau) + \frac{xz}{cr^2(r+z)} f'(\tau), \qquad H_x = -\frac{y}{cr(r+z)} f'(\tau),$$

$$E_y = \frac{y}{r^3} f(\tau) + \frac{yz}{cr^2(r+z)} f'(\tau), \qquad H_y = \frac{x}{cr(r+z)} f'(\tau),$$

$$E_z = \frac{z}{r^3} f(\tau) - \frac{r-z}{cr^2} f'(\tau), \qquad H_z = 0,$$

$$\left(\tau = t - \frac{r}{c}\right),$$

which specify the field of a point charge at O whose charge is variable on account of the emission along the negative z-axis of electric poles which travel with the speed of light. After differentiation the resulting field may be simplified by subtracting the field of a Hertzian dipole whose varying moment is in the direction Ox. The final expressions still contain both $f'(\tau)$ and $f''(\tau)$ and so do not specify a *simple* radiant field. In working with fields in which electric entities move with the speed of light it is best always to start with combinations of simple radiant fields in which electric poles are projected from moving or stationary singularities. By cancelling moving poles whose charges balance and coincide, a clear physical picture can be formed of the mathematical operations. The field components given above were obtained by a process of this type and a passage to the limit.

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*On the Zeros of Mathieu Functions**

EINAR HILLE.

The object of the paper is to study the distribution in the complex plane of the zeros and extrema of *Mathieu functions, i.e.* solutions of Mathieu's equation

$$w'' + (B - A\cos 2z)w = 0, \qquad \text{(M)}$$

where A and B are real constants and $A > 0$. Two different aspects of this problem are studied, namely (1) the approximate location of the zeros

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* Communicated to the meeting of March 8, 1923.

and extrema of large moduli, and (2) the structure of the point set formed by the zeros and extrema of a given solution. The latter aspect is emphasized.

The set in question is found to consist of infinitely many subsets, here called *strings*, namely in general one for each of the half-lines $x = \frac{1}{2}k\pi$, $y > 0$ or $y < 0$, and in addition one for the real axis. The string associated with one of the half-lines consists of infinitely many points whose abscissae and ordinates form monotonic sets. The distance from the points of the string to the line approaches zero very rapidly when the ordinates increase indefinitely. In the string the zeros and the extrema alternate. The point nearest to the real axis is in general an extremum. For every half-line, there are two particular solutions which lack the associated string. These "*truncated*" solutions have remarkable asymptotic properties as well.

When $B > A$ the subset associated with the real axis contains an infinite number of zeros and extrema which alternate.

Considerable attention is given to solutions which are real on one or several of the lines where $B - A\cos 2z$ is real. The most important of these solutions are the *functions of the elliptic cylinder*, for which we can solve the distribution problem completely.

Finally, we undertake a study of the movement of the point set when either the initial conditions or the parameter B are varied.

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*A Contribution to the Theory of Fourier Transforms**

E. C. TITCHMARSH.

The theory of the Fourier cosine transform of a function $f(x)$, whose pth power $(1 < p \leqslant 2)$ is integrable over $(0, \infty)$, is developed from the work of Hausdorff† on trigonometrical series. The transform is given by the formula

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{\sin xt}{t} f(t)\, dt.$$

It is proved that this and the reciprocal formula obtained by interchanging $f(x)$ and $F(x)$ are valid for almost all values of x. It is shown that $F(x)$ has its $p/(p-1)$th power integrable over $(0, \infty)$, and other properties of the transformation are obtained.

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* Communicated to the meeting of June 14, 1923.

† F. Hausdorff, *Math. Zeitschrift*, 16 (1923), 163–169.

On a Case of Integrability of Darboux's Differential Equation of Dimension Two

ARTHUR BERRY.

An algebraic differential equation in two variables of order and degree 1 can be written, in Darboux's notation,

$$L(y\,dz-z\,dy)+M(z\,dx-x\,dz)+N(x\,dy-y\,dx)=0,$$

where L, M, N are homogeneous polynomials in x, y, z of the same order m. If $m=0$ or $m=1$, the equation has known elementary integrals.

It is proved that, in the next higher case $m=2$, if the equation has a singular point of the type called a dicritic point, a singularity which plays an important part in the general theory, the equation can in all cases be integrated completely in terms of elementary functions and certain known transcendental functions. In all cases necessary and sufficient conditions are given that the complete integral should be algebraic, or that there should be one or more particular algebraic integrals.

A Note on Generalized Hypergeometric Series

G. N. WATSON.

In a recently published paper,* entitled "A Chapter from Ramanujan's Note-Book," Prof. Hardy has given a catalogue of all the known cases in which a series of the type ${}_3F_2$ with last element unity is expressible in terms of Gamma functions. Incidentally he quoted a formula which I had discovered, namely that

$${}_3F_2\left(\begin{matrix}-n,\ \lambda,\ 2\lambda+2\mu+n-1\\ 2\lambda,\ \lambda+\mu\end{matrix}\right)=\frac{n!\ \Gamma(\lambda+\frac{1}{2}n)\,\Gamma(\mu+\frac{1}{2}n)\,\Gamma(2\lambda)\,\Gamma(\lambda+\mu)}{(\frac{1}{2}n)!\ \Gamma(\lambda+\mu+\frac{1}{2}n)\,\Gamma(2\lambda+n)\,\Gamma(\lambda)\,\Gamma(\mu)}$$

when n is an even integer (when n is an odd integer the sum of the hypergeometric series is zero). Prof. Hardy pointed out that it was a special case of a formula discovered by Ramanujan, and I did not attach any

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* *Proc. Camb. Phil. Soc.*, 21 (1923), 492–503.

particular importance to it. But since the formula plays a moderately important part in Mr. Whipple's paper, it seems worth while to supplement his paper by giving my own proof of it. This proof was suggested to me by the proof* which I constructed of a formula concerning the square of a Bessel function discovered by Prof. Jolliffe.

If we use the notation

$$(a)_n \equiv a(a+1)\dots(a+n-1),$$

and put

$$a_n = \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)} \cdot \frac{(2\lambda)_n}{n!} \cdot {}_3F_2\left(\begin{matrix}-n,\ \lambda,\ 2\lambda+2\mu+n-1\\ 2\lambda,\ \lambda+\mu\end{matrix}\right),$$

then, by expanding the hypergeometric function in the integrand in powers of t, it is easily seen that†

$$a_n = \frac{(2\lambda)_n}{n!} \int_0^1 t^{\lambda-1}(1-t)^{\mu-1} \cdot {}_2F_1(-n,\ 2\lambda+2\mu+n-1\,;\ 2\lambda\,;\ t)\,dt.$$

Now this hypergeometric function is one of Jacobi's polynomials, and it is expressible in the form‡

$${}_2F_1(-n,\ 2\lambda+2\mu+n-1\,;\ 2\lambda\,;\ t) = \frac{t^{1-2\lambda}(1-t)^{1-2\mu}}{(2\lambda)_n} \frac{d^n}{dt^n}\left\{t^{2\lambda+n-1}(1-t)^{2\mu+n-1}\right\}.$$

Making this substitution and integrating by parts n times, we find that

$$a_n = \frac{1}{n!} \int_0^1 t^{-\lambda}(1-t)^{-\mu} \frac{d^n}{dt^n}\left\{t^{2\lambda+n-1}(1-t)^{2\mu+n-1}\right\} dt$$

$$= \frac{(-)^n}{n!} \int_0^1 t^{2\lambda+n-1}(1-t)^{2\mu+n-1} \frac{d^n}{dt^n}\left\{t^{-\lambda}(1-t)^{-\mu}\right\} dt.$$

By Taylor's theorem it follows that, for sufficiently small values of $|z|$,

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^1 t^{2\lambda-1}(1-t)^{2\mu-1} \sum_{n=0}^{\infty} \frac{(-)^n\{zt(1-t)\}^n}{n!} \frac{d^n}{dt^n}\left\{t^{-\lambda}(1-t)^{-\mu}\right\} dt$$

$$= \int_0^1 t^{2\lambda-1}(1-t)^{2\mu-1}\left\{(t+w)^{-\lambda}(1-t-w)^{-\mu}\right\} dt,$$

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* See my *Theory of Bessel Functions*, § 16.3.

† If the real part of λ or μ is negative, the work has to be modified by using a double-circuit integral in place of a definite integral.

‡ *Journal für Math.*, 56 (1859), 156.

where $$w = -zt(1-t).$$

Hence $$\sum_{n=0}^{\infty} a_n z^n = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\{1-z(1-t)\}^{-\lambda}(1+zt)^{-\mu}\,dt.$$

Now put $$t = \frac{u(1-z)}{1-zu}, \quad 1-t = \frac{1-u}{1-zu},$$

$$1-z(1-t) = \frac{1-z}{1-zu}, \quad 1+zt = \frac{1-z^2u}{1-zu},$$

and we get $$\sum_{n=0}^{\infty} a_n z^n = \int_0^1 u^{\lambda-1}(1-u)^{\mu-1}(1-z^2u)^{-\mu}\,du.$$

When this integral is expanded in ascending powers of z, only even powers occur ; and, by equating coefficients, we see that a_n is zero when n is odd, while when n is even

$$a_n = \frac{(\mu)_{\frac{1}{2}n}}{(\frac{1}{2}n)!}\int_0^1 u^{\lambda+\frac{1}{2}n-1}(1-u)^{\mu-1}\,du.$$

If we compare this result with the definition of a_n, we obtain at once the formula which had to be proved.

Thursday, December 13th, 1923.

Prof. L. N. G. FILON, Vice-President, in the Chair.

There were present twenty-one members.

The Minutes of the last meeting were read and confirmed.

The following were elected as members of the Society :—Messrs. S. T. Aston, E. T. Copson, M. H. A. Newman, and R. Stoneley.

Messrs. C. H. C. Cansdale and F. H. Murray were nominated for membership.

The Auditor's (Lt.-Col. Cunningham's) report was read and accepted.

Prof. Watson, as Secretary, reported that the number of members on November 1st, 1922, was 388. Twenty-three new members (including one under By-law 20 B) were elected in the Session 1922–1923 ; eight

members had died, and twenty-one had resigned. The number of members on November 1st, 1923, was therefore 382.

Mr. R. G. Cooke read a paper "An Integral involving Legendre Polynomials."

Prof. G. H. Hardy gave an account of a paper by Mr. E. C. Titchmarsh "A System of Linear Equations in an Infinite Number of Variables"; and also of a paper written in collaboration with Mr. J. E. Littlewood "Some Problems of Partitio Numerorum (VI): Further Researches in Waring's Problem."

The following papers were communicated by title from the chair :—

*On the Trinomial Quintic : H. B. C. Darling.

On the Representation of Numbers by Quadratic Forms: H. D. Kloosterman.

Some Inequalities concerning Ovals and Ovaloids: Prof. Tadahiko Kubota.

*On the Occurrence of a Linear Determinantal System of Points : W. L. Marr.

*Properties of Prime Numbers deduced from the Calculus of Symmetric Functions : P. A. MacMahon.

*Note on the Converse of Abel's Continuity Theorem : A. E. Ingham

*On Complexes of Conics in Space of Four Dimensions : C. G. F. James.

Perspective Space Transformations : Miss G. D. Sadd.

*Note on the Reduction of a Certain Invariant of Three Quaternary Quadrics : H. W. Turnbull.

NOTES AND ABSTRACTS.

On the Representation of Numbers by Quadratic Forms

H. D. KLOOSTERMAN.

The object of the paper is to apply the general analytical methods of Hardy and Littlewood to the problem of the representation of integers n

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* Printed in this volume.

by quadratic forms $x^2+y^2+cz^2+ct^2$, where c is a given positive integer. The paper consists of three parts. In the first part an asymptotic expression $S(n)$ (the singular series) is obtained for $r(n)$, the number of representations. In the second part the singular series is summed and the following problem is considered: for which values of c are all sufficiently large integers representable by the quadratic form?

In the third part exact formulae for $r(n)$ are obtained for special values of c. All of Liouville's results ($c = 2, 3, 4, 5, 8, 16$) follow as special cases. It is shown that for small values of c ($c = 2, 3, 4$) $r(n)$ is exactly equal to $S(n)$. For other values of c other arithmetical functions must be added to $S(n)$.

A Reciprocal Relation between Two Sequences of Numbers

E. C. TITCHMARSH.

I have observed the following reciprocal relation between two sequences of numbers. Suppose that the numbers a_n are such that the series

$$\sum_{n=-\infty}^{\infty} a_n^2 \tag{1}$$

is convergent; and let

$$b_n = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \frac{a_m}{m+n+\frac{1}{2}} \tag{2}$$

for all integer values of n. Then conversely

$$a_n = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \frac{b_m}{m+n+\frac{1}{2}} \tag{3}$$

for all integer values of n; and also

$$\sum_{n=-\infty}^{\infty} b_n^2 = \sum_{n=-\infty}^{\infty} a_n^2. \tag{4}$$

More generally, the reciprocal relations (2) and (3) are true if the series

$$\sum_{n=-\infty}^{\infty} |a_n|^p \tag{5}$$

is convergent for any given value of p greater than unity; and the b_n series always has the same convergence exponent as the a_n series.

The proofs which I have constructed of these results are of a direct nature.

Note on the Reduction of a Certain Invariant of Three Quaternary Quadrics

H. W. TURNBULL.

The invariant $(Abc)(B\gamma a)(AC)(CB)b_\gamma c_a$

of three quaternary quadrics was included in the complete system given in *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 20, p. 483. This type has now been reduced, leaving a system of at most 41 invariants.

On the Occurrence of a Linear Determinantal System of Points

W. L. MARR.

Given two triads of points $U_1U_2U_3$ and $V_1V_2V_3$ on the lines m, n respectively, two and only two cubics through these six points possess the property that their tangents at all the six points are concurrent.

If H, K be the two points of concurrency thus defined, and if the point in which the line joining U_r and V_ρ cuts the line HK be denoted by (U_rV_ρ), the configuration of points

$$\begin{array}{lll} (U_1V_1) & (U_1V_2) & (U_1V_3) \\ (U_2V_1) & (U_2V_2) & (U_2V_3) \\ (U_3V_1) & (U_3V_2) & (U_3V_3) \end{array}$$

constitutes a determinantal system on the lines AK, the pair of points harmonically separating the Hessian-points of all the triads defined by the expansion of the above determinantal form being the poles H, K.

An Integral involving Legendre Polynomials

RICHARD G. COOKE.

The formula

$$(1) \qquad \int_{-1}^{1} (1+z)^{m+n} P_m(z) P_n(z)\, dz = \frac{2^{m+n+1}\{(m+n)!\}^4}{(m!\ n!)^2(2m+2n+1)!},$$

where m and n are positive integers, was deduced by Mr. E. C. Titchmarsh from the theory of Bessel Functions. It may be proved directly as follows. We have the formulæ

$$(2) \qquad P_m(z) = (-1)^m \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r}{(r!)^2} \cdot \frac{(1+z)^r}{2^r},$$

where $(\kappa)_r = \kappa(\kappa+1)(\kappa+2)\ldots(\kappa+r-1)$; and

$$(3) \qquad \int_{-1}^{1} (1+z)^p P_n(z)\, dz = \frac{2^{p+1}(p!)^2}{(p-n)!\,(p+n+1)!},$$

where p and n are positive integers, and $p > n$. Hence

$$(4) \qquad \int_{-1}^{1} (1+z)^{m+n} P_m(z) P_n(z)\, dz$$

$$= (-1)^m \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r}{(r!)^2 2^r} \cdot \frac{2^{m+n+r+1}\{(m+n+r)!\}^2}{(m+r)!\ (m+2n+r+1)!}$$

$$= \frac{(-1)^m 2^{m+n+1}\{(m+n)!\}^2}{m!\ (m+2n+1)!} \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r \{(m+n+1)_r\}^2}{(r!)^2 (m+1)_r (m+2n+2)_r}.$$

The latter summation is

$$(5) \qquad {}_3F_2(-m,\ m+n+1,\ m+n+1\ ;\ 1,\ m+2n+2\ ;\ 1).$$

Now Saalschütz* has proved that, if m be a positive integer,

$$(6) \qquad {}_3F_2(-m,\ \beta,\ \delta+\epsilon+m-\beta-1\ ;\ \delta\ ;\ \epsilon\ ;\ 1)$$

$$= \frac{\Gamma(\delta)\,\Gamma(\epsilon)\,\Gamma(\delta-\beta+m)\,\Gamma(\epsilon-\beta+m)}{\Gamma(\delta+m)\,\Gamma(\epsilon+m)\,\Gamma(\delta-\beta)\,\Gamma(\epsilon-\beta)}.$$

In this result, write

$$\beta = m+n+1+\theta, \qquad \delta = 1, \qquad \epsilon = m+2n+2+2\theta,$$

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* *Zeitschrift für Math.*, Vol. 35 (1890), pp. 186–188.

and we obtain

$$ {}_3F_2(-m,\ m+n+1+\theta,\ m+n+1+\theta;\ 1;\ m+2n+2+2\theta;\ 1) $$
$$ = \frac{\Gamma(1)\,\Gamma(m+2n+2+2\theta)\,\Gamma(-n-\theta)\,\Gamma(m+n+1+\theta)}{\Gamma(m+1)\,\Gamma(2m+2n+2+2\theta)\,\Gamma(-m-n-\theta)\,\Gamma(n+1+\theta)}. $$

Also
$$ \frac{\Gamma(-n-\theta)}{\Gamma(-m-n-\theta)} = (-1)^n \frac{(\theta+1)_{m+n}}{(\theta+1)_n}. $$

Hence, making $\theta \to 0$, we have, as the sum of (5),

$$ \frac{(-1)^m \{(m+n)!\}^2 (m+2n+1)!}{m!\ (n!)^2\ (2m+2n+1)!}. $$

Combining this result with (4) we obtain (1).

Some Problems of "Partitio Numerorum" (VI): Further Researches in Waring's Problem

G. H. HARDY *and* J. E. LITTLEWOOD.

This paper contains a continuation of the researches of the first, second, and fourth memoirs of the series [*Göttinger Nachrichten* (1920), 33–54; *Mathematische Zeitschrift*, 9 (1921), 14–27, and 12 (1923), 161–188].

We say that almost all numbers (positive integers) n possess a property P, if the number of numbers less than n, for which P is false, is $o(n)$ when n is large.

We prove first

THEOREM 1.—*If* $k \neq 4$, *then almost all numbers are sums of*

$$ (\tfrac{1}{2}k-1)2^{k-1}+3 $$

(*positive*) *k-th powers, and in particular of* 5 *cubes,* 27 *fifth powers, and* 67 *sixth powers.*

The case $k = 4$ is abnormal. In this case we prove

THEOREM 2.—*Almost all numbers are sums of* 15 *biquadrates.*

Here 15 is the "correct" number, and cannot be replaced by any smaller number.

Combining Theorem 1 with a new elementary idea, we are able to make a substantial improvement in the known upper bound for the Waring number $G(k)$.

THEOREM 3.—*If* $k > 3$, *and*

$$\rho_k = \frac{(k-2)\log 2 - \log k + \log(k-2)}{\log k - \log(k-1)}$$

then all large numbers are sums of

$$(\tfrac{1}{2}k-1)2^{k-1}+k+5+[\rho_k]$$

(*positive*) *k-th powers; and in particular of* 19 *biquadrates*, 41 *fifth powers, and* 87 *sixth powers.*

The case of biquadrates is particularly interesting. It is known that $g(4) \geqslant 19$ (since $79 = 4\cdot 2^4 + 15\cdot 1^4$ actually needs 19 biquadrates) The problem of the determination of the exact value of $g(4)$ is thus reduced (in theory at any rate) to a problem of computation.

Finally, we examine the bearing on Waring's problem of a plausible but unproved hypothesis, known to be true when $k = 2$: viz. that *the number of representation of n by k k-th powers is* $O(n^\epsilon)$ *for every positive* ϵ. We prove that, if this hypothesis is true, and $k \neq 4$, then almost all numbers are sums of

$$\text{Max}\left(k+1, \Gamma(k)\right)$$

k-th powers, and all large numbers are sums of

$$\text{Max}\left(2k+1, \Gamma(k)\right)$$

k-th powers. Here $\Gamma(k)$ is the least value of s for which the "singular series" of our investigations is always positive, and may be determined, with sufficient labour, for any value of k.

In particular, if it be true that the number of representations of n by 3 cubes is $O(n^\epsilon)$, then all large numbers are sums of 7 cubes, and almost all are sums of 4.

Thursday, January 17th, 1924.

Prof. L. N. G. FILON, Vice-President, in the Chair.

There were present eighteen members.

The Minutes of the last meeting were read and confirmed.

Messrs. C. H. C. Cansdale and F. H. Murray were elected members of the Society.

Mr. B. Puri was nominated for election.

The Chairman announced that Mr. E. A. Milne would deliver a lecture on February 14th on "The Star and the Atom."

Mr. E. C. Titchmarsh made a communication on "Problems connected with Fourier's Double Integral."

Prof. G. H. Hardy communicated some theorems of Dr. M. Riesz on conjugate trigonometrical series.

Dr. F. B. Pidduck made an informal communication on magnetic induction.

The following papers were communicated by title from the chair :—

A contribution to the Theory of Bounded Matrices and Quadratic Forms, with an Infinity of Variables: J. Hyslop. (Communicated by E. W. Hobson.)

Note on the Converse of Abel's Continuity Theorem: A. E. Ingham.

*On Inequalities in the Theory of Functions: J. E. Littlewood.

A Certain Relation between Two Quadrics: S. L. Green.

*On the Rank of the Double-binary Form: R. Vaidyanathaswami.

NOTES AND ABSTRACTS.

Extensions of Fourier's Integral Formula to Formulæ involving Bessel Functions

E. C. TITCHMARSH.

It is well known that Fourier's integral formulæ

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty \sin ut\, f(t)\,dt, \tag{1}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin ux\,du \int_0^\infty \sin ut\, f(t)\,dt, \tag{2}$$

are true if $f(t)$ is (i) absolutely integrable over $(0, \infty)$, and (ii) of bounded variation in an interval containing the point $t = x$. At a point of ordinary discontinuity the value of the integral is $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

There are a number of extensions of this result to Bessel functions, each being true with conditions corresponding to (i) and (ii). The

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* To be published in Vol. 24.

simplest is Hankel's well-known formula

$$(3) \qquad f(x) = \int_0^\infty J_\nu(ux)u\,du \int_0^\infty J_\nu(ut)\,t f(t)\,dt.$$

This is true for $\nu \geqslant -\frac{1}{2}$ if $\sqrt{t}\,f(t)$ is integrable over $(0, \infty)$, and for $-1 < \nu < -\frac{1}{2}$ if we are given further that $t^{\nu+1}f(t)$ is integrable over $(0, \delta)$. It reduces to Fourier's cosine formula for $\nu = -\frac{1}{2}$, and to Fourier's sine formula for $\nu = \frac{1}{2}$.

A somewhat similar result* is

$$(4) \qquad f(x) = \int_0^\infty Y_\nu(xu)\,u\,du \int_0^\infty \mathbf{H}_\nu(ut)\,t f(t)\,dt,$$

where Y_ν is the Bessel function of the second kind and $\mathbf{H}_\nu$ is Struve's function. This holds for $-\frac{3}{2} \leqslant \nu < \frac{1}{2}$ if $\sqrt{t}\,f(t)$ is integrable over $(0, \infty)$. It reduces to Fourier's sine formula for $\nu = -\frac{1}{2}$, and for $\nu = -\frac{3}{2}$ it is the same as Hankel's formula for $\nu = \frac{3}{2}$. It may be regarded as in some sense conjugate to (3) as (2) is conjugate to (1).

The formula

$$(5) \qquad f(x) = -2\pi \int_0^\infty J_\nu(xu)\,Y_\nu(xu)\,u\,du \int_0^\infty \frac{d}{du}\{u J_\nu^2(tu)\}\,t f(t)\,dt$$

was discovered by Bateman† in the case $\nu = 0$. It is true for $\nu \geqslant 0$ if $tf(t)$ is integrable over $(0, \infty)$, and for $-\frac{1}{2} < \nu < 0$ if we are also given that $t^{2\nu+1}f(t)$ is integrable over $(0, \delta)$. It reduces to Fourier's sine formula in the cases $\nu = \pm\frac{1}{2}$.

We have finally the formula‡ of Weber and Orr,

$$(6) \qquad f(x)$$

$$= \int_0^\infty \frac{J_\nu(xu)\,Y_\nu(u) - Y_\nu(xu)\,J_\nu(u)}{J_\nu^2(u) + Y_\nu^2(u)}\,u\,du \int_1^\infty \{J_\nu(tu)\,Y_\nu(u) - Y_\nu(tu)\,J_\nu(u)\}\,t f(t)\,dt,$$

which is true for all values of ν if $\sqrt{t}\,f(t)$ is integrable over $(0, \infty)$. It reduces to Fourier's sine formula for $\nu = \pm\frac{1}{2}$.

We may evidently obtain from (4) the associated formula

$$(4') \qquad f(x) = \int_0^\infty \mathbf{H}_\nu(xu)\,u\,du \int_0^\infty Y_\nu(ut)\,t f(t)\,dt,$$

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* See *London Math. Soc. Records*, May 17, 1923.

† H. Bateman, *Proc. London Math. Soc.* (2), 4 (1906), 482.

‡ See my paper *Proc. London Math. Soc.* (2), 22 (1923), 15–28, for a proof, and further references concerning formulæ (6) and (6').

but this is not true under such general circumstances as the previous formulæ. On the other hand the formula associated with (6),

$$(6') \quad f(x) = \int_1^\infty \{J_\nu(xu)\, Y_\nu(x) - Y_\nu(xu)\, J_\nu(x)\}\, u\, du \int_0^\infty \frac{J_\nu(ut)\, Y_\nu(t) - Y_\nu(ut)\, J_\nu(t)}{J_\nu^2(t) + Y_\nu^2(t)}\, t f(t)\, dt,$$

is true for all values of ν if $t f(t)$ is integrable. I have not considered how generally the formula associated with (5) holds.

Sur les séries trigonométriques conjuguées

M. RIESZ.*

Voici les grandes lignes de la démonstration desirée, qui va paraître dans le *Math. Zeitschrift.*

I. *Deux seriés trigonométriques conjuguées sont en même temps les séries de Fourier de fonctions de la classe* L^p $(p > 1)$.†

Ce théorème est évidemment équivalent au théorème suivant :—

II. *Soit* $F(z) = F(re^{i\theta}) = U(r, \theta) + iV(r, \theta)$ *une fonction holomorphe pour* $r < 1$. *Si*

$$\int_0^{2\pi} |\, U(r, \theta)\,|^p\, d\theta$$

reste borné lorsque r tend vers 1, *il en sera de même de*

$$\int_0^{2\pi} |\, V(r, \theta)\,|^p\, d\theta\,;$$

ou plus précisément

III. *Posons* $F(0) = u + iv$. *Si l'on a*

$$(1) \qquad \frac{1}{2\pi} \int_0^{2\pi} |\, U(r, \theta)\,|^p\, d\theta \leqslant 1 \quad (r < 1),$$

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* Extracted from a letter to Prof. G. H. Hardy.

† That is to say, functions whose p-th power is integrable (in the sense of Lebesgue).

on aura $$\frac{1}{2\pi}\int_0^{2\pi} |V(r,\theta)|^p\, d\theta \leqslant (M_p + |v|)^p,$$

où M_p ne dépend que de p.

. . . . . .

Démontrons maintenant le théorème III pour p entier pair. On a, d'après Minkowski,

(2) $$\left(\int |f+g|^p\, d\theta\right)^{1/p} \leqslant \left(\int |f|^p\, d\theta\right)^{1/p} + \left(\int |g|^p\, d\theta\right)^{1/p}.$$

Définissons encore le nombre q par $1/p+1/q=1$, c'est à dire que $q = p/(p-1)$. On a alors, d'après Hölder,

(3) $$\left|\int fg\, d\theta\right| \leqslant \left(\int |f|^p\, d\theta\right)^{1/p}\left(\int |g|^q\, d\theta\right)^{1/q}.$$

Nous pourrons, grace à (2), supposer que $v=0$. Pour simplifier, admettons un instant que aussi $u=0$. On a alors, d'après le théorème de Cauchy,

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{\left(F(z)\right)^p}{z}\, dz = \frac{1}{2\pi}\int_0^{2\pi} (U+iV)^p\, d\theta = 0;$$

d'où $$\int_0^{2\pi}\left\{V^p - \binom{p}{2} V^{p-2}U^2 + \binom{p}{4} V^{p-4}U^4 - \ldots \pm U^p\right\} d\theta = 0,$$

et par conséquent

(4) $$\int_0^{2\pi} V^p\, d\theta < \int_0^{2\pi}\left\{\binom{p}{2} V^{p-2}U^2 + \binom{p}{6} V^{p-6}U^6 + \ldots\right\} d\theta.$$

Si l'on remplace en (3) p et q par $p/(p-2k)$ et $p/2k$ respectivement, on trouve, pour $2k \leqslant p$,

$$\int V^{p-2k}U^{2k}\, d\theta \leqslant \left(\int V^p\, d\theta\right)^{(p-2k)/p}\left(\int U^p\, d\theta\right)^{(2k)/p}.$$

En posant ici $$\frac{1}{2\pi}\int_0^{2\pi} V^p\, d\theta = \chi^p,$$

on obtient, d'après (1) et (4),

$$\chi^p < \binom{p}{2}\chi^{p-2} + \binom{p}{6}\chi^{p-6} + \ldots;$$

c'est à dire que χ est plus petit que la plus grande racine positive de l'équation

$$\chi^p - \binom{p}{2}\chi^{p-2} - \binom{p}{6}\chi^{p-6} - \ldots = 0,$$

et alors plus petit qu'une constante ne dépendant que de p.

Si l'on ne suppose plus que $u = 0$, il sera très peu à changer au calcul ci-dessus. . . . . .

Note on the Converse of Abel's Continuity Theorem

A. E. INGHAM.

A well known theorem of Littlewood* asserts that, *if*

$$a_n = O\left(\frac{1}{n}\right) \tag{1}$$

and

$$\Sigma a_n x^n \to A, \tag{2}$$

when $x \to 1$ *from below, then* Σa_n *converges to* A. Littlewood also proved that the conclusion does not follow when (1) is replaced by any condition of the type

$$a_n = O\left(\frac{\chi(n)}{n}\right), \tag{3}$$

where $\chi(n) \to \infty$ with n. He conjectured further that (2) and (3) do not even involve the summability of Σa_n by any Cesàro mean; but proved this only in the particular case in which $\chi(n) = (\log n)^\alpha$ $(\alpha > 0)$.

In this paper the correctness of Littlewood's conjecture is established in full generality. It is shown also that, for every form of $\chi(n)$, series exist which satisfy (2) and (3) and possess any assigned non-negative index of Cesàro summability.

The generalised problem for Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$ and Riesz's methods of summation by typical means (of type λ_n) is discussed, and it is shown that an analogous result holds, provided that $\lambda_n \sim \lambda_{n-1}$. This last condition (or some restriction of a similar character) is, as was pointed out by Littlewood, quite essential for the truth of negative results of the kind here considered.

* J. E. Littlewood, "The converse of Abel's Theorem on power series", *Proc. London Math. Soc.* (2), 9 (1911), 434–448.

Thursday, February 14th, 1924.

Prof. A. E. JOLLIFFE, Vice-President, in the Chair.

There were present thirty-three members.

The Minutes of the last meeting were read and confirmed.

Mr. B. Puri was elected a member of the Society.

Mr. V. D. Gokhale was nominated for election.

Mr. E. A. Milne delivered a lecture on "The Star and the Atom."

The following papers were communicated by title from the chair:—

*Bipolar Solutions of Einstein's Gravitational Equations: H. E. J. Curzon.

†A Class of Expansions in Non-Orthogonal Functions: J. Proudman.

Time-Relations in Meteorological Effects on the Sea: J. Proudman and A. T. Doodson.

The Applications of the Absolute Differential Calculus to the Theory of Elasticity: J. L. Synge.

A Relation between the Incomplete Beta and Gamma Functions: R. G. Cooke.

A Simple Proof of Jacobi's Transformation: W. L. Ferrar.

NOTES AND ABSTRACTS.

Čech's Transformations

H. P. HUDSON.

An example used in a recent paper‡ shows that there is an omission in the list, published some years ago,§ of particular cases of the bilinear space transformation T_{3-3}, in which the fundamental sextic breaks up into six lines, some of which coincide.

The more general of the two transformations used by Čech (p. 243) is expressed by

$$(W.1_3)\quad Zw = Wz,\quad Xz+Yw = Zx+Wy,\quad Xy-Yx = Zw;$$

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* Published in this volume.

† To be published in Vol. 24.

‡ Čech, "Sur l'élément du troisième ordre d'une courbe et d'une surface dans l'espace projectif", *Casopis*, 50 (1921), 219–250.

§ "On the 3-3 birational transformation in three dimensions" (second paper), *Proc. London Math. Soc.* (2), 10 (1911), 15; this is quoted below as X, and its notation is used.

the general homaloid in either space is of the form

$$\Phi \equiv \phi(c_1x+c_2y+c_3z+c_4w)+zw(c_1w-c_2z),$$

where $$\phi = xz+yw.$$

The homaloids have four-point contact with each other, and osculate the quadric ϕ, at every point of the line l, $z = w = 0$. They also have the two simple fundamental lines, $x = w = 0$ and $y = z = 0$, skew to each other and meeting l. The reverse transformation is of the same nature. The diagram is therefore the same as $W.1_1$ and $W.1_2$ (X, p. 45), but in each of these the homaloid touches a fixed plane along l, instead of a proper quadric as here.

The two simple lines are generators of ϕ of the opposite system to l; since ϕ is proper, they cannot meet, nor coincide with l, but they may coincide with each other. This gives a fresh transformation X_4, also used by Čech, p. 246; we replace the third bilinear equation by $Xy-Yx = Ww$, and in the last term of Φ we replace zw by w^2.

These cases belong to type (iii) of the tables of X, p. 18. The number of independent groups admissible for four-point contact along l is 4, instead of 3 as given for the general case of this type in the second table.

An additional group exists because, at the variable point of l given by $c_1x+c_2y = 0$, Φ has a binode, of the type B_4, which reduces the class by 4, a number greater than the order of contact, whereas the general case has a fixed B_3. There is a still more special case with a B_5 or B_6, when the additional cubic group has five-point contact with Φ, but as there are no other groups admissible for that order, this does not give a transformation.

A variable B_s generally causes an additional group to be admissible for s-point contact*; but the argument of XI assumes that the degree of the family is high enough, compared to s, to include all admissible groups. Here we can only admit those of degree $\leqslant 3$. In X, those of degrees 1 and 2 are considered; those of degree 3 are overlooked. A cubic group exists in two cases, the first leading to Čech's transformations.

The second does not lead to a new T_{3-3}. It occurs with five-point contact along l of type (vi), that is, when there is a fixed tangent plane and an osculating quadric cone along l. In the general type (vi), there is a fixed binode B_3, with its edge distinct from l; in our particular case, this is replaced by a fixed B_5, and l is its edge, a circumstance expressly excluded

* "Curves of contact of any order on algebraic surfaces", *Proc. London Math. Soc.* (2), 11 (1912), 398, quoted as XI.

in XI, p. 399. Such cases, and also that of a unode, have therefore been investigated on their merits, with the results stated. The freedom of the family is 5 instead of 4, but it no longer contains a simple fundamental line, without a further condition, which just destroys the additional freedom, and leads back to the transformation Y_2 of X, p. 30.

Thus Čech's two are the only fresh cases of T_{3-3} that arise in this way.

Bipolar Solutions of Einstein's Gravitation Equations

H. E. J. CURZON.

Let A, B be two stationary points, P any other point, and let the lengths AB, AP, BP be denoted by $2a$, r_1, and r_2 respectively. Let ϕ denote the angle made by the plane APB with an initial plane, Δ the area of APB, and c the velocity of light.

Then a rigorous solution of Einstein's equations $G_{mn} = 0$ is given by

$$ds^2 = -e^{\lambda}\left(r_1^2 r_2^2 dr_1^2 - r_1 r_2 (r_1^2 + r_2^2 - 4a^2)\, dr_1 dr_2 + r_1^2 r_2^2 dr_2^2\right)\Big/ 4\Delta^2$$
$$- e^{\nu}\Delta^2 d\phi^2/a^2 + e^{\rho} c^2 dt^2,$$

where

$$\lambda = 2m_1/r_1 + 2m_2/r_2 - m_1^2 \sin^2\theta_1/r_1^2 - 2m_1 m_2 \sin^2 \tfrac{1}{2}\psi/a^2 - m_2^2 \sin^2\theta_2/r_2^2,$$

$$\nu = -\rho = 2m_1/r_1 + 2m_2/r_2,$$

m_1 and m_2 are constants of integration, and θ_1, θ_2, ψ denote the angles BAP, ABP, APB respectively.

A rigorous isotropic solution of $G_{mn} = a g_{mn}$ for empty space is given by

$$\lambda = \nu = -2\log(1+v), \qquad \rho = 2\log\{(1-v)/(1+v)\},$$

where $$v = r_1^2/a_1 + r_2^2/a_2 - 4a^2/(a_1+a_2), \qquad 1/a_1 + 1/a_2 = \tfrac{1}{12}a.$$

A rigorous solution of the equations $G_{mn} = -8\pi E_{mn}$ for the electric field is given by

$$\lambda = \log\{r^2/(r_1 r_2)\}, \quad \rho = -\nu = \log(1 - a^2/r^2),$$

where $$2r = r_1 + r_2.$$

A Simple Proof of Jacobi's Transformation

W. L. FERRAR.

If $\mu = \cos\theta$, the formula in question is

$$\frac{d^{n-1}\sin^{2n-1}\theta}{d\mu^{n-1}} = (-1)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}\sin n\theta. \qquad (1)$$

In view of the complicated algebraic work involved in most of the known proofs,* the following may be of interest.

We have

$$\frac{d\sin^3\theta}{d\mu} = -\tfrac{3}{2}\sin 2\theta. \qquad (2)$$

Assume now that (1) is true for a particular value of n. Then writing

$$K = (-1)^{n-1}\frac{1.3.5\ldots(2n-1)}{n},$$

we have

$$\frac{d^n\sin^{2n-1}\theta}{d\mu^n} = -K\frac{n\cos n\theta}{\sin\theta},$$

$$\frac{d^{n+1}\sin^{2n-1}\theta}{d\mu^{n+1}} = -K\frac{n^2\sin n\theta\sin\theta + n\cos n\theta\cos\theta}{\sin^3\theta}.$$

Hence

$$\frac{d^{n+1}\sin^{2n+1}\theta}{d\mu^{n+1}} = \frac{d^{n+1}\{(1-\mu^2)\sin^{2n-1}\theta\}}{d\mu^{n+1}}$$

$$= K\{-n^2\sin n\theta - n\cos n\theta\cot\theta + 2n(n+1)\cos n\theta\cot\theta$$

$$-n(n+1)\sin n\theta\}$$

$$= Kn(2n+1)\cos(n+1)\theta/\sin\theta.$$

Integrating with regard to μ,

$$\frac{d^n\sin^{2n+1}\theta}{d\mu^n} = (-1)^n\frac{1.3.5\ldots(2n+1)}{n+1}\sin(n+1)\theta + C. \qquad (3)$$

The first two terms of (3) are odd functions and so $C = 0$.

Thus we have a proof by induction of an extremely simple character.

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* G. N. Watson, *Theory of Bessel Functions*, 26–28.

A Relation between the Incomplete Beta and Gamma Functions

RICHARD G. COOKE.

The Incomplete Beta Function

$$B(m, n, t) \equiv \int_0^t u^{m-1}(1-u)^{n-1}\,du \quad (0 < t < 1),$$

and the Incomplete Gamma Function

$$\Gamma(n, t) \equiv \int_0^t e^{-u}\,u^{n-1}\,du \quad (t > 0),$$

are employed in statistics; the latter has been tabulated recently.

No simple relation such as connects the Complete Functions appears to exist, and there does not seem to be a published relationship of any kind between the two functions.*

We have the formula†

$$(-1)^n(e^{-x}-1)^n = x^n - \frac{\Delta^n 0^{n+1}}{(n+1)!}\,x^{n+1} + \frac{\Delta^n 0^{n+2}}{(n+2)!}\,x^{n+2} - \ldots,$$

where n is a positive integer.

The function $(e^{-x}-1)^n$ is an integral function, so that the series is uniformly convergent over any finite interval, and we may multiply by e^{-mx} $(m > 0)$ and integrate term by term, the resulting series being uniformly convergent over any finite range.

We obtain the formula

$$B(m, n+1) - B(m, n+1, e^{-z}) = \sum_{s=1}^{\infty} (-1)^{s-1}\,\frac{\Delta^n 0^{n+s-1}}{(n+s-1)!} \cdot \frac{\Gamma(n+s, mz)}{m^{n+s}},$$

on integrating between limits 0 and z, n being a positive integer, and $m > 0$.

As a particular case let $z \to \infty$. We know that, given a power series convergent for all values of x, we may multiply by e^{-mx} and integrate between 0 and ∞, provided only the resulting series is convergent.‡ We thus obtain the formula

$$B(m, n+1) = \frac{\Gamma(n+1)}{m^{n+1}} - \frac{\Delta^n 0^{n+1}}{m^{n+2}} + \frac{\Delta^n 0^{n+2}}{m^{n+3}} - \ldots.$$

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* See "The Numerical Evaluation of the Incomplete Beta Function", by H. E. Soper, *Tracts for Computers*, No. 7, Camb. Univ. Press, which, however, deals with the problem from the purely practical point of view of obtaining a formula for $B(m, n, t)$ suitable for computation.

† Boole, *Finite Differences*, 3 ed., 24.

‡ See G. H. Hardy, "Further researches in the theory of divergent series and integrals", *Trans. Camb. Phil. Soc.*, 21 (1908), 1–48 (5).

Now $$\mathrm{B}(m, n+1) = \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$$

is a rational function in $x = 1/m$, since n is a positive integer; it may be written

$$\frac{A m^{-(n+1)}}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\ldots\left(1+\frac{n}{m}\right)}$$

or $$\frac{A x^{n+1}}{(1+x)(1+2x)\ldots(1+nx)},$$

where A is independent of x. Of the poles of this function, the nearest to the origin is $x = -1/n$. The series is consequently convergent if $|x| < 1/n$; *i.e.* if $m > n$.

Thursday, March 13th, 1924.

Prof. A. E. JOLLIFFE, Vice-President, in the Chair.

There were present twenty-one members and two visitors.

The Minutes of the last meeting were read and confirmed.

Mr. V. D. Gokhale was elected a member of the Society.

Mr. A. Oppenheim and Mr. O. G. Sutton were nominated for election.

The following gentlemen were nominated by the Council for honorary membership:—Prof. L. Bianchi, Prof. A. Einstein, M. Jacques Hadamard, Prof. E. Landau, M. Henri Lebesgue, Prof. T. Levi-Civita, Prof. L. Prandtl, and Prof. A. Sommerfeld.

Mr. H. E. J. Curzon read a paper entitled "Bipolar Solutions of Einstein's Gravitation Equations."

Prof. A. E. Jolliffe read a paper entitled "Some Properties of Syzygetic Cubics."

Prof. G. N. Watson communicated a paper by Dr. B. von Kerékjártó, "On the Variations of the Tangent of a Simple Closed Curve."

The following papers were communicated by title from the Chair:—

*A Certain Series of Bessel Functions: T. J. I'A. Bromwich.

*Functions Discontinuous for Rational Values of the Variable: J. L. Burchnall and T. W. Chaundy.

Extensions of Abel's and Tauber's Theorems: A. Kienast (communicated by G. H. Hardy).

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* To be published in Vol. 24.

*Sur les formules d'inversion de Fourier et de Hankel: M. Plancherel (communicated by G. H. Hardy).

Sur une inégalité de M. Littlewood dans la théorie des fonctions: F. Riesz.

*Conjugate Trigonometrical Integrals: E. C. Titchmarsh.

On Pseudo-orthogonal Systems of Functions arising as Solutions of an Integral Equation: B. M. Wilson.

NOTES AND ABSTRACTS.

Functions Discontinuous for Rational Values of the Variable

J. L. BURCHNALL and T. W. CHAUNDY.

If x, lying between 0 and 1, is expressed as an arithmetical continued fraction

$$x = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\ldots\,\frac{1}{a_n+}\,\ldots,$$

the function $y(x)$ defined by

$$y = \frac{1}{1+a_1+}\,\frac{1}{1+a_2+}\,\ldots\,\frac{1}{1+a_n+}\,\ldots$$

is discontinuous for all rational values of x in (0, 1) and continuous for all irrational values. The theory of functions of this kind is explored, and an example is added of a function continuous for all transcendental values of its argument, but discontinuous for all algebraic values.

Some Properties of Syzygetic Cubics

A. E. JOLLIFFE.

The plain cubic curves, whose equations are obtained by varying k in the equation $x^3+y^3+z^3+6kxyz = 0$, are said to form a syzygy.

Another syzygy of class-cubics is obtained by varying k in the tangential equation $k(l^3+m^3+n^3) = 3lmn$.

* To be published in Vol. 24.

Typical members of the two syzygies are referred to as "the cubic k of the point syzygy" and "the cubic k of the tangential syzygy."

The cubic k of the point syzygy meets an arbitrary straight line in three points, and the tangents at these points meet the cubic again in three points on a straight line, referred to as the satellite of the first straight line with respect to the cubic. The intersection of a straight line with its satellite is referred to as the satellite point of the line with respect to the cubic.

If the points of the triad in which a straight line meets the cubic k of the point syzygy are represented by the roots of a cubic equation, there are two points on the straight line represented by the roots of the Hessian of this cubic equation. They are the intersections of the straight line with its polo-conic with respect to the cubic k, that is to say the conic which is the envelope of the polar lines of points on the line and the locus of points whose polar conics touch the line. They are referred to as the Hessian points of the line with respect to the cubic k. The Hessian points and the satellite point of a line with respect to a cubic k of the point syzygy form a triad through which a single cubic of the point syzygy can be drawn.

A straight line is said to meet a cubic k of the point syzygy in a triad apolar to the cubic k' of the point syzygy when the dipolar with respect to k' of any two of the points coincides with the tangent to k at the third point. The envelope of such a straight line is that cubic of the tangential syzygy which has for its equation

$$(2k+k')(l^3+m^3+n^3)+3(1-4k^2k')lmn = 0.$$

The following are some properties of the two syzygies:—

(1) If a straight line is a tangent to the cubic k of the tangential syzygy, the polar line of its point of contact with respect to the cubic k_1 of the point syzygy is its satellite with respect to the cubic k_2 of the point syzygy, where

$$4kk_1k_2+2k^2(k_1+k_2)+1 = 0.$$

In this relation k_1 and k_2 can be interchanged, the relation being involutory, the double elements corresponding to the two cubics of the point syzygy which have the same Cayleyan as the cubic k.

A particular case is that the satellite with respect to the cubic k of a tangent to the cubic k of the tangential syzygy is the polar line of its point of contact with respect to the Hessian of the cubic k of the point syzygy.

(2) If a straight line meets the cubic k_1 of the point syzygy in a triad apolar to the cubic k, the polar line with respect to k_1 of its point of contact with its envelope is its satellite with respect to the cubic k_2, where

$$4kk_1k_2+2k^2(k_1+k_2)+1=0,$$

the same relation as above.

(3) If a straight line meets the cubic k in a triad apolar to the cubic k', its satellite with respect to k is the dipolar with respect to k' of its Hessian points with respect to k.

There is an apparent exception to this when the straight line is a tangent to the Cayleyan.

(4) If a straight line meets the cubic k in a triad apolar to k', its point of contact with its envelope and its satellite point with respect to k' are harmonic conjugates with respect to the Hessian points of the line with respect to k.

(5) If C_1 and C_2 are the equations of the cubics k_1 and k_2 of the point syzygy, the equation of the cubic $\lambda C_1+\mu C_2=0$ is

$$\lambda^4K_1+\lambda^3\mu\Theta_1+\lambda^2\mu^2\Phi+\lambda\mu^3\Theta^2+\mu^4K_2=0,$$

where $K_1=0$ and $K_2=0$ are the tangential equations of $C_1=0$ and $C_2=0$ and Θ_1, Φ, Θ_2 are tangential forms of the sixth degree.

$\Theta_1=0$ is the envelope of straight lines such that the triad formed by their Hessian points and satellite point with respect to $C_1=0$ lies on $C_2=0$. Except for special values of k_1 and k_2, it degenerates only when $C_2=0$ is the Hessian of $C_1=0$, and then becomes the square of the Cayleyan of $C_1=0$.

$\Phi=0$ is the envelope of straight lines such that their Hessian points with respect to the cubics k_1 and k_2 are harmonic conjugates, and in this case the two pairs of Hessian points and the two satellite points are in involution. If the cubics k_1 and k_2 have the same Cayleyan as the cubic k, this envelope degenerates into this Cayleyan and the cubic k of the tangential syzygy.

(6) A curious analytical result is the following :—If the three cubics, k, k_1, k_2, have the same Hessian θ, and θ, θ_1, θ_2 have the same Hessian, the cubics k, k_1, k_2 have the same 36 common tangents. Of these, 18

touch θ_1 and the cubic θ_2 of the tangential syzygy. The other 18 touch θ_2 and the cubic θ_1 of the tangential syzygy.

Of the four cubics which touch an arbitrary real straight line, two are real and two are imaginary. Three cubics which have the same real cubic for Hessian are either all real or one real and two imaginary, so that two real cubics which have the same Hessian can have no real common tangent.

Sur une inégalité de M. Littlewood dans la théorie des fonctions

FRÉDÉRIC RIESZ.

1. Les recherches récentes se rattachant au théorème de M. Picard ont fait valoir l'importance de certains " principes " généraux, permettant de ramener l'étude de classes entières de fonctions à celui d'une seule fonction " majorante." Tel est par exemple le lemme bien connu de Schwarz et sa généralisation par représentation conforme, due a M. Lindelöf.* Dans une note préliminaire, M. Littlewood vient d'ajouter à ces principes une inégalité nouvelle de haute importance, comprenant le principe de M. Lindelöf comme cas limite.

Posons

$$M_\lambda(\rho) = M_\lambda(\rho, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\phi})|^\lambda \, d\phi\right\}^{1/\lambda} \quad (\lambda > 0),$$

$$M_\infty(\rho) = \operatorname*{Max}_{|z|=\rho} |f(z)|,$$

$$M_0(\rho) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\phi})|\, d\phi\right\},$$

et soient $f(z)$ et $f^*(z)$ deux fonctions holomorphes pour $|z| < R$, liées par la relation $f^*(z) = f[w(z)]$, où $w(z)$ est une fonction holomorphe telle que $|w(z)| \leqslant |z|$ pour $|z| < R$. Alors on a, d'après M. Littlewood,

(L) $$M_\lambda(\rho, f^*) \leqslant M_\lambda(\rho, f) \quad (\rho < R, \quad 0 \leqslant \lambda \leqslant \infty).\dagger$$

C'est le cas limite $\lambda = \infty$ qui correspond au principe de M. Lindelöf.

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* E. Lindelöf, " Mémoire sur certaines inégalités dans la théorie des fonctions monogènes ", *Acta Soc. Sci. Fenn.*, 35 (7) (1908), 3–35.

† J. E. Littlewood, " On inequalities in the theory of functions ", *Proc. London Math. Soc.* (2), 23 (1923) (*Records of Proceedings at Meetings*, November 8th, 1923).

2. L'inégalité (L) peut être aussi considérée comme la généralisation d'un théorème important de M. Hardy, d'après lequel *la valeur moyenne* $M_\lambda(\rho, f)$ *est une fonction croissante de* ρ.* En effet, pour voir que

$$M_\lambda(\rho_1, f) \leqslant M_\lambda(\rho, f)$$

lorsque $\rho_1 < \rho$, on n'aura qu'à poser $w(z) = \rho_1 z/\rho$. Or, dans une note précédente, j'ai montré que le théorème de M. Hardy est compris dans un théorème relatif à une classe étendue de fonctions de deux ou de plusieurs variables réelles, parmi lesquelles on trouve, par exemple, les puissances $|u|^\lambda$ $(\lambda \geqslant 1)$ du module d'une fonction harmonique $u(x, y, \ldots)$ quelconque.† Il est donc bien naturel de se demander si l'inégalité de M. Littlewood rentre de sa part dans cet ordre d'idées qui nous a fait sortir du cadre de la Théorie des fonctions proprement dite. Nous allons répondre à cette question en déduisant l'inégalité (L) d'un principe extrêmement simple. Ce "*principe de la fonction harmonique majorante*", dont je me suis servi dans la note citée, fut aussi appliqué indépendemment, peu après, par MM. F. et R. Nevanlinna dans un mémoire très intéressant, concernant l'allure des fonctions analytiques au voisinage des lignes et des points singuliers.‡

3. En essence, il ne s'agira que du fait presque évident que voici. Soit $g(x, y)$ une fonction réelle et continue définie pour $x^2+y^2 < R^2$ et soit $w(z) = w(x+iy) = u(x, y)+iv(x, y)$ une fonction holomorphe à l'intérieur de cette circonférence et telle que $|w(z)| \leqslant |z|$. Nous pouvons donc former la fonction

$$g^*(x, y) = g[u(x, y),\ v(x, y)].$$

Supposons maintenant que, pour une certaine valeur de $\rho < R$, *la fonction* $g(x, y)$ *jouisse de la propriété suivante. En désignant par* $G(x, y)$ *la fonction harmonique égale à* $g(x, y)$ *sur la circonférence*

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* G. H. Hardy, "The mean value of the modulus of an analytic function", *Proc. London Math. Soc.* (2), 14 (1915), 269-277. La seconde partie du théorème, d'après laquelle $\log M_\lambda$ est une fonction convexe de $\log \rho$, n'est pas en relation immédiate avec l'inégalité (L).

† F. Riesz, "Sur les valeurs moyennes du module des fonctions harmoniques et des fonctions analytiques", *Acta Universitatis Franc.-Jos.* (Szeged), sectio math., 1 (1922), 27-32.

‡ F. u. R. Nevanlinna, "Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie", *Acta Soc. Sci. Fenn.*, 50 (5) (1922), 3-46.

$x^2+y^2=\rho^2$, *on a, pour tout point intérieur,* $g(x, y) \leqslant G(x, y)$. Je dis que, *sous cette hypothèse, la valeur moyenne de* $g^*(x, y)$, *formée sur la circonférence considérée, est inférieure ou égale à celle de* $g(x, y)$. Pour le voir, rappelons que, lorsque $x^2+y^2 \leqslant \rho^2$, on a aussi $u^2+v^2 \leqslant \rho^2$; nous pouvons donc former la fonction auxiliaire

$$G^*(x, y) = G[u(x, y),\ v(x, y)],$$

et nous aurons évidemment, dans le cercle considéré, $g^* \leqslant G^*$.

Alors il vient

$$\frac{1}{2\pi}\int_0^{2\pi} g^*(\rho\cos\phi,\ \rho\sin\phi)d\phi \leqslant \frac{1}{2\pi}\int_0^{2\pi} G^*(\rho\cos\phi,\ \rho\sin\phi)d\phi = G^*(0, 0)$$

$$= G(0, 0) = \frac{1}{2\pi}\int_0^{2\pi} G(\rho\cos\phi,\ \rho\sin\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} g(\rho\cos\phi,\ \rho\sin\phi)d\phi,$$

c'est-à-dire l'inégalité que nous avions en vue.

Observons encore que l'hypothèse de la continuité de $g(x, y)$ n'est nullement essentielle et que, en particulier, notre conclusion reste vraie lorsque la fonction $g(x, y)$ devient infinie négative en certains points isolés.

4. L'hypothèse faite sur la fonction $g(x, y)$ et que l'on pourra appeler *l'hypothèse de la fonction harmonique majorante*, est remplie pour une classe très générale de fonctions, et cela non seulement pour les domaines circulaires que nous venons de considérer, mais pour un domaine quelconque, si l'on sait déjà que la fonction harmonique correspondante $G(x, y)$ existe. Une *condition nécessaire et suffisante* se trouve établie dans ma note citée plus haut ; cette condition consiste en ce que, *pour tout point* x_0, y_0 *du domaine, la valeur* $g(x_0, y_0)$ *soit inférieure ou égale à la valeur moyenne de* $g(x, y)$ *formée sur la circonférence* $(x-x_0)^2+(y-y_0)^2=r^2$ *et cela pour toute valeur suffisamment petite de* r. Pour arriver à l'inégalité de M. Littlewood, il n'est point nécessaire de recourir à un théorème d'une telle généralité. En effet, considérons la fonction

$$g(x, y) = |f(z)|^\lambda \quad (z = x+iy,\ \lambda > 0),$$

$f(z)$ étant supposée holomorphe pour $|z| < R$. Pour voir que, dans ce cas, l'hypothèse du § 3 est remplie quelque soit $\rho < R$, on peut raisonner par exemple comme il suit. Tout d'abord, sans restreindre la généralité, on peut supposer que $f(z)$ ne s'annule pas pour $|z| < \rho$. En effet, dans le cas contraire, on n'aura qu'à poser $f(z) = h(z)f_1(z)$, $h(z)$ étant la fonction

rationnelle bien connue admettant, pour $|z|<\rho$, les mêmes zéros que $f(z)$, compté chacun avec sa multiplicité, et telle que, pour $|z|=\rho$, on a $|h(z)|=1$. Alors on aura, pour $|z|<\rho$, $|h(z)|<1$ et par conséquent $|f(z)|<|f_1(z)|$, tandis que, pour $|z|=\rho$, on aura $|f(z)|=|f_1(z)|$; enfin, la fonction $f_1(z)$ ne s'annulera pas pour $|z|<\rho$.

Supposons donc que $f(z)\neq 0$ pour $|z|<\rho$; alors l'une quelconque des déterminations de $[f(z)]^\lambda$ étant holomorphe à l'intérieur du cercle $|z|=\rho$ et continue sur cette circonférence, elle peut être exprimée par ses valeurs periphériques sous la forme de l'intégrale de Poisson, tandis que la même expression, formée avec les modules de ces valeurs periphériques, fournira la fonction $G(x,y)$ correspondante. Il s'ensuit immédiatement que $|f(z)|^\lambda \leqslant G(x,y)$ pour $|z|=|x+iy|<\rho$, et l'inégalité (L) est démontrée pour $0<\lambda<\infty$. Le même raisonnement tient dans le cas $\lambda=0$, c'est-à-dire pour $\log|f(z)|$. Le cas $\lambda=\infty$ est évident.

5. Ajoutons encore qu'une inégalité analogue tient pour d'autres valeurs moyennes importantes, savoir pour celles de $\log^+|f^*(z)|$ et $\log^-|f^*(z)|$, ces symboles désignant les fonctions égales à ces logarithmes lorsqu'ils sont positifs et égales à zéro ailleurs. Pour obtenir cette inégalité, il suffit de remarquer que, si l'hypothèse du § 3 est satisfaite pour chacune des deux fonctions $g_1(x,y)$ et $g_2(x,y)$, elle le sera aussi pour la fonction $g(x,y)$ égale, en tout point x_0, y_0, à la plus grande des valeurs $g_1(x_0,y_0)$, $g_2(x_0,y_0)$, et d'appliquer cette remarque aux fonctions $g_1=\log|f(z)|$ et $g_2=0$.

On the Variation of the Tangent of a Simple Closed Curve

B. VON KERÉKJÁRTÓ.

In his paper "A Problem of Analysis Situs",* Prof. Watson proves the following theorem:

Let j be a simple closed curve in the plane which has a continuously turning tangent at every point P; then the variation of the direction of

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* *Proc. London Math. Soc.* (2), 15 (1916), 227-242.

the tangent as the point P describes the curve j once in the positive sense is equal to 2π.

The above theorem follows immediately from the following proposition*:

Let t be a continuous (1, 1) *transformation of the curve j into itself, without invariant point; and let us denote by P' the image of the point P. Then the variation of the vector* $\overrightarrow{PP'}$ *as P describes j once in the positive sense is equal to* 2π.

If j is a circle, the proposition is obvious. In the general case, we take a circle k which contains j in its interior, and we transform the region bounded by k and j into a ring-shaped surface, bounded by two concentric circles k' and j', the transformation being continuous and (1, 1), including the boundary.† In the ring-shaped surface, we deform continuously the circle j' into k' so that every point moves along a radius vector and at every moment of the deformation the image of j' is a concentric circle. To this deformation corresponds a continuous deformation of the curve j into the circle k, such that any two distinct points of j remain distinct during the whole deformation. By this deformation, the total variation of the vector $\overrightarrow{PP'}$ changes continuously, and on the other hand it is equal to an integer multiple of 2π, so that it is constant. This is equal, for the terminal position, that is to say for the circle k, to 2π, so that for the initial position, that is for the curve j, it is also equal to 2π.

Conjugate Trigonometrical Integrals

E. C. TITCHMARSH.

The integral $$\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty \sin u(x-t) f(t)\, dt$$

is called the integral conjugate to Fourier's integral

$$\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty \cos u(x-t) f(t)\, dt.$$

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* B. von Kerékjártó, *Vorlesungen über Topologie*, 1 (Berlin, 1923), 86.

† The possibility of such a transformation follows from the theory of conformal representation; for a topological proof, see, for instance, Kerékjártó, *loc. cit.*, 71–72.

The convergence and summability of conjugate integrals for various types of functions is discussed. Applications are then made to the theory of the reciprocal formulæ

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt, \quad f(x) = -\frac{1}{\pi}\int_0^\infty \frac{g(x+t)-g(x-t)}{t}\,dt.$$

Cases in which $f(x)$ satisfies a Lipschitz condition, and in which the integrals are not absolutely convergent, are discussed particularly.

Principal Value Fourier Series.

E. C. TITCHMARSH.

Let $f(x)$ be a function such that the integrals

$$(1) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt,$$

exist for all values of n as Lebesgue integrals except in the neighbourhood of a finite number of points, where they may be defined as "principal values" in Cauchy's sense. Then the series

$$(2) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

will be called the Principal Value Fourier Series associated with the function $f(x)$.

Examples of such series are

$$\sum_{n=2}^{\infty}\frac{\sin nx}{\log n}, \qquad \sum_{n=0}^{\infty}\sin(2n+1)x.$$

The former is the example usually given of a convergent trigonometrical series which is not a Fourier series in the ordinary sense. The latter is the series associated with the function $\frac{1}{2}\operatorname{cosec} x$, and it is summable $(C, 1)$ to the sum $\frac{1}{2}\operatorname{cosec} x$ for all values of x except $x = 0$.

In studying the properties of these series it is clearly sufficient to suppose that there is only one point, say $x = \xi$, where the principal value need be taken. Then if the given function is $f(x)$ and $\phi(u) = f(\xi+u)$,

the integrals (1) exist if the limits

$$\lim_{\epsilon\to 0}\int_\epsilon^\delta \big(\phi(u)+\phi(-u)\big)\,du, \qquad \lim_{\epsilon\to 0}\int_\epsilon^\delta u\big(\phi(u)-\phi(-u)\big)\,du$$

exist. We shall consider here only the special case in which the integrals

$$\int_0^\delta |\,\phi(u)+\phi(-u)\,|\,du, \qquad \int_0^\delta u\,|\,\phi(u)-\phi(-u)\,|\,du \tag{3}$$

exist. *In this case we have* $a_n = o(n)$, $b_n = o(n)$, *and no more than this is generally true.* The first part of the theorem is easily proved directly, and the second may be established by means of an example. It is a particular case of the following theorem:—*If* $h(t)$ *increases steadily to infinity, and* $t\,h(t)$ *is non-increasing, as* $t\to 0$, *and* $f(t)$ *is any integrable function, then*

$$\int_0^1 \sin nt\,h(t)f(t)\,dt = o\big(h(1/n)\big);$$

and, however slowly $\lambda(n)\to 0$, *there is an integrable function* $f(t)$ *such that*

$$\int_0^1 \sin nt\,h(t)f(t)\,dt \neq o\{\lambda(n)\,h(1/n)\}.$$

The function last mentioned may be taken equal to $n_p^2\,\lambda(n_p)$ in the intervals

$$(n_p+1)^{-1} < t < n_p^{-1} \quad (p = 1, 2, \ldots),$$

and zero elsewhere; and the result follows if $n_p\to\infty$ sufficiently rapidly as $p\to\infty$.

It is plain, from the example already given of the series associated with the function $\frac{1}{2}$ cosec x, that one singularity may make the series divergent in general, so that the convergence of the series (2) does not depend on the nature of $f(t)$ in the neighbourhood of $t = x$ only. Consider the series for a value $x\neq\xi$. Let $0<\delta<|\,\xi-x\,|$, and let

$$f_1(t) = f(t) \quad (\xi-\delta < t < \xi+\delta),$$

$f_1(t) = 0$ elsewhere, and $f(t) = f_1(t)+f_2(t)$. Then we consider separately the series associated with $f_1(t)$ and $f_2(t)$. The latter is simply an ordinary Fourier series, whose properties may be supposed known. The former has the property that *it converges to zero if its coefficients tend to zero.* It follows that *the convergence of the series associated with* $f(t$

depends only on the nature of the function in the neighbourhood of $t = x$, *provided its coefficients tend to zero.*

The condition that this should be so is

$$\lim_{n\to\infty} \int_0^\delta \big(\phi(u)-\phi(-u)\big) \sin nu\, du = 0.$$

Since we have assumed the existence of the second of the integrals (3), the problem of this limit is substantially the same as the convergence problem for ordinary Fourier series.

When we turn from convergence to summability $(C, 1)$ the singularity ceases to make any difference, since the series associated with $f_1(t)$ is summable $(C, 1)$ to the sum zero. Hence the theory of summability for ordinary Fourier series may be extended at once to the series considered here.

On Pseudo-orthogonal Systems of Functions arising as Solutions of an Integral Equation.

B. M. WILSON.

We define a system of functions $\phi'(s)$ as those solutions of the non-homogeneous equation,

$$\phi(s) = f(s)+\lambda \int_0^1 k(s,\ t)\, \phi(t)\, dt,$$

in which the nucleus $k(s, t)$ is symmetric, and either (a) closed or (b) positive-definite, which satisfy the supplementary condition

$$w(\lambda) = \int_0^1 f(s)\, \phi(s)\, ds = a\lambda+\beta,$$

where a, β are given real constants. These functions $\phi'(s)$ satisfy the non-orthogonal relation

$$\int_0^1 \phi_m(s)\, \phi_n(s)\, ds = a \quad (m \neq n).$$

In the present paper the case when β is zero is alone considered ; the cases arising when β has a value different from zero will be studied in a second paper, to be presented to the Society shortly.

When β is zero the functions $\phi'(s)$ are shown to satisfy a certain homogeneous equation

$$\phi(s) = \lambda \int_0^1 K(s,\ t)\, \phi(t)\, dt$$

with unsymmetrical nucleus. This equation however has, in general, additional solutions $\phi''(s)$, and if the functions $\phi'(s)$, $\phi''(s)$ are grouped together in a single system $\phi(s)$, the functions $\phi(s)$ satisfy the pseudo-orthogonal condition

$$\int_0^1 \phi_m(s)\,\phi_n(s)\,ds = 0 \text{ or } \alpha \quad (m \neq n).$$

The nucleus $K(s, t)$ is shown (1) subject to a restriction on α when $k(s, t)$ is closed, (2) always when $k(s, t)$ is positive-definite, to be symmetrisable; and the reality and distribution of the characteristic values of λ is discussed, and the possibility of expressing an "arbitrary" function by means of the functions $\phi(s)$ is deduced from general theorems in the theory of integral equations with symmetrisable nuclei.

Thursday, April 24th, 1924.

Dr. W. H. YOUNG, President, in the Chair.

There were present twenty-two members.

The Minutes of the last meeting were read and confirmed.

Prof. L. Bianchi (Pisa), Prof. A. Einstein (Berlin), M. Jacques Hadamard (Paris), Prof. E. Landau (Göttingen), M. Henri Lebesgue (Paris), Prof. T. Levi-Civita (Rome), Prof. L. Prandtl (Göttingen), and Prof. A. Sommerfeld (Munich) were elected honorary members of the Society.

Mr. A. Oppenheim and Mr. O. G. Sutton were elected members of the Society.

Mr. J. B. Bretherton and Dr. F. D. Murnaghan were nominated for election.

Mr. A. Buxton was admitted into the Society.

Prof. H. Hilton read a paper, written in collaboration with Miss M. E. Green, entitled "On a Type of Plane Rational Quartic Curve".

Prof. M. J. M. Hill, Miss F. M. Wood, and Mr. R. G. Cooke read a paper entitled "Approximate Determination of the Portion of the Unit Length which is covered by the Borel Intervals which it contains."

Mr. J. Brill read a paper entitled "On the Treatment of Rotation in the Special Theory of Relativity".

The following papers were communicated by title from the Chair:—

Note on a theorem of Hilbert concerning series of positive terms: G. H. Hardy.

Properties of integrals of fractional order: G. H. Hardy and J. E. Littlewood.

NOTES AND ABSTRACTS.

Note on a theorem of Hilbert concerning series of positive terms

G. H. HARDY.

It was first proved by Hilbert that *if* Σa_m^2 *and* Σb_n^2 *are convergent then*

$$\Sigma\Sigma \frac{a_m b_n}{m+n}$$

is convergent. Of the considerable number of published proofs of this theorem, the simplest is that (due partly to myself and partly to Marcel Riesz) which I gave in the *Math. Zeitschrift* in 1920.*

Dr. Riesz has pointed out to me that, in this note, I proved (though without enunciating the result as an explicit theorem) a more general proposition, viz. that (1) *is convergent whenever*

$$\Sigma a_m^p, \quad \Sigma b_n^q \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

are convergent. For I proved (i) that (1) is convergent whenever

$$\Sigma a_m \frac{B_m}{m}, \quad \Sigma b_n \frac{A_n}{n},$$

where $A_m = a_1+a_2+\ldots+a_m, \quad B_m = b_1+b_2+\ldots+b_m,$

are convergent, and (ii) that

$$\Sigma\left(\frac{A_n}{n}\right)^p$$

is convergent if Σa_n^p is convergent; and the theorem stated follows at once from these two propositions. It may also be proved in other ways, for example by the elegant, though less elementary, method of Schur,† referred to in my former note.

All these theorems are capable of wide generalization. For example we have the following theorem. *Suppose that the series*

$$\Sigma(a_{n_1}^{(1)})^{p_1}, \quad \Sigma(a_{n_2}^{(2)})^{p_2}, \quad \ldots, \quad \Sigma(a_{n_r}^{(r)})^{p_r},$$

where $p_1 > 1, \ p_2 > 1, \ \ldots, \ p_r > 1, \quad \varpi = \Sigma\frac{1}{p} \geqslant 1,$

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* G. H. Hardy, "Note on a theorem of Hilbert", *Math. Zeitschrift*, 6 (1920), 314–317.

† I. Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen", *Journal für Math.*, 140 (1912), 1–28 (23–28).

are convergent. Then the series

$$\Sigma\Sigma\ldots\Sigma\frac{a_{n_1}^{(1)}a_{n_2}^{(2)}\ldots a_{n_r}^{(r)}}{(n_1+n_2+\ldots+n_r)^{r-\varpi}}$$

is convergent. This was proved by Wiener* in the special case in which

$$p_1=p_2=\ldots=p_r=r,\quad \varpi=1.$$

I should add that Riesz, in the course of his beautiful investigations concerning conjugate trigonometrical series,† has gone further in another direction, and has proved (generalizing another result of Hilbert) that, if Σa_m^p and Σb_n^q converge, and $1/p+1/q=1$, then

$$\sum_{m\neq n}\frac{a_m b_n}{m-n}$$

is convergent. In this case, however, the convergence is not necessarily absolute, and the theorem lies essentially deeper.

The Relation between Two Quadrics which are such that the Edges of an Octahedron or Icosahedron inscribed in one are Tangents to the other

S. L. GREEN.

This paper is the result of an attempt at establishing for two quadrics a theorem analogous to that of Poncelet for two conics which have inscribed and circumscribed polygons.

The names "octahedron" and "icosahedron" will be applied to figures analogous to the regular figures known by those names; the former having 6 vertices and 8 triangular faces such that the 3 diagonals are concurrent, and the latter having 12 vertices and 20 triangular faces such that the 5 vertices, which are the other extremities of the edges that meet in any one vertex, are coplanar.

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* F. W. Wiener, "Elementarer Beweis eines Reihensatzes von Herrn Hilbert", *Math. Annalen*, 68 (1910), 361–366.

† See his note "Sur les séries trigonométriques conjuguées", printed in the January *Records*, xxiv.

Two tangent cones to the same quadric T intersect in two conics, which, if distinct, have two and only two common points. If there be an octahedron all of whose edges touch T and whose vertices lie on a quadric S, then it is clear that the conics of intersection of the tangent cones to T drawn from a pair of opposite vertices must coincide, and that, in consequence, each cone must cut S in a plane section. If there be an icosahedron inscribed in S and with its edges touching T, then the tangent cone to T drawn from a vertex of the icosahedron cuts S in a plane section.

Let the quadrics be given by the equations

$$S \equiv x^2+y^2+z^2+w^2 = 0, \qquad T \equiv ax^2+by^2+cz^2+dw^2 = 0.$$

The tangent cone to T whose vertex is the point (X, Y, Z, W) on S is given by

$$\Sigma ax^2(bY^2+cZ^2+dW^2)-2\Sigma abXYxy = 0.$$

The equation of the cone whose vertex is the point (X, Y, Z, W), and whose base is the section of S by the plane $lx+my+nz+pw = 0$, can be written

$$\Sigma x^2(-lX+mY+nZ+pW)-2\Sigma xy(mX+lY) = 0.$$

By identifying these two cones we deduce that 3 of the 4 numbers a, b, c, d must be equal, *i.e.* the quadrics have ring-contact and so are projectively equivalent to two concentric spheres for which the polyhedra would obviously be regular. Hence we deduce the following properties of the polyhedra inscribed in, and circumscribed to, two quadrics :—

(1) If there be one such figure, then there is an infinite number of such figures.

(2) Their diagonals are concurrent in a vertex of the common self-conjugate tetrahedron of the quadrics.

(3) The faces of the polyhedra envelope a third quadric having ring-contact with the two quadrics.

By reciprocation similar results may be obtained for the figures which are the analogues of the cube and the dodecahedron.

On a theorem of Carleman

O. G. SUTTON.

Carleman has recently* given a proof of the following theorem, which is a generalization of one given previously by Hardy and Littlewood.†

THEOREM.—*If* $2\phi(t) = f(x+t)+f(x-t)-2f(x)$,

and if, for some particular x,

$$\int_0^\delta |\phi(t)|\,dt = o(\delta), \tag{1}$$

$$\int_0^\delta \phi^2(t)\,dt = O(\delta), \tag{2}$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^k = 0$$

for every positive number k, *where* s_ν *denotes the* ν*-th partial sum of the Fourier series of* $f(x)$.

These conditions are satisfied almost everywhere by a function of summable square. The object of the present note is to establish a more general form of the theorem, with a simpler proof.

THEOREM 1.—*If, with the above notation,*

$$\int_0^\delta |\phi(t)|^q\,dt = o(\delta) \quad (0 < q < 1), \tag{3}$$

$$\int_0^\delta |\phi(t)|^p\,dt = O(\delta) \quad (1 < p < 2), \tag{4}$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^{p/(p-1)} = 0.$$

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* T. Carleman, "A theorem concerning Fourier series", *Proc. London Math. Soc.* (2), 21 (1923), 483.

† G. H. Hardy and J. E. Littlewood, "Sur la série de Fourier d'une fonction à carré sommable", *Comptes Rendus*, April 28th, 1913.

We may remark with regard to these conditions that individually (3) and (4) are more general than (1) and (2). Taken together, however, they imply (1) but not (2), so that the generalization is due to the replacement of (2) by (4). If (4) is satisfied by a particular p it is satisfied by any smaller p greater than 1, and $p/(p-1)$ may be made as large as we please by taking p near to 1. Hence Theorem 1 includes Carleman's theorem.

We require Hausdorff's completed form of Parseval's theorem,* viz. that if a_k be the typical Fourier constant of a function $f(x)$ whose r-th power is summable, and if $1/r+1/s = 1$, $s > r$, then

$$\left[\sum_{k=0}^{\infty} |a_k|^s\right]^{1/s} \leqslant \left[\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^r\right]^{1/r}.$$

We take, as in Carleman's proof, the point considered as $x = 0$, and find that

$$(5)\quad \left\{\sum_{\nu=0}^{n} |s_\nu - f(0)|^{p/(p-1)}\right\}^{(p-1)/p}$$
$$\leqslant \left\{\sum_{\nu=0}^{n} |\alpha_\nu|^{p/(p-1)}\right\}^{(p-1)/p} + \left\{\sum_{\nu=0}^{n} |\beta_\nu|^{p/(p-1)}\right\}^{(p-1)/p} + \left\{\sum_{\nu=0}^{n} |\gamma_\nu|^{p/(p-1)}\right\}^{(p-1)/p},$$

where

$$\alpha_\nu = \int_0^\delta \sin \nu t \cot \tfrac{1}{2}t\,\phi(t)\,dt, \quad \beta_\nu = \int_\delta^\pi \sin \nu t \cot \tfrac{1}{2}t\,\phi(t)\,dt, \quad \gamma_\nu = \int_0^\pi \cos \nu t\,\phi(t)\,dt.$$

Using the first mean value theorem, we obtain almost immediately

$$(6)\qquad \left\{\sum_{\nu=0}^{n} |\alpha_\nu|^{p/(p-1)}\right\}^{(p-1)/p} < n^{2-(1/p)}\delta\epsilon(\delta),$$

where $\epsilon(\delta)$ tends to zero with δ; and, since γ_ν is the Fourier constant of a function whose p-th power is summable, we have, using Hausdorff's theorem,

$$\left\{\sum_{\nu=0}^{n} |\gamma_\nu|^{p/(p-1)}\right\}^{(p-1)/p} < K_1,$$

where K_1 is a constant.

The case of β_ν is slightly more complicated. We write, with Carleman,

$$c_\nu(s) = \int_0^s \sin \nu x\,\phi_s(x)\,dx,$$

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* F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163–169.

where $\phi_s(x) = \phi(x) \quad (0 < x < s), \qquad \phi_s(x) = 0 \quad (s < x < \pi),$

$$\phi_s(-x) = -\phi_s(x).$$

Then since $c_\nu(s)$ is the typical Fourier constant of $\phi_s(x)$, whose p-th power is summable, we have, again employing Hausdorff's theorem,

$$(8) \qquad \left\{\sum_1^\infty |\, c_\nu(s)\,|^{p/(p-1)}\right\}^{(p-1)/p} < \left\{\int_0^{2\pi} |\, \phi_s(x)\,|^p\right\}^{1/p} < K_2 s^{1/p},$$

whose K_2 is a constant, by (4). It is easily deducible* that

$$\left\{\sum_1^\infty |\, \beta_\nu\,|^{p/(p-1)}\right\}^{(p-1)/p}$$

$$\leqslant \tfrac{1}{2}\cot\tfrac{1}{2}\delta\left\{\sum_1^\infty |\, c_\nu(\delta)\,|^{p/(p-1)}\right\}^{(p-1)/p} + \tfrac{1}{4}\int_\delta^\pi \operatorname{cosec}^2\tfrac{1}{2}t\left\{\sum_1^\infty |\, c_\nu(t)\,|^{p/(p-1)}\right\}^{(p-1)/p} dt.$$

From (8) we then deduce

$$(9) \qquad \left\{\sum_1^\infty |\, \beta_\nu\,|^{p/(p-1)}\right\}^{(p-1)/p} < \frac{K_3}{\delta}\delta^{1/p} + K_4\int_\delta^\pi t^{(1/p)-2}dt < K_5\delta^{(1/p)-1},$$

where K_3, K_4, K_5 are constants.

Combining (5), (6), (7), and (9), we obtain

$$\left\{\sum_{\nu=0}^n |\, s_\nu - f(0)\,|^{p/(p-1)}\right\}^{(p-1)/p} < n^{2-(1/p)}\delta\epsilon(\delta) + K_5\delta^{(1/p)-1} + K_1,$$

$$\left\{\frac{1}{n+1}\sum_{\nu=0}^n |\, s_\nu - f(0)\,|^{p/(p-1)}\right\}^{(p-1)/p} < n\delta\epsilon(\delta) + K_5(n\delta)^{(1/p)-1} + K_1 n^{(1/p)-1}.$$

Take $\delta = b/n$, b having been fixed so that

$$K_5 b^{(1/p)-1} < \tfrac{1}{2}\epsilon,$$

and then choose n large enough to make

$$b\epsilon(\delta) + K_1 n^{(1/p)-1} < \tfrac{1}{2}\epsilon.$$

The result then follows.

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* Carleman, *loc. cit.*

A modification of the above argument gives us a generalization of a well-known theorem due to Lebesgue.* If σ_ν denote the arithmetic mean of the first ν terms of the Fourier series of $f(x)$, we have :

THEOREM 2.—*If we replace, in the enunciation of Theorem* 1, *condition* (3) *by the more general condition*

$$\int_0^\delta \phi(t)\,dt = o(\delta)$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |\sigma_\nu - f(x)|^{p/(p-1)} = 0.$$

Systems of Pseudo-Orthogonal Functions arising as Solutions of an Integral Equation (Part 2)

B. M. WILSON.

Continuing the investigations of an earlier paper,† a discussion is given of the solutions of the pair of equations

$$\phi(s) = f(s) + \lambda \int_0^1 k(s, t)\phi(t)\,dt,$$

$$w(\lambda) = \int_0^1 f(s)\phi(s)\,ds = \alpha + \beta/\lambda,$$

where α and β are real, and it is now supposed that β is not zero. It is shown that if either (1) $k(s, t)$ is positive-definite and β is positive or (2) $k(s, t)$ is closed and α is negative, the problem is reducible to that treated in the earlier paper; the roots of the equation for λ are in these cases all real, and expansion-theorems in terms of the functions $\phi(s)$ may be formulated.

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* Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, 61 (1905), 251–280.

† *Supra*, xliii.

When $k(s, t)$ is closed, but neither of the above conditions is satisfied, the reduction may still be performed and the same theorems are true provided that the problem possesses a real solution $\psi(s)$ for which

$$\int_0^1 \big(\psi(s)\big)^2 ds = \gamma^2 < a.$$

It is shown that at most one real solution $\psi(s)$ can satisfy this inequality, and the inequality itself can be further analysed by considering the graph, for real values of λ, of the function $\lambda w(\lambda)$. An application of the theory developed in this paper is made to Proudman's special problem, out of which both the present paper and the earlier one arose.

In a subsequent paper, which will be presented shortly to the Society, a discussion of the solutions will be given when α and β have real values such that no real solution $\psi(s)$ satisfies the condition for the reducibility of the problem.

Thursday, May 15th, 1924.

Dr. W. H. YOUNG, President, in the Chair.

There were present twenty-one members.

The Minutes of the last meeting were read and confirmed.

Mr. J. B. Bretherton and Dr. F. D. Murnaghan were elected members of the Society.

Mr. S. Narayan was nominated for election.

Mr. F. J. W. Whipple read a paper entitled "The Distribution of Potential near two Cylindrical Conductors," and also made an informal communication on "The Conformal Representation of the part of a Plane External to a given Polygon."

Prof. G. H. Hardy made a communication concerning work by himself and Mr. J. E. Littlewood on "Properties of fractional integrals."

The following papers were communicated by title from the Chair :—

Functions arising as Solutions of an Integral Equation : B. M. Wilson.

The Lines of Electric Force due to a moving Point Charge : Maxime Kourensky.

On the Discontinuous Motion produced in an Infinite Stream by two Plane Obstacles (second paper) : E. G. C. Poole.

NOTES AND ABSTRACTS.

Note on the General Case of the Discontinuous Motion produced in an Infinite Stream by Two Plane Obstacles

E. G. C. POOLE.

Immediately on the appearance of my first paper on this problem,* my attention was drawn to an investigation of the same problem by R. Thiry.† This dissertation contains a complete study of the problem when both obstacles are normal to the stream at infinity, and covers my Cases III, IV, and the general case. The restriction of the problem to the cases of normal incidence has enabled Thiry to carry the integrations a stage further, and to calculate lengths and pressures; he also considers in detail the cases where one obstacle becomes infinitely small.

In view of Thiry's work, I have withdrawn my own solution of the general case, as I find that his discussion is more thorough than my own. It may, however, be of interest to make a few comments on the two methods.

Thiry subdivides the general case into two cases, according as the finite dead water has two or four points of inflexion in the boundary. The w-region is a plane area bounded by one semi-infinite cut and one finite cut parallel to it. The Ω-region, in the case with two inflexions, is drawn on a two-sheeted Riemann surface, with one winding point, which is doubly connected. The boundary consists of an open rectangle in the lower sheet and of an open rectangle in the upper sheet, in which part of the boundary is described three times, forming a "loop" which lies in the lower sheet. To effect the conformal representation of both regions on a rectangle, Thiry begins by solving the problem of conformal representation of a ring between two polygons, and obtains a formula involving theta-functions, which has a close analogy with the Schwarz-Christoffel formula. By applying this to the w- and Ω-regions, Thiry obtains the solution, the restriction of which to the case of normal incidence is only apparent. I had found the same result by a different

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* *Proc. London Math. Soc.* (2), 22 (1924), 425–453.

† R. Thiry, "Sur les solutions multiples des problèmes d'hydrodynamique relatifs aux mouvements glissants", *Thèse* (Gauthier-Villars, 1921). This also appeared in the *Annales Scientifiques de l'École Normale Supérieure* for 1921.

method, and had verified that it contained five independent parameters, corresponding to the five degrees of freedom of the problem.

In my own solution I did not solve the problem of the polygonal ring, as I found that the problem of the w-region was a particular case of two very much simpler problems, namely the condenser problem investigated by Prof. Love* and the two-slit problem investigated by Mr. Hodgkinson† and myself. The whole difficulty of the problem was therefore reduced to the discussion of the Ω-region. This I approached by a heuristic method, beginning with the case of symmetry. Finding that the Ω-region appeared in places to have three sheets, I abandoned the attempt at a conformal representation of the Ω-region and dealt directly with the z- and w-regions, determining the poles and zeros of $(d\Omega/dw)^2$ by Schwarz's principle of symmetry. After some algebraic transformations, I arrived at the same result as Thiry.

In this investigation, the case of symmetry was not studied in detail, but was merely used as a stepping-stone to the general configuration. But a subsequent and minute examination revealed the impossibility of constructing an elliptic function which would have *all* the properties required of $d\Omega/dw$, so that the difficulty in constructing an Ω-region to fit the case of symmetry was fully explained. This case must therefore be regarded as impossible; the proper solution is given in Case III or in the degenerate case, which was mentioned in the introduction of my previous paper, where the second obstacle is surrounded on all sides by the dead-water behind the first.

I had also forgotten that the figure of Case II appears on page 60 of Villat's book,‡ which I quoted, though in connexion with another problem. It is there given merely as an illustration of a "solution" of the wedge problem, which does not satisfy the boundary conditions, and is not investigated directly. In any case, it is clear that I cannot claim priority in noticing any of the cases which I have discussed, though I believe that mine is the first systematic investigation in which an attempt is made to enumerate all the cases and to classify them according to their degrees of freedom. The two most interesting features of the investigation, namely the fact that, in the general case, the region is doubly connected, and the use of Riemann surfaces, were anticipated by Thiry, who makes great use, in dealing with the latter, of a memoir of Schläfli,§ which was also unknown to me.

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* A. E. H. Love, *Proc. London Math. Soc.* (2), 22 (1924), 337–369.

† J. Hodgkinson and E. G. C. Poole, *Proc. London Math. Soc.* (2), 23 (1924), 396–422.

‡ *Aperçus théoriques sur la résistances des fluides* (Gauthier-Villars, 1920).

§ L. Schläfli, *Journal für Math.*, 78 (1874), 63–80.

On Pseudo-Orthogonal Systems of Functions arising as Solutions of an Integral Equation (Part 3)

B. M. WILSON.

The present communication completes the investigations, begun in two earlier papers* with the same title, of the solutions of the pair of equations

$$\phi(s) = f(s) + \lambda \int_0^1 k(s, t)\, \phi(t)\, dt, \tag{1}$$

$$w(\lambda) \equiv \int_0^1 f(s)\, \phi(s)\, ds = \alpha + \frac{\beta}{\lambda}, \tag{2}$$

in which $k(s, t)$ is a closed symmetric nucleus, and α, β are real constants. We are now concerned with the case in which (2) has imaginary roots. It is shown that (2) cannot have more than two imaginary roots, and that a necessary and sufficient condition that it may have any imaginary roots is that

$$\int_0^1 \big(\phi(s)\big)^2\, ds > \alpha > 0$$

for all real solutions $\phi(s)$ of (1) and (2). Supposing this condition to be satisfied, we show that the real solutions $\phi(s)$ of (1) and (2), together with a certain linear combination of the real and imaginary parts of the conjugate complex solutions of these equations, form the complete set of solutions of a pair of equations which satisfy conditions imposed in Part I of this paper; expansion theorems can therefore be deduced, valid in the case now considered, by an application of theorems already proved. The variation of the complex roots of (2), as α and β vary continuously, is then discussed, and a geometrical interpretation of these results is given.

The Relation between the Distributions of Potential in the neighbourhood of a Cylindrical Conductor, when it is charged and when it is placed in a Uniform Field of Force

F. J. W. WHIPPLE.

1. The following proposition has recently† been demonstrated by Mr. H. W. Richmond :—

A cylindrical conductor has for normal section a curve with two perpendicular axes of symmetry. If the field when the conductor is

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* *Supra*, xliii, li.

† H. W. Richmond, *Proc. London Math. Soc.* (2), 22 (1924), 483-494 (490).

freely charged is known, it is possible to deduce the field when the same conductor, having any charge, is introduced with any orientation into a uniform field of force.

The object of the following note is to show that the limitation, to the case of a symmetrical cross section, is unnecessary.

2. In the known solution let the charge on the cylinder per unit length be Q and its potential zero. Let the potential at any point outside the cylinder be $-2Qu$, and let the complementary function be $-2Qv$. If the rectangular coordinates of the point, regarded as in an assigned plane containing a normal section of the cylinder, are x and y, and if

$$x+iy = z$$

and

$$u+iv = w,$$

then there is a functional relation

$$z = F(w), \tag{1}$$

F being a periodic function, such that

$$F(w+2i\pi) = F(w). \tag{2}$$

Since the field at great distances from the origin is independent of the form of the cylinder, we have, for large values of z,

$$w \sim \log z$$

and therefore

$$z = F(w) \sim \exp w.$$

In the case of the circular cylinder of unit radius $F(w) = \exp w$.

Since the spaces outside different cylinders may be conformally represented in the w plane in similar ways, it follows that the space outside one cylinder may be conformally represented by that outside any other cylinder. In particular, the space outside any cylinder, for which the function F is known, may be conformally represented by the space outside a circular cylinder of unit radius.

Now the problem of the uniform field can be solved for a circular cylinder; when the cylinder is uncharged, the potential U and the conjugate function V are given by

$$\begin{aligned} U+iV &= C(ze^{-i\alpha} - z^{-1}e^{i\alpha}) \\ &= C(e^{w}e^{-i\alpha} - e^{-w+i\alpha}) \\ &= 2C\sinh(w-i\alpha), \end{aligned}$$

where C and α are constants.

The same solution,

$$U+iV = 2C \sinh (w-i\alpha), \tag{3}$$

will be valid in the general case in which

$$z = F(w).$$

As a verification we examine the forms which the solution takes on the conductor and at great distances from the conductor.

On the conductor we have $u = 0$, and therefore (3) implies (i) that $U = 0$, *i.e.* that the potential on the cylinder is zero; and (ii) that V is unaltered by the addition of 2π to v, so that the total charge on the cylinder is zero.

For large values of u,

$$U+iV \sim 2C \exp (w-i\alpha) \sim 2Cze^{-i\alpha}.$$

Accordingly the field is uniform at great distances from the cylinder, and the solution is valid.

The extension to the case when the cylinder carries a charge is obvious; the solution is

$$U+iV = 2C \sinh (w-i\alpha)-2Qw. \tag{4}$$

3. Two results of physical significance follow, as deductions from the facts that putting $v = \alpha$ or $\alpha+\pi$ in equation (3) gives $V = 0$, and putting $v = \alpha \pm \frac{1}{2}\pi$ gives $U = 0$.

When the cylinder carries a free charge, it has in any principal plane a pair of lines of force with their common asymptote in any assigned direction. When a field of force with that particular direction is imposed, then those particular lines remain lines of force.

On the other hand, when a field of force perpendicular to that particular direction is imposed (the cylinder being uncharged), those lines become equipotentials.

4. Finally it may be mentioned that the problem of finding the field due to any number of line charges on lines parallel to a cylinder can be solved, by a similar method, provided that the field when the cylinder is freely charged is known.

Thursday, June 12th, 1924.

Prof. L. N. G. FILON, Vice-President, in the Chair.

There were present sixteen members.

The Minutes of the last meeting were read and confirmed.

Mr. S. Narayan was elected a member of the Society.

Mr. D. D. Chelliah was nominated for election.

Dr. H. Cramér and Mr. C. G. F. James were admitted into the Society.

The following communications were made :—

Sur quelques points du calcul des probabilités : H. Cramér.

On Double Fourier Series : E. C. Titchmarsh.

On Generalized Hypergeometric Series : F. J. W. Whipple.

These communications were followed by discussions, in which the Chairman, Prof. Hardy, and Prof. Watson took part.

The following paper was communicated by title from the Chair :—

The allied series of a Fourier series : G. H. Hardy and J. E Littlewood.

NOTES AND ABSTRACTS.

Sur quelques points du calcul des probabilités

HARALD CRAMÉR.

1. Faisons n tirages successives d'une urne renfermant des boules blanches et noires, la probabilité d'amener une boule blanche étant toujours égale à p. Quelle est la probabilité $a_{n,\mu}$ d'avoir dans le cours de ces n tirages une suite de μ tirages ne donnant que des boules blanches ?

Ce problème a été traité par De Moivre, Condorcet et Laplace, qui ont trouvé la formule de récurrence

$$(1) \qquad a_{n,\mu} = a_{n-1,\mu} + p^{\mu} q (1 - a_{n-\mu-1,\mu}), \qquad (q = 1-p)$$

et la fonction génératrice

$$F_{\mu}(x) = \sum_{n=0}^{\infty} \beta_{n,\mu} x^n = \frac{1-p^{\mu} x^{\mu}}{1-x+p^{\mu} q x^{\mu+1}} \quad (\beta_{n,\mu} = 1 - a_{n,\mu}).$$

Plus récemment, notre problème a été longuement discuté par divers

savants (on peut consulter sur ce sujet un ouvrage de M. Bortkiewicz: *Die Iterationen*, Berlin 1917); cependant les théorèmes suivants ne semblent pas avoir été observés.

Quelque petit que soit $\epsilon > 0$, *on peut choisir* n_0 *tel que, pour tout* $n > n_0$ *et pour toute valeur de* μ, *l'expression*

$$1-e^{-np^\mu q}$$

représente la probabilité $\alpha_{n,\mu}$ *avec une erreur dont la valeur absolue ne surpasse pas* ϵ.

En partant de l'égalité

$$F_\mu(x) = (1-p^\mu x^\mu) \sum_{n=0}^{\infty} \gamma_{n,\mu} x^n,$$

où

$$\gamma_{n,\mu} = \sum_{\nu=0}^{n/(\mu+1)} \frac{(n-\mu\nu)!}{\nu!\,(n-\mu\nu-\nu)!} (-p^\mu q)^\nu$$

$$= \sum_{\nu=0}^{n/(\mu+1)} \frac{(-np^\mu q)^\nu}{\nu!} \left(1-\frac{\mu\nu}{n}\right) \dots \left(1-\frac{\mu\nu+\nu-1}{n}\right),$$

on obtient d'abord

$$|\beta_{n,\mu} - e^{-np^\mu q}| < \epsilon$$

pour tout n assez grand et pour toute valeur de μ telle que

$$np^\mu q < \tfrac{1}{2} \log n.$$

Pour achever la démonstration, il suffit de remarquer que, n restant fixe, $\beta_{n,\mu}$ doit être une fonction non décroissante de μ. Notre théorème est donc démontré; en voici quelques conséquences immédiates.

La probabilité pour que, dans n *tirages, la plus longue des suites ne donnant que des boules blanches comprend précisément* μ *tirages est pour tout* n *suffisamment grand, à moins que* ϵ *près égale à* $e^{-np^{\mu+1}q} - e^{-np^\mu q}$.

On peut déterminer un nombre h *dépendant uniquement de* ϵ *de manière que la probabilité des inégalités*

$$\frac{\log n}{\log (1/p)} - h < \mu < \frac{\log n}{\log (1/p)} + h$$

sera, pour tout n *assez grand, supérieure à* $1-\epsilon$.

Ainsi on aura, par exemple, dans le cas $p = q = \frac{1}{2}$, $n = 2^\nu$,

$$\nu - 3 \leqslant \mu \leqslant \nu + 3$$

avec une probabilité qui finit par rester supérieure à 0·95, lorsque ν tend vers l'infini.

2. Parmi les nombres entiers de 1 jusqu'à x, il y a comme on sait asymptotiquement $\int_2^x \frac{dt}{\log t}$ nombres premiers, distribués d'une manière très irrégulière. On pourrait interpréter ce fait en disant que, pour un nombre n pris au hasard, la probabilité d'être premier est égale à $1/\log n$. Considérons donc le schéma suivant en quelque sorte analogue : les urnes $U_1, U_2, \ldots, U_n, \ldots$ renferment des boules blanches et noires, la probabilité d'extraire une boule blanche de U_n étant $1/\log n$. Pour U_1 et U_2 cette définition doit être modifiée ; on peut supposer, par exemple, que U_1 contient seulement des boules noires, U_2 seulement des boules blanches. En tirant une boule de chacune des n premières urnes, désignons le nombre de boules blanches obtenues par

$$\varpi(n) = \int_2^n \frac{dt}{\log t} + \lambda \sqrt{\left(\frac{n}{\log n}\right)}.$$

Alors, la probabilité pour que $|\lambda| < h$ tendra, pour n infini, vers la limite

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^h e^{-\frac{1}{2}t^2}\, dt.$$

Rapprochons cela des théorèmes connus sur la distribution des nombres premiers. En désignant par $\pi(n)$ le nombre des nombres premiers inférieurs à n, on a, la célèbre hypothèse de Riemann étant admise,

$$\pi(n) = \int_2^n \frac{dt}{\log t} + \theta \sqrt{n} \log n,$$

tandis qu'on n'a pas

$$\pi(n) = \int_2^n \frac{dt}{\log t} + \theta \frac{\sqrt{n}}{\log n},$$

θ désignant d'une manière générale une quantité qui reste bornée lorsque n tend vers l'infini.

Considérons maintenant la différence la plus grande entre deux nombres premiers inférieurs à n. Même en admettant l'hypothèse de Riemann, on sait seulement que cette différence n'est pas d'un ordre supérieur à $\sqrt{n} \log n$, ce qui peut s'exprimer par la relation

$$p_{\nu+1} - p_\nu = \theta \sqrt{p_\nu} \log p_\nu,$$

p_ν désignant le ν-ième nombre premier. D'autre part, en étudiant dans notre schéma les suites de boules noires, nous aurons d'abord une formule de récurrence analogue à (1) ; par cette formule nous pouvons prouver que, avec une probabilité tendant vers l'unité quand n tend vers l'infini, le nombre des boules noires dans la plus longue suite obtenue dans les n premiers tirages sera inférieur à $(\log n)^2$.

Some formulae in the theory of Bessel functions

G. H. HARDY.

The formulae

$$(1)\quad J_\mu(x)\,Y_\nu(x)-J_\nu(x)\,Y_\mu(x)=\frac{4\sin(\mu-\nu)\pi}{\pi^2}\int_0^\infty K_{\nu-\mu}(2x\sinh t)\,e^{-(\mu+\nu)t}\,dt,$$

$$(2)\quad J_\mu(x)\,J_\nu(x)+Y_\mu(x)\,Y_\nu(x)$$

$$=\frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2x\sinh t)\left\{e^{-(\mu+\nu)t}\cos(\mu-\nu)\pi+e^{(\mu+\nu)t}\right\}dt,$$

are due to Watson, and are quoted by him on page 445 of his *Theory of Bessel Functions*.* The second formula is a corollary of the first. The most important case is that in which $\mu=\nu$, when (2) shows, for example, that $J_\nu^2+Y_\nu^2$ is monotonic. This case of the formula was first found by Nicholson. †

Watson gives a proof of (2) when $\mu=\nu$ and comments on its difficulty. It may therefore be worth while to give a proof of (1), in which the difficulties are overcome by an appeal to a general theorem.

We suppose in the first instance that all the variables are real and

$$(3)\qquad -2\mu<0<\nu-\mu<s<1;$$

and we denote the integral in (1) by $f(x)$. We have then

$$(4)\quad \int_0^\infty x^{s-1}f(x)\,dx=\int_0^\infty e^{-(\mu+\nu)t}\,dt\int_0^\infty x^{s-1}K_{\nu-\mu}(2x\sinh t)\,dt$$

$$=\tfrac{1}{4}\Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s-\mu+\nu}{2}\right)\int_0^\infty e^{-(\mu+\nu)t}(\sinh t)^{-s}\,dt$$

$$=\frac{2^s}{8\pi}\,\Gamma(1-s)\sin\tfrac{1}{2}\pi(s-\mu-\nu)\,\mathrm{II}(s),$$

where $\mathrm{II}(s)=\Gamma\left(\frac{s+\mu+\nu}{2}\right)\Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s-\mu+\nu}{2}\right)\Gamma\left(\frac{s-\mu-\nu}{2}\right)$.

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* I have corrected a slip in Watson's statement of the formulae, in which the exponentials $e^{(\mu+\nu)t}$ and $e^{-(\mu+\nu)t}$ are interchanged.

† J. W. Nicholson, "The asymptotic expansions of Bessel functions", *Phil. Mag.* (6), 19 (1910), 228–249 (234).

The inversion presents no difficulty, everything being positive, and the conditions (3) ensuring the convergence of all the integrals which occur.

The integral (4) is convergent if the real part of s lies between 1 and $\nu-\mu$. We may therefore use Mellin's inversion formula,* when we obtain

$$(5) \qquad f(x) = \frac{1}{8\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\tfrac{1}{2}x)^{-s}\,\Gamma(1-s)\,\sin\tfrac{1}{2}\pi(s-\mu-\nu)\,\Pi(s)\,ds,$$

where $\nu-\mu<c<1$. If we replace s by $-2s$, we obtain an integral identical, save for a constant factor, with that given by Watson† for $J_\mu Y_\nu - J_\nu Y_\mu$.

If we are content with a verification of (1), the proof may be shortened further, as we have only to calculate

$$\frac{\pi^2}{4\sin(\mu-\nu)\pi}\int_0^\infty \{J_\mu(x)\,Y_\nu(x)-J_\nu(x)\,Y_\mu(x)\}\,x^{s-1}\,dx,$$

and to verify its identity with (4). In this case Mellin's formula is not required, the ultimate appeal being to one of the variations of what is usually called "Lerch's Theorem."

I take this opportunity of recording a new inversion formula which presents some interest, viz. :—

$$(6.1) \qquad \phi(x) = \int_0^\infty \{\cos a\pi\, J_\nu(xt)+\sin a\pi\, Y_\nu(xt)\}\,tf(t)\,dt,$$

$$(6.2) \qquad f(x) = \int_0^\infty F(xt)\,t\,\phi(t)\,dt,$$

where‡
$$F(x) = \sum_{p=0}^{\infty} \frac{(-1)^p\,(\frac{1}{2}x)^{\nu+2a+2p}}{\Gamma(a+1+p)\,\Gamma(\nu+a+1+p)}.$$

When $a=0$, the formulae (6) reduce to Hankel's familiar formulae. When $a=\frac{1}{2}$, they give

$$(7) \qquad \phi(x) = \int_0^\infty Y_\nu(xt)\,tf(t)\,dt, \qquad f(x) = \int_0^\infty \mathbf{H}_\nu(xt)\,tf(t)\,dt,$$

* See G. H. Hardy, "Notes on some points in the integral calculus (49): On Mellin's inversion formulae", *Messenger of Math.*, 47 (1918), 178–184, for a sufficiently general set of conditions for the validity of the formula.

† *Loc. cit.*, 223.

‡ $F(x)$ is a "Lommel's function": see Watson, *loc. cit.*, 345.

an inversion theorem discovered recently by Titchmarsh.* Here $\mathbf{H}_\nu$ is "Struve's function." Another interesting case is that in which $\nu = \frac{1}{2}$. We have then

$$\psi(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \sin(xt - a\pi)\, g(t)\, dt, \tag{8}$$

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty C_{2a+1}(xt)\, \psi(t)\, dt.$$

Here $C_q(t)$, a specialized Lommel's function, is a generalization of the cosine, studied recently by Young† and useful in the theory of the summation of Fourier series.

* E. C. Titchmarsh, "A pair of inversion formulae", *Proc. London Math. Soc.* (2), 22 (1924), xxxiv-xxxv (*Records* for 17 May, 1923).

† W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 43 (1912), 161–177.

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OBITUARY NOTICES

CHARLES LEUDESDORF.

Charles Leudesdorf, Senior Fellow and Vicegerent of Pembroke College, and Registrar of the University of Oxford, who died on August 10th, 1924, in his seventy-second year, was active as a member of our Society in the second and early third decades of its history; but for the last thirty-five years his life has been one of devotion to administrative academical duties, for the first half of them as Secretary to the Boards of Faculties and for the second half as Registrar, so that he has let himself be lost sight of as a mathematician. He was elected a member of the Society on January 14th, 1875, and served for a good many years on the Council. Our *Proceedings* contain some half a dozen papers from his pen, and the *Messenger of Mathematics* about as many. His translation of Cremona's *Projective Geometry*, which was made in order to satisfy a need of Oxford students, among whom a taste for something like specialization in Geometry had been fostered by H. J. S. Smith's captivating lectures, has gone through a number of editions, and still commands a sale. He was the very first contributor of a paper to the Oxford Mathematical Society on its foundation under Sylvester in 1888.

As we turn over the pages of what he wrote for us and the *Messenger*, thought is forced on us about differences in attitude and environment between young mathematicians who were feeling their way forty or fifty years ago, in the days of the problem lover, and their successors of the present time, when training for research pervades teaching in Universities, and digging about the roots of old growths in Analysis is familiar exercise. It is not unfair to Leudesdorf to say that he began by writing carefully and precisely about isolated problems of no great intrinsic importance. But his close attention to completeness in argument, as to a theorem of Pure Geometry and its converse, for instance, where acceptance of half knowledge as adequate was not unusual, greatly helped teachers. He first awakened more general interest, perhaps, by two notes in the *Messenger* on plane kinematics; and a corollary to a

result in one of these, stated by Mr. A. B. Kempe, finds a place in text-books as Kempe's theorem. To me, naturally, and I hope to some others, a paper of real interest was one in our *Proceedings* on "Multiple Frullanian Integrals," in which he gently pointed out and put to rights certain deficiencies in what I had myself written on the subject. More elaborate were three fairly long papers on the interchange of variables in differential expressions, with applications to the Sylvesterian Theory of Reciprocants.

He and I were the closest of friends; and this must end on a personal note. Oh the delights of the infection when from time to time "reciprocant fever," as we used to call it, spread from one of us to the other! Oh the merit which we seemed to be acquiring together in our monthly 5 a.m. breakfasts at Paddington before the only train down which would deliver us in time for our 9 o'clock lectures, in the days when the Society met at 8 p.m., and the last train for Oxford left at 8.10!

F. B. E.

GRACE DOROTHY SADD.

THE sudden death of Miss Sadd from influenza while on a holiday at Lucerne has deprived the Society of one of the comparatively few promising younger mathematicians of her sex. Her opportunities for advanced mathematical study were confined to the last few years, when she was on the staff of Bedford College for Women, and subsequently of University College, Exeter. She had in preparation Notes on the Theory of Cremona Transformations, and a paper of hers on "Plane Unicursal Sextic Curves" has appeared in the *Proceedings*.

H. H.

JOHN EDWARD CAMPBELL.

The death of JOHN EDWARD CAMPBELL last October after an illness of only an hour was a great shock to his many friends. He was born in 1862, educated at the Methodist College and University College, Belfast, and later at Hertford College, Oxford. He was made a Fellow of Hertford College in 1887, and remained there as tutor, bursar, or vice-principal for the rest of his life. Of his work as tutor it is difficult for one to write adequately who owes him such a deep debt of gratitude. In the first term under his care an undergraduate learnt more than in the whole of his previous work. No time or effort was ever grudged that could possibly help a pupil, and his unselfish industry and kindly interest in his students were a wonderful example and inspiration.

His mathematical researches were published almost entirely in the *Proceedings* of the London Mathematical Society and in the *Messenger of Mathematics*. They date from 1891 till the time of his death, with the exception of the period 1914 to 1920.* The loss of a son early in the war was a terrible blow, and this, combined with anxiety on behalf of two other sons on service, seemed to destroy entirely for some years all his interest in Mathematics, though characteristically he diverted his energies into toil for others in connexion with philanthropic "war work" and administrative duties as bursar at Hertford College. His election as president of the London Mathematical Society was the means of restoring his interest in mathematical work, and helped him to recover from his trouble. The methodical and tactful way in which he filled the office will be fresh in the memory of the members of the Council.

His earlier papers were on miscellaneous topics, including one on the theory of Higher Plane Curves, a subject for which he retained his fondness for many years, and on which he lectured at times, though he did not publish further work thereon. By 1897 he was writing on matters

* His published work will be found in *Messenger of Mathematics*, 21 (1892), 78, 158; 23 (1894), 130, 144; 28 (1899), 97. *Proc. London Math. Soc.* (1): 24 (1892), 67; 28 (1896), 381; 29 (1897), 14, 249; 31 (1899), 235; 33 (1900), 285. (2): 1 (1903), xxx; 5 (1907), 6; 6 (1908), 178; 8 (1910), 383; 9 (1911), 410; 10 (1912), 406; 13 (1914), 372; 20 (1922), 1; 21 (1923), 317; 22 (1924), 92. *Bull. Amer. Math. Soc.* (2), 4 (1898), 407. *Trans. Amer. Math Soc.*, 1 (1900), 243. *Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups*, Clarendon Press (1903), v + 416.

connected with Lie's transformation groups, which were his main theme for several years, culminating in his "Lie's Theory of Finite Continuous Transformation Groups," published by the Clarendon Press in 1903. The geometrical applications of Lie's contact transformations gradually led him to direct his thoughts to the differential geometry of surfaces, and papers on Bäcklund's transformation, the application of quaternions to the deformation of a surface, cyclic congruencies, etc., were written by him during the next ten years. When his mathematical enthusiasm revived after the war, it was the tensor calculus and the applications of differential geometry to Einstein's work that appealed to him. The appeal was from the geometrical side. Neither mathematical physics nor experimental science ever seemed really to interest him. His presidential address in 1920 was on this subject, and a treatise on the tensor calculus was in its final stage at the time of his sudden death. It is to be hoped that it may prove possible to publish it, even though the author is no longer here to put the final touches to the task.

His method of work was intensive. He preferred to take some really first-rate book and know it by heart rather than to skim through a large amount of material perfunctorily. The treatises of Engel on continuous groups, of Darboux and Bianchi on surfaces, and of Weyl on the tensor calculus were his favourites. These books he absorbed and reproduced with their contents marked with an originality all his own. Unfortunately, his argument was not always easy to follow. Even in his lectures the middle and end of an example would appear before the beginning, and hearers unused to his methods must have found him at times a little puzzling. Nevertheless, his lectures were some of the best given in Oxford; and similarly in his papers the reader was always well repaid for the trouble which perusal involved.

He was a Fellow of the Royal Society, and held an honorary D.Sc. of the Queen's University of Belfast. Perhaps, if he had pushed his own claims more, he might have attained to other distinctions. But such assertion was entirely alien to his character, and his whole life was spent in the service of others and forgetfulness of self.

H. H.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

SOLUTIONS OF SOME INTEGRAL EQUATIONS CONSIDERED BY BATEMAN, KAPTEYN, LITTLEWOOD, AND MILNE

By G. H. Hardy *and* E. C. Titchmarsh.

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1. *Introduction and Summary.*

1.1. The integral equations discussed in this paper are all very similar in form, but fall sharply into two classes as regards the character of their solutions. The three with which we are concerned primarily are

(1.11) (B)
$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(t-x)}{t-x} f(t)\,dt = \frac{1}{\pi}\int_0^{\infty} \frac{\sin t}{t}\{f(x+t)+f(x-t)\}\,dt,$$

(1.12) (K)
$$f'(x) = \tfrac{1}{2}\int_0^{\infty} \frac{J_1(t)}{t}\{f(x+t)-f(x-t)\}\,dt,$$

(1.13) (M)
$$f'(x) = \tfrac{1}{2}\int_0^{\infty} \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt.$$

These equations have all been considered before. Thus (B) occurs in a paper of Bateman,* and was discussed later in considerable detail by Hardy.† The second equation, in the slightly different form

(1.141) (K′)
$$f'(x) = \tfrac{1}{2}\int_0^{\infty} \frac{J_1(t)}{t}\{f(t+x)+f(t-x)\}\,dt,$$

* H. Bateman, "The inversion of a definite integral", *Proc. London Math. Soc.* (2), 4 (1907), 461–498 (483).

† G. H. Hardy, "On an integral equation", *Proc. London Math. Soc.* (2), 7 (1909), 445–472.

was given by Kapteyn,* as the condition that an odd function $f(x)$ should be expansible in a "Neumann" series

$$\Sigma a_{2n+1} J_{2n+1}(x) \tag{1.142}$$

in which the coefficients are determined by the formula

$$a_{2n+1} = (4n+2)\int_0^\infty f(t) J_{2n+1}(t)\,\frac{dt}{t}. \tag{1.143}$$

This method of determining the coefficients had been suggested previously by Webb,† but his conclusions as regards the generality of the expansion were, as Kapteyn showed, erroneous.

It is obvious that all solutions of (K′) are odd, and that, if $f(x)$ is odd, (K′) and (K) are identical. A solution of (K), however, is not necessarily odd, so that (K) presents a slightly more general problem. Kapteyn's equation was discussed later by Bateman,‡ who was, however, unable to reach very definite conclusions as to the class of functions which satisfy the equation.

The equation (M) originates with Milne,§ who was led to it in an investigation of the distribution of temperature in the interior of a star. It was discussed later from the pure mathematical standpoint by Littlewood.‖ It is formally equivalent to the linear differential equation

$$\tfrac{1}{3}f^{(3)}(x)+\tfrac{1}{5}f^{(5)}(x)+\ldots = 0, \tag{1.15}$$

with constant coefficients and of infinite order. Such equations have been discussed in an important memoir by Schürer,¶ but the conclusions concerning the equation (M), which can be deduced from Schürer's theorems, are much less general than those which we propose to prove.

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* W. Kapteyn, "On an expansion of an arbitrary function in a series of Bessel functions", *Messenger of Math.*, 35 (1906), 122–125. See G. N. Watson, *Theory of Bessel functions*, 533–535.

† H. A. Webb, "The expansion of an arbitrary function in a series of Bessel functions", *Messenger of Math.*, 33 (1904), 55–58.

‡ H. Bateman, "On the expansion of an arbitrary function in a series of Bessel functions", *Messenger of Math.*, 36 (1907), 31–37.

§ E. A. Milne, "Radiative equilibrium in the outer layers of a star", *Monthly Notices of the R.A.S.*, 81 (1921), 361–375.

‖ J. E. Littlewood, "On an integral equation", *Proc. Camb. Phil. Soc.*, 21 (1922), 205–215.

¶ F. Schürer, "Eine gemeinsame Methode zur Behandlung gewisser Funktionalgleichungsprobleme", *Leipziger Berichte*, 70 (1918), 185–246. Extensions of Schürer's work will be found in memoirs by Hilb in volumes 82 and 84 of the *Math. Annalen*.

There are three other equations, which form, with (B), (K), and (M), a symmetrical set of six. These equations are

$$(1.16)\quad (B_1)\qquad f'(x)=\frac{1}{\pi}\int_0^\infty \frac{\sin t}{t^2}\{f(x+t)-f(x-t)\}\,dt,$$

$$(1.17)\quad (K_1)\qquad f(x)=\tfrac{1}{2}\int_0^\infty J_1(t)\{f(x+t)+f(x-t)\}\,dt,$$

$$(1.18)\quad (M_1)\qquad f'(x)=\frac{1}{\pi}\int_0^\infty K_1(t)\{f(x+t)-f(x-t)\}\,dt,$$

where $K_1(t)$ is the function commonly so designated in mathematical physics.* The theory of these equations is substantially the same as that of the first three,† and we shall refer to them only incidentally.

We assume throughout that the formulae are significant and true for every real x. Hence, if $f'(x)$ occurs in an equation, any solution has necessarily a finite derivative for every x. It is of course not immediately obvious (though it may be true in fact) that this derivative is necessarily bounded in the neighbourhood of any x. The integrals are always Lebesgue integrals. Our notation, so far as Bessel functions are concerned, is that of Watson's treatise.

1.2. It appears at once that the theory of (M) and (M_1) is quite different from that of the other equations. Each has the solution

$$(1.21)\qquad f(x)=ax^2+2bx+c,$$

and there are no other obvious solutions. The simplest solution of each of the other equations is

$$(1.22)\qquad f(x)=A\cos ax+B\sin ax,$$

where $0\leqslant a\leqslant 1$; this function is not a solution if $a>1$. But these equations have much more general solutions involving arbitrary functions; thus any function of the form

$$(1.23)\qquad f(x)=\int_0^1 \{\lambda(u)\cos xu+\mu(u)\sin xu\}\,du,$$

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* Watson, *loc. cit.*, 78.

† The theory of (B_1) is practically the same as that of (K), and that of (M_1) the same as that of (M). The equation (K_1) is rather more troublesome than (B); in particular, the results of §2 do not hold for (K_1) without reservation.

where λ and μ are subject only to very general conditions of inequality, is a solution. The reason underlying this difference will appear in § 2.

In § 2 we discuss the equations by the methods of the theory of functions of a real variable, and in particular by the use of the theory of "Fourier transforms" due to Plancherel,* and its generalisation to "Hankel transforms".† The use of this theory involves a restriction on the function $f(x)$, namely, that its square is integrable over the range $(-\infty, \infty)$; and it is plain that this assumption rules out some of the most obvious solutions, such as (1.21) and (1.22). But, within its range, the solution of the problem afforded by the theory of transforms is complete. Thus we prove:—

THEOREM 1.—*Suppose that the square of $f(x)$ is integrable over $(-\infty, \infty)$. Then, in order that $f(x)$ should be a solution of* (B), (B_1), *or* (K), *it is necessary and sufficient that $f(x)$ should be of the form* (1.23), *λ and μ being functions whose squares are integrable over* (0, 1).

We cannot prove so comprehensive a result for the equation (K_1), owing to the comparative slowness with which the kernel $J_1(t)$ tends to zero; and since additional hypotheses would be required, we have not worked out the results in detail.

THEOREM 2.—*There are no solutions of* (M) *or* (M_1) *of integrable square, other than $f(x) = 0$.*

Our methods give something even when the hypothesis that the square of $f(x)$ is integrable is abandoned; thus we can prove

THEOREM 3.—*Any function of the form* (1.23) *is a solution of* (B), (B_1), *or* (K); *and if $f(x)$ is of the form*

$$f(x) = \int_0^\infty \{\lambda(u) \cos xu + \mu(u) \sin xu\}\, du, \tag{1.24}$$

then λ and μ are zero for $u > 1$, so that $f(x)$ is of the form (1.23).

The example (1.22), however, shows that a solution is not necessarily of this form. This remark leads us, in §§ 2.6–7, to discuss alternative forms of solution, and in particular one suggested by Bateman.

1.3. Our analysis brings out clearly the reason which underlies the difference between the theory of (B), (K), (B_1), or (K_1) on the one hand and

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* M. Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289–335.

† E. C. Titchmarsh, "Hankel Transforms", *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473.

that of (M) or (M_1) on the other. If we consider generally the equation

$$f'(x) = \tfrac{1}{2}\int_0^\infty \frac{\chi(t)}{t}\{f(x+t)-f(x-t)\}\,dt, \tag{1.31}$$

it will not generally be true that

$$\int_0^\infty \frac{\chi(t)}{t}\sin tu\,dt = u, \tag{1.32}$$

except possibly for isolated values of u. In this case there will be no solutions of the equation, or a small number of exceptional solutions. But if it should happen that (1.32) holds over whole intervals of values of u, as it does for $0<u<1$ when $\chi(t)$ is $J_1(t)$ or $2\sin t/\pi t$, then there is a multitude of solutions defined by definite integrals taken over these intervals.

The methods of § 2 may be applied to many other equations, such as

$$f''(x) = \int_0^\infty \frac{\chi(t)}{t^2}\{f(x+t)+f(x-t)-2f(x)\}\,dt. \tag{1.33}$$

If $\chi(t)$ is $J_1(t)$ or $2\sin t/\pi t$, and f is of integrable square, the solution of this equation is the same as that of (B) or (K); if $\chi(t)$ is e^{-t}, there are no non-zero solutions of integrable square. In the last case the only obvious solution is a cubic polynomial.

They may also be applied to such equations as

$$f'(x) = g(x)+\tfrac{1}{2}\int_0^\infty \frac{J_1(t)}{t}\{f(x+t)-f(x-t)\}\,dt, \tag{1.34}$$

$$f'(x) = g(x)+\tfrac{1}{2}\int_0^\infty \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt, \tag{1.35}$$

and here again there is a complete difference between the natures of the solutions. If $f(x)$ and $g(x)$ are of integrable square and $x^{-3}g(x)$ is integrable at the origin, the equation (1.35) has a unique solution; while (1.34) has no solution unless $g(x)$ is a function of special form, and then an infinity, any two of which differ by a solution of (K). We have not included formal proofs of these results.

1.4. The theory of transforms gives, up to a certain point, very simple and satisfactory results, but it has serious limitations, inasmuch as it will not give an account of such obvious solutions as (1.21) and (1.22). In § 3 we consider the theory of (K) and (M) by more powerful methods depending on the theory of analytic functions. Our main results are:—

THEOREM 6.—*Suppose that* $f(x)$ *is real and*

$$f(x) = O(|x|^{\frac{1}{2}-\delta}) \tag{1.41}$$

where $\delta > 0$, *for large real values of* x. *Then, in order that* $f(x)$ *should be a solution of* (K), *it is necessary and sufficient that* $f(z)$ *should be an integral function of* $z = x+iy$, *and*

$$|f(z)| < A|z|^{\frac{1}{2}-\delta}e^{|y|}, \tag{1.42}$$

where A *is a constant, for all sufficiently large values of* $|z|$.

THEOREM 7.—*Suppose that* $f(x)$ *is real and*

$$f(x) = O(e^{A|x|}), \tag{1.43}$$

where $A < 1$, *for real values of* x, *and that* $f(x)$ *is a solution of* (M). *Then* $f(x)$ *is a quadratic.*

These theorems give a reasonably, though not entirely, complete account of the equations, since the orders of magnitude prescribed by (1.41) and (1.43) are very near those demanded by the convergence of the integrals in the equations. The most that has been proved before is Littlewood's result that, roughly,* (M) *has no solution, other than a quadratic, whose second derivative is bounded.* This he proves by entirely elementary reasoning.

We conclude by proving a theorem, of a much more special character than Theorem 7, but interesting both for the method of proof and because, in one way, it goes further, encroaching, so to say, on the case $A = 1$.

THEOREM 8.—*Suppose that a solution of* (M) *is of the form*

$$f(x) = \Sigma a_n \frac{x^n}{n!},$$

and that $a_0+a_2+\ldots+a_{2n}+\ldots$, $a_1+a_3+\ldots+a_{2n+1}+\ldots$ *are convergent. Then* $f(x)$ *is a quadratic.*

This includes the case of Theorem 7 in which it is assumed that $f(z) = O(e^{A|z|})$, with $A < 1$, for all values of z; and some cases which are not covered by Theorem 7.

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* Littlewood gives theorems more general than this in other directions, but not in regard to the order of magnitude of the function. We quote only the simplest theorem.

2. *Solutions by the methods of the theory of functions of a real variable.*

2.1. The theorems in the theory of Fourier transforms which will be made use of are as follows:—

(2.11) If the square of a function $f(x)$ is integrable over the range $(0, \infty)$, then the function

$$F_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt f(t)\, dt$$

converges in mean* to a function $F(x)$, whose square is also integrable over the range $(0, \infty)$. In these circumstances $F(x)$ is called the Fourier cosine transform of $f(x)$.

(2.12) $$\int_0^x F(t)\, dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin xt}{t} f(t)\, dt.$$

(2.13) Reciprocally, $f(x)$ is the transform of $F(x)$.

(2.14) If the square of $g(x)$ is integrable over $(0, \infty)$, then

$$\lim_{a\to\infty} \int_0^\infty g(x)\{F(x) - F_a(x)\}\, dx = 0.$$

(2.15) Similar results hold if $\sin xt$ or $\sqrt{(xt)}J_0(xt)$ be substituted for $\cos xt$.

2.2. *Proofs of Theorems* 1 *and* 2.—The equations (K), (B_1), (M), and (M_1) can be conveniently considered together as particular cases of (1.31), the function $\chi(t)$ having the following forms:—

(2.21) (K) $J_1(t)$, (B_1) $\frac{2}{\pi}\frac{\sin t}{t}$, (M) e^{-t}, (M_1) $\frac{2}{\pi} tK_1(t)$.

If $f(x)$ is a solution, so is each of the functions $\frac{1}{2}\{f(x) \pm f(-x)\}$, so that any solution is the sum of an odd and an even solution. Suppose first that $f(x)$ is an even solution whose square is integrable over $(-\infty, \infty)$. Then, for almost all x,

(2.22) $$f(x) = \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} \lambda(u)\, du,$$

---

* That is to say

$$\int_0^\infty \{F_a(x) - F(x)\}^2\, dx \to 0,$$

when $a \to \infty$.

where λ^2 is integrable over $(0, \infty)$. Hence

$$f(x+t)-f(x-t) = 2\frac{d}{dx}\int_0^\infty \frac{\cos xu \sin tu}{u}\lambda(u)\,du,$$

and the equation takes the form

$$f'(x) = \int_0^\infty \frac{\chi(t)}{t}\left\{\frac{d}{dx}\int_0^\infty \frac{\cos xu \sin tu}{u}\lambda(u)\,du\right\}dt.$$

Now in each case it is easily seen* that the integral with respect to t converges uniformly over a finite range of values of x, so that the order of integration with respect to t and differentiation with respect to x may be inverted. Thus

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_0^\infty \frac{\chi(t)}{t}\,dt\int_0^\infty \frac{\cos xu \sin tu}{u}\lambda(u)\,du \\ &= \frac{d}{dx}\int_0^\infty \frac{\cos xu}{u}\lambda(u)\,du\int_0^\infty \frac{\sin tu}{t}\chi(t)\,dt, \end{aligned}$$

on inverting the order of integration.†

The integral $$\int_0^\infty \frac{\sin tu}{t}\chi(t)\,dt$$

has the following values in the four cases under consideration:

(K) u $(u<1)$, $\{u+\sqrt{(u^2-1)}\}^{-1}$ $(u>1)$;

(B_1) u $(u<1)$, 1 $(u>1)$;

(M) $\arctan u$; (M_1) $u(u^2+1)^{-\frac{1}{2}}$.

The difference between (K), (B_1) on the one hand, and (M), (M_1) on the other, becomes clear at this point. Take (K) as typical of the former class. This equation is now reduced to

$$f'(x) = \frac{d}{dx}\left\{\int_0^1 \cos xu\,\lambda(u)\,du+\int_1^\infty \frac{\cos xu\,\lambda(u)}{u\{u+\sqrt{(u^2-1)}\}}\,du\right\},$$

and, integrating,

$$f(x) = \int_0^1 \cos xu\,\lambda(u)\,du+\int_1^\infty \frac{\cos xu\,\lambda(u)}{u\{u+\sqrt{(u^2-1)}\}}\,du.$$

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* From the original form (1 . 31).

† The associated double integral is absolutely convergent, since

$$|\sin tu| \leqslant t^\alpha u^\alpha \quad (0<\alpha<\tfrac{1}{2}).$$

The constant of integration is zero; for both terms on the right tend to zero when x tends to infinity, and $f(x)$ cannot tend to a limit other than zero. But also, by (2.22),

$$f(x) = \int_0^1 \cos xu\, \lambda(u)\, du + \frac{d}{dx}\int_1^\infty \frac{\sin xu}{u}\, \lambda(u)\, du;$$

so that

$$\frac{d}{dx}\int_1^\infty \frac{\sin xu}{u}\left\{1 - \frac{1}{u\{u+\sqrt{(u^2-1)}\}}\right\}\lambda(u)\, du = 0, \tag{2.23}$$

Hence if

$$\phi(u) = \left\{1 - \frac{1}{u\{u+\sqrt{(u^2-1)}\}}\right\}\lambda(u) \quad (u > 1), \qquad 0 \quad (u < 1),$$

then the transform of $\phi(u)$ is a null function. Thus $\phi(u)$ is a null function, and so $\lambda(u)$ is a null function for $u > 1$, since the other factor does not vanish. Hence

$$f(x) = \int_0^1 \cos xu\, \lambda(u)\, du.$$

Thus any even solution of (K), of integrable square, is of the form prescribed by the theorem. The proof for (B_1) is substantially the same; and it is plain that our argument shows the sufficiency of the condition as well as its necessity. The equation (M), which is typical of the second class, has been reduced to

$$f'(x) = \frac{d}{dx}\int_0^\infty \frac{\cos xu}{u} \arctan u\, \lambda(u)\, du,$$

and it follows as before that

$$\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}\left(1 - \frac{\arctan u}{u}\right)\lambda(u)\, du = 0.$$

Hence in this case $\lambda(u)$ is a null function for $u > 0$, *i.e.* $f(x)$ vanishes. The same conclusion evidently holds for (M_1).

If $f(x)$ is an odd solution of (1.31), whose square is integrable over $(0, \infty)$, then

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{1-\cos xu}{u}\, \mu(u)\, du$$

where μ^2 is integrable over $(0, \infty)$, for almost all x. It follows as before that for (K) and (B_1)

$$f(x) = \int_0^1 \sin xu\, \mu(u)\, du,$$

while for (M) and (M_1) $f(x)$ vanishes. This completes the proof of Theorems 1 and 2, except as regards equation (B).

The discussion of (B) proceeds on the same lines, except that the inversion of the order of integration of the repeated integral* cannot be so easily justified. In this case the integral is (for an even solution)

$$\int_0^\infty \frac{\sin t}{t}\,dt \int_0^\infty \frac{\sin xu \cos tu}{u}\,\lambda(u)\,du.$$

The integral with respect to u converges uniformly over any finite range of values of t, so that

$$\int_0^A \frac{\sin t}{t}\,dt \int_0^\infty \frac{\sin xu \cos tu}{u}\,\lambda(u)\,du = \int_0^\infty \frac{\sin xu}{u}\,\lambda(u)\,du \int_0^A \frac{\sin t \cos tu}{t}\,dt.$$

Hence it is sufficient to prove that

$$\lim_{A\to\infty} \int_0^\infty \frac{\sin xu}{u}\,\lambda(u)\,du \int_A^\infty \frac{\sin t \cos tu}{t}\,dt = 0. \tag{2.24}$$

But here the integral with respect to t is bounded for all values of A and u, and its limit is zero for all values of u. It follows from a classical theorem of Lebesgue† that (2.24) is true, and so the inversion is justified.‡

2.4. *Proof of Theorem* 3.—Here λ and μ are integrable instead of λ^2 and μ^2. The proof given under the previous hypotheses remains valid, except that, having obtained the formula (2.23), we must integrate again and use Fourier's theorem directly.

2.5. *The equation* (K′) *and the theory of Neumann series.*—Before we proceed to a discussion of (K′), with a less stringent condition for the convergence of the integral than that of Theorem 1, we must say something about the connection of this equation with the theory of Neumann series. It is convenient to refer to Watson's account of this subject,§ in which it is proved that, in order that $f(x)$ should satisfy (K′), it is necessary and sufficient that the expansion

$$f(x) = \sum_{n=0}^{\infty} (4n+2)\,J_{2n+1}(x) \int_0^\infty \frac{J_{2n+1}(t)}{t}\,f(t)\,dt \tag{2.51}$$

* See p. 8, f.n.†

† See, for example, de la Vallée Poussin, *Cours d'analyse* (ed. 3, 1, 1914), 264.

‡ We shall refer to this argument as "inversion by bounded convergence".

§ Watson, *loc. cit.*, 533–535.

should be valid for all real values of x. Watson assumes that $f'(x)$ is continuous, and that $f(x)$ is absolutely integrable over the range $(0, \infty)$.

If the integrals are taken in the sense of Lebesgue, it is unnecessary to assume more about $f'(x)$ than that it exists and is finite everywhere. It is clear, in fact, from Watson's proof that all that is necessary is that $f(x)$ should be the integral of $f'(x)$. Now if it is assumed that the integral

$$\int^{\infty} f(t)\, t^{-\frac{3}{2}}\, dt \tag{2.52}$$

is absolutely convergent, then it follows from (K′) that

$$|f'(x)| \leqslant \tfrac{1}{2}\int_0^{\infty} \frac{|J_1(t)|}{t}\, |f(t+x)+f(t-x)|\, dt.$$

The integral

$$\int_0^{\infty} \frac{|J_1(t)|}{t}\, dt \int_{x_1}^{x_2} |f(t+x)+f(t-x)|\, dx$$

is plainly convergent, so that $f'(x)$ is integrable over any finite interval (x_1, x_2); and so $f(x)$ is the integral of $f'(x)$.*

Further, *the theorem stated by Watson is true, provided only that the integral* (2.52) *is absolutely convergent.* To prove this, it is necessary to sum the series (2.51) by changing the order of summation and integration. This is justified by uniformity of convergence if the integrals are only taken over a finite range; and it will be justified when the range is infinite if the integral

$$\int_k^{\infty} |f(t)| \sum_{n=0}^{\infty} (4n+2) \left| J_{2n+1}(x)\, \frac{J_{2n+1}(t)}{t} \right| dt \quad (k > 0) \tag{2.53}$$

is convergent. Let

$$\sum_{n=0}^{\infty} (4n+2) |J_{2n+1}(x)\, J_{2n+1}(t)| = \sum_{n<\theta t} + \sum_{n>\theta t} = \Sigma_1 + \Sigma_2, \tag{2.54}$$

where θ is a constant less than unity. In Σ_2

$$|J_{2n+1}(t)| < 1, \qquad |J_{2n+1}(x)| < A\delta^n \quad (\delta = \delta(\theta) < 1).$$

Hence

$$\Sigma_2 < A \sum_{n>\theta t} n\delta^n < At\delta^{\theta t} = O(t^{-\frac{1}{2}}). \tag{2.55}$$

In Σ_1 we use the asymptotic formula†

$$J_n(n \sec \beta) \sim \frac{\cos\{n(\tan\beta - \beta) - \frac{1}{4}\pi\}}{\sqrt{(\frac{1}{2}n\pi \tan\beta)}} \quad (\beta > 0),$$

* See E. W. Hobson, *Theory of Functions of a Real Variable* (ed. 2, 1, 1921), 554.

† Watson, *loc. cit.*, 234, 244.

which shows that $J_n(t) = O(t^{-\frac{1}{2}})$ uniformly for $n < \theta t$. It follows that

$$\Sigma_1 = O(t^{-\frac{1}{2}}) \Sigma(4n+2) |J_{2n+1}(x)| = O(t^{-\frac{1}{2}}), \tag{2.56}$$

and from (2.54), (2.55), and (2.56), we deduce

$$\sum_{n=0}^{\infty} (4n+2) |J_{2n+1}(x) J_{2n+1}(t)| = O(t^{-\frac{1}{2}}).$$

Hence (2.53) is convergent whenever (2.52) is absolutely convergent. The remainder of Watson's discussion requires no modification.

2.6. *An alternative solution of* (K′).—We now prove

THEOREM 4.—*Suppose that the square of* $x^{-\frac{1}{2}}f(x)$ *is integrable over* $(0, \infty)$. *Then in order that* $f(x)$ *should be a solution of* (K′), *it is necessary and sufficient that it should be of the form*

$$f(x) = x \int_0^1 u^{\frac{1}{2}} J_0(ux) \lambda(u)\, du, \tag{2.61}$$

where λ^2 *is integrable over* $(0, 1)$.*

Since $\{f(x)\}^2/x$ is integrable over $(0, \infty)$, the integral (2.52) is absolutely convergent. Hence $f(x)$, if it is a solution of (K′), is expansible in a Neumann series in accordance with the formulæ (1.142) and (1.143); and conversely.

To verify the sufficiency of the condition, we can use Bateman's expansion †

$$xJ_0(ux) = \sum_{n=0}^{\infty} (4n+2) P_n(1-2u^2) J_{2n+1}(x),$$

which is valid and uniformly convergent for $0 \leqslant u \leqslant 1$. Thus if $f(x)$ is of the form (2.61),

$$f(x) = x \int_0^1 u^{\frac{1}{2}} J_0(ux) \lambda(u)\, du = \sum_{n=0}^{\infty} (4n+2) J_{2n+1}(x) \int_0^1 P_n(1-2u^2) u^{\frac{1}{2}} \lambda(u)\, du.$$

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* This is the form of solution suggested by Bateman.

† See Bateman, *loc. cit.*, 32; and "Certain definite integrals and expansions connected with the Legendre and Bessel functions", *Messenger of Math.*, 38 (1904), 182–188 (this reference is given incorrectly in Bateman's later paper).

Hence $f(x)$ is expansible in a Neumann series convergent for all real values of x. Also

$$\int_0^\infty J_{2n+1}(x) f(x) \frac{dx}{x} = \int_0^\infty J_{2n+1}(x)\, dx \int_0^1 u^{\frac{1}{2}} J_0(ux)\, \lambda(u)\, du$$

$$= \int_0^1 u^{\frac{1}{2}} \lambda(u)\, du \int_0^\infty J_{2n+1}(x)\, J_0(ux)\, dx^*$$

$$= \int_0^1 P_n(1-2u^2)\, u^{\frac{1}{2}} \lambda(u)\, du,$$

since† $\int_0^\infty J_{2n+1}(x)\, J_0(ux)\, dx = P_n(1-2u^2) \quad (u<1) \qquad = 0 \quad (u>1).$

Hence the coefficients are of the Webb-Kapteyn form; and so $f(x)$ is a solution of (K′). The sufficiency of the condition can also be verified without difficulty by direct substitution.

We have next to prove the necessity of the condition. Suppose then that $f(x)$ satisfies (K′). It follows from the theory of Hankel transforms‡ that there is a function $\lambda(u)$, whose square is integrable over $(0, \infty)$, such that the function

$$x^{-\frac{1}{2}} f_a(x) = \int_0^a x^{\frac{1}{2}} u^{\frac{1}{2}} J_0(xu)\, \lambda(u)\, du$$

converges in mean to $x^{-\frac{1}{2}} f(x)$, when $a \to \infty$; and that

$$\lim_{a\to\infty} \int_0^\infty x^{-\frac{1}{2}} J_{2n+1}(x) \{x^{-\frac{1}{2}} f(x) - x^{-\frac{1}{2}} f_a(x)\}\, dx = 0,\S$$

or

$$\int_0^\infty J_{2n+1}(x) f(x) \frac{dx}{x} = \lim_{a\to\infty} \int_0^\infty J_{2n+1}(x) f_a(x) \frac{dx}{x}.$$

Hence

$$a_{2n+1} = (4n+2) \lim_{a\to\infty} \int_0^\infty J_{2n+1}(x)\, dx \int_0^a u^{\frac{1}{2}} J_0(ux)\, \lambda(u)\, du$$

$$= (4n+2) \lim_{a\to\infty} \int_0^a u^{\frac{1}{2}} \lambda(u)\, du \int_0^\infty J_{2n+1}(x)\, J_0(ux)\, dx$$

$$= (4n+2) \int_0^1 P_n(1-2u^2)\, u^{\frac{1}{2}} \lambda(u)\, du.$$

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* The inversion may be justified by bounded convergence.

† The proof of these formulae is included implicitly in Bateman's analysis. See also Watson, *loc. cit.*, 408–410.

‡ Titchmarsh, *loc. cit.*, 470.

§ Since the square of $x^{-\frac{1}{2}} J_{2n+1}(x)$ is integrable.

It follows that the coefficients in the Neumann series, and so also $f(x)$, are independent of the values of $\lambda(u)$ for $u > 1$. Let $\phi(u)$ be equal to $\lambda(u)$ when $u < 1$ and to zero otherwise. Then* $\phi(u)$ possesses a Hankel transform $x^{-\frac{1}{2}}g(x)$, where

$$g(x) = \frac{d}{dx}\left\{x\int_0^\infty u^{-\frac{1}{2}}J_1(ux)\,\phi(u)\,du\right\} = x\int_0^1 u^{\frac{1}{2}}J_0(ux)\,\lambda(u)\,du.$$

This function is, as we have seen, a solution of (K′) : and the coefficients in its Neumann expansion are the same as those of $f(x)$, so that it is identical with $f(x)$. Hence $\lambda(u)$ and $\phi(u)$ have the same Hankel transform, and so $\lambda(u) = \phi(u)$ almost everywhere. That is to say, $f(x)$ is of the form (2 . 61).

A result analogous to Theorem 3 can be obtained in a similar way, viz. that *if*

$$f(x) = x\int_0^\infty u^{\frac{1}{2}}J_0(ux)\,\lambda(u)\,du,$$

then it is necessary and sufficient that $\lambda(u) = 0$ *for* $u > 1$.

2 . 7. In Theorems 1 and 4 two forms for the solution of (K′) have been obtained, viz.

$$(\alpha)\ \int_0^1 \sin xu\,\lambda(u)\,du, \qquad (\beta)\ x\int_0^1 J_0(xu)\,u^{\frac{1}{2}}\lambda(u)\,du,$$

$\lambda(u)$ being a function whose square is integrable. It is clear that these cannot be precisely equivalent, for a function of type (α) has its square integrable, while a function of type (β) is the product of $\sqrt{x}$ by a function of integrable square ; thus (α) is more restricted at infinity, (β) more restricted at $x = 0$. The relation between the two types is as follows :—

If $f(x)$ *is of type* (α), *then, in order that it should be of type* (β), *it is necessary and sufficient that the square of* $x^{-\frac{1}{2}}f(x)$ *should be integrable over* $(0, \infty)$.

If $f(x)$ *is of type* (β), *then, in order that it should be of type* (α), *it is necessary and sufficient that the square of* $f(x)$ *should be integrable over* $(0, \infty)$.

In each case the necessity of the condition is obvious, from the theory of transforms. To prove the sufficiency of the condition in the first pro-

* Titchmarsh, *loc. cit.*, 471.

position, suppose $f(x)$ of the form (a). Since $\{f(x)\}^2/x$ is integrable, we have

$$f(x) = \frac{d}{dx}\int_0^\infty xu^{-\frac{1}{2}} J_1(ux)\,\mu(u)\,du,$$

where μ^2 is integrable; and

$$x^{\frac{1}{2}}\mu(x) = \frac{d}{dx}\int_0^\infty \frac{xJ_1(ux)}{u} f(u)\,du = \frac{d}{dx}\left\{x\int_0^\infty \frac{J_1(ux)}{u}\,du\int_0^1 \sin ut\,\lambda(t)\,dt\right\}$$

$$= \frac{d}{dx}\left\{x\int_0^1 \lambda(t)\,dt\int_0^\infty \frac{J_1(ux)}{u}\sin ut\,du\right\} = 0,$$

if $x > 1$, since $$\int_0^\infty \frac{J_1(ux)}{u}\sin ut\,du = \frac{t}{x} \quad (x > t).$$

Hence $$f(x) = \frac{d}{dx}\int_0^1 xu^{-\frac{1}{2}} J_1(ux)\,\mu(u)\,du = x\int_0^1 u^{\frac{1}{2}} J_0(ux)\,\mu(u)\,du,$$

which is the result stated. The second part of the theorem may be proved in a similar way.

2.8. *An alternative solution of* (B_1).—We can also prove

THEOREM 5.—*If the square of* $f(x)/x$ *is integrable over* $(-\infty, \infty)$, *then, in order that* $f(x)$ *should be a solution of* (B_1), *it is necessary and sufficient that* $f(x)/x$ *should be of the form* (1.23), λ *and* μ *being functions whose squares are integrable over* (0, 1).

The discussion proceeds on the lines of § 2.2. It is rather long, and will be omitted.

2.9. It is plain that some of the simplest solutions of our equations do not belong to the classes hitherto considered. They do not include, for example, the obvious quadratic solutions of (M) or (M_1). Again, $\sin ax$ $(a \leqslant 1)$ is a solution of (B), (K'), and (B_1). It is a case of Theorem 5, but not of Theorems 1 or 4. On the other hand

$$\sin ax = x\int_0^a J_0(xt)(a^2-t^2)^{-\frac{1}{2}}\,t\,dt,$$

so that it is of the form mentioned at the end of § 2.6, when $\lambda(u)$, but not its square, is integrable. If the integrals are taken to be non-absolutely convergent integrals, or integrals generalised in the "Cesàro" or other senses, new solutions appear; thus any polynomial, or any polynomial multiplied by $\sin ax$ $(a \leqslant 1)$, is in this sense a solution of (K).

3. *Solutions by the methods of the theory of functions of a complex variable.*

3.1. *Proof of Theorem 6.*

3.11. To prove the sufficiency of the condition (1.42), consider the integral

$$\int \frac{H_1^{(1)}(z-a)}{z-a} f(z)\,dz,$$

where a is real and the contour consists of the real axis, with an indentation above the point $z=a$, and a large semi-circle above it; and $H_1^{(1)}$ is the Bessel function of the third kind. On the large semi-circle

$$H_1^{(1)}(z-a) = O(|z-a|^{-\frac{1}{2}} e^{-y}), \qquad f(z) = O(|z|^{\frac{1}{2}-\delta} e^{y}),$$

so that this part of the integral tends to zero when the radius tends to infinity. The integral round the indentation at $z=a$ tends to $-2f'(a)$, since

$$H_1^{(1)}(z-a) = -\frac{2i}{\pi(z-a)} + O(z-a).$$

Hence

$$\int_{-\infty}^{\infty} \frac{H_1^{(1)}(z-a)}{z-a} f(z)\,dz = 2f'(a).$$

Taking the real parts of this equation, we see that $f(z)$ is a solution of (K). The proof that the condition is necessary is more elaborate.

3.12. LEMMA *a.—Suppose that $f(z)$ is an integral function which satisfies the following conditions:—*

(3.121) $$|f(x)| < C_1 |x|^{\alpha} \quad (\alpha > 0),\ |x| > x_0;$$

(3.122) $$|f(iy)| < C_2(\epsilon)\, e^{|y|(1+\epsilon)}$$

for all positive values of ϵ;

(3.123) $$|f(z)| < C_3\, e^{K|z|}$$

for some value of K and all values of z. Then

(3.124) $$|f(z)| \leqslant C |z|^{\alpha} e^{|y|}$$

for all values of z of sufficiently large modulus.

Here K and the C's are absolute constants, except C_2, which may depend upon ϵ. The lemma is contained in substance in well known

theorems due to Phragmén and Lindelöf.* Let

$$g(z) = (z+ik)^{-\alpha} e^{i(1+\delta)z} f(z) \quad (k > 0, \ \delta > \epsilon). \tag{3.125}$$

Then $|g(z)| < C_1$ on the real axis. Also $g(iy) \to 0$ when $y \to \infty$, so that $|g(iy)|$ has a maximum C_4, and $|g(z)| \leqslant C_4$ on the positive imaginary axis. Finally $|g(z)| < C_5 e^{K|z|}$ for all values of z. Hence

$$|g(z)| \leqslant C_6 = \text{Max}\,(C_1, C_4),$$

for $y \geqslant 0$. This would not in itself suffice to prove the lemma, since C_4 depends *prima facie* on δ. But it is easily proved that $C_4 \leqslant C_1$.† For suppose $C_4 > C_1$, and consider the angle bounded by the lines $\theta = 0$, $\theta = \frac{1}{2}\pi + \beta$, where $\theta = \arg z$ and $0 < \beta < \frac{1}{2}\pi$. Then $|g(z)| \leqslant C_4$ on the boundary of the angle, and attains the value C_4 at a point inside it; from which it follows that $g(z)$ is constant, and that $|g(z)| \leqslant C_1$ in any case. Thus we may take $C_6 = C_1$.

It follows then from (3.125) that

$$|f(z)| \leqslant C\,|z|^{\alpha} e^{(1+\delta)y} \quad (y > 0, \ |z| > A)$$

for all positive values of δ, and so

$$|f(z)| \leqslant C|z|^{\alpha} e^{y}.$$

A similar result can be obtained for $y < 0$. Thus the lemma is completely proved.

3.13. Suppose now that $f(x)$ is a solution of (K) which satisfies the condition (1.41). Then it follows from § 2.5 that $f(x)$ is expansible in a Neumann series (1.142) with Webb-Kapteyn coefficients. The series is convergent for all real values of x, and so by the general theory of Neumann series, for all complex values; hence $f(z)$ is an integral function.

It remains to prove that condition (1.42) is satisfied, *i.e.* that (3.124) is satisfied with $\alpha = \frac{1}{2} - \delta$. Since (1.41) is equivalent to (3.121) with $\alpha = \frac{1}{2} - \delta$, it is sufficient to prove that (3.122) and (3.123) are satisfied.

The proof of the last assertion is immediate; for‡

$$|J_{2n+1}(z)| \leqslant \frac{|\frac{1}{2}z|^{2n+1}}{(2n+1)!} e^{|y|}, \tag{3.131}$$

* E. Phragmén and Ernst Lindelöf, "Sur une extension d'un principe classique de l'analyse", *Acta Math.*, 31 (1908), 381–406 (385).

† Cf. Phragmén and Lindelöf, *loc. cit.*, 387.

‡ Watson, *loc. cit.*, 49, § 3.31 (1).

and it is evident that

$$a_{2n+1} = O(n). \tag{3.132}$$

Hence

$$\Sigma |\, a_{2n+1} J_{2n+1}(z) \,| < C |\, z \,|^C e^{2|z|} < C e^{3|z|}. \tag{3.133}$$

To prove that (3.122) is satisfied, we may use the inequality*

$$|\, J_m(iy) \,| < \frac{y^m e^{\sqrt{(m^2+y^2)}}}{\{m+\sqrt{(m^2+y^2)}\}^m} \quad (y > 0). \tag{3.134}$$

It may be verified that, for a given value of y, the expression on the right decreases as m increases; and for $m = 0$ it reduces to e^y. Let

$$\sum_{n=0}^{\infty} a_{2n+1} J_{2n+1}(iy) = \sum_{2n+1 \leqslant 3y} + \sum_{2n+1 > 3y} = \Sigma_1 + \Sigma_2.$$

Then

$$|\, \Sigma_1 \,| < C e^y \sum_{n < Cy} n < C y^2 e^y.$$

If $m > 3y$, (3.134) gives

$$|\, J_m(iy) \,| < \left(\frac{1}{3+\sqrt{10}}\right)^m e^{\frac{1}{3}m\sqrt{10}},$$

which is the general term of a convergent geometrical progression. Thus $|\, \Sigma_2 \,| < C$ and so

$$\sum_{n=0}^{\infty} a_{2n+1} J_{2n+1}(iy) = O(y^2 e^y) = O(e^{(1+\epsilon)y}) \quad (\epsilon > 0),$$

which completes the proof.

3.2. *Proof of Theorem* 7.

3.21. It is convenient to assume at first that $f(x)$ has derivatives of all orders for all real values of x, a restriction which is to be removed later.

LEMMA β.—*If $f(x)$ is an indefinitely differentiable solution of* (M), *and*

$$|\, f(x) \,| < C e^{A|x|} \quad (A < 1) \tag{3.211}$$

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* Watson, *loc. cit.*, 268, § 8.7 (1).

for all real values of x, then

$$|f^{(p)}(x)| < C\left(\frac{8e^{\frac{1}{2}A}}{1-A}\right)^p e^{A|x|} \tag{3.212}$$

for all values of p and all real values of x.

It is sufficient to prove that (3.212) holds for $p=1$; for then the integral obtained by differentiating under the integral sign is uniformly convergent, so that $f'(x)$ is also a solution of the equation, and the argument can be repeated. Since $f'(x)$ is continuous, we have

$$f'(x) = \tfrac{1}{2}\int_0^\infty \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt$$

$$= \int_0^\delta e^{-t} f'(x+\theta t)\,dt + \tfrac{1}{2}\int_\delta^\infty \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt,$$

where $|\theta|<1$; and

$$|f'(x)| \leqslant \int_0^\delta e^{-t}|f'(x+\theta t)|\,dt + \frac{1}{\delta}\int_\delta^\infty e^{-t}.\,Ce^{A|x|+A|t|}.\,dt$$

$$\leqslant \mu\delta + \frac{Ce^{A|x|}}{\delta(1-A)},$$

where

$$\mu = \operatorname*{Max}_{|\xi-x|\leqslant\delta} |f'(\xi)|.$$

But $|f'(\xi)|$, being continuous, attains its upper bound for some value x_1 of ξ such that $|x_1-x| \leqslant \delta$. Hence

$$|f'(x)| \leqslant \delta|f'(x_1)| + \frac{Ce^{A|x|}}{\delta(1-A)}.$$

In exactly the same way a chain of inequalities can be formed, the $(n+1)$-th being

$$|f'(x_n)| \leqslant \delta_n|f'(x_{n+1})| + \frac{Ce^{A|x_n|}}{\delta_n(1-A)} \tag{3.213}$$

$$(\delta_0=\delta, \quad x_0=x, \quad |x_{n+1}-x_n| \leqslant \delta_n, \quad n=0,\,1,\,2,\,\ldots).$$

Let $\delta_n = 2^{-n-2}$. Then

$$|x_n| \leqslant |x| + 2^{-2}+2^{-3}+\ldots+2^{-n-1} < |x|+\tfrac{1}{2},$$

and (3.213) gives, for all values of n,

$$|f'(x_{n+1})| > 2^{n+2}|f'(x_n)| - \frac{2^{2n+4}Ce^{A|x|+\frac{1}{2}A}}{1-A}. \tag{3.214}$$

If the lemma is false, for $p=1$, there is a value of x for which

$$|f'(x)| > \frac{8Ce^{A|x|+\frac{1}{2}A}}{1-A}. \tag{3.215}$$

But it follows from (3.214) that if, for any value of n,

$$|f'(x_n)| > \frac{2^{n+3}Ce^{A|x|+\frac{1}{2}A}}{1-A},$$

then this is also true with $n+1$ instead of n. Hence, if it is true for $n=0$, as in (3.215), it is true for all values of n; and it follows that $f'(x)$ is not bounded in the interval $(x-\frac{1}{2}, x+\frac{1}{2})$, contrary to the hypothesis that $f'(x)$ is continuous. This proves Lemma β.

3.22. LEMMA γ.—*If $f(z)$ is an integral function such that*

$$|f(x)| < Ce^{A|x|}, \quad |f(z)| < Ce^{K|z|} \quad (K > A),$$

then

$$|f(z)| < Ce^{A|x|+K|y|}. \tag{3.221}$$

This is another case of the theorems of Phragmén and Lindelöf already referred to. In fact, if

$$g(z) = e^{-(A-iK)z}f(z),$$

then $|g(z)| < C$ when z lies on the positive halves of the real and imaginary axes, and $|g(z)| < Ce^{K|z|}$ for all values of z in the first quadrant. Hence $|g(z)| < C$ for all values of z in the first quadrant, and (3.221) follows. The other quadrants may be treated in a similar way.

3.23. LEMMA δ.—*Under the hypotheses of Lemma α, $f(z)$ is an integral function, and satisfies* (3.221).

For $|f^{(p)}(0)| < CK^p$, by Lemma β. Hence the series

$$\sum_{p=0}^{\infty} \frac{z^p}{p!} f^{(p)}(0)$$

is convergent for all values of z, *i.e.* $f(z)$ is an integral function; and

$$|f(z)| < C \sum_{p=0}^{\infty} \frac{|z|^p}{p!} K^p = Ce^{K|z|}.$$

Lemma δ now follows from Lemma γ.

3.24. LEMMA ϵ.—*The function $f(z)$ can be expressed in the form*

$$f(z) = \frac{1}{2\pi i} \int_C e^{zu} \chi(u)\, du, \tag{3.241}$$

where $\chi(u)$ is regular at all points (including infinity) outside the rectangle whose vertices are $(\pm A, \pm K)$, and

$$\chi(u) = \frac{b_1}{u} + \frac{b_2}{u^2} + \ldots \tag{3.242}$$

for sufficiently large values of u. The contour C is a closed curve surrounding the rectangle, but lying between the lines $\mathfrak{R}(u) = \pm 1$.

Let $$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \phi(z) = \sum_{n=0}^{\infty} n!\, a_n z^n.$$

The function $\phi(z)$ is regular for sufficiently small values of z, since

$$|n!\, a_n| = |f^{(n)}(0)| < CK^n.$$

We proceed to prove it regular in a more extended domain. In the first place,

$$\phi(z) = \int_0^{\infty} e^{-u} f(zu)\, du$$

for sufficiently small values of z. The integral is convergent if

$$A|x| + K|y| < 1;$$

so that $\phi(z)$ is regular inside the rhombus **R** whose vertices are $(\pm A^{-1}, 0)$, $(0, \pm K^{-1})$. The domain can be extended still further by using the formula

$$\phi(z) = e^{i\lambda} \int_0^{\infty} e^{-ue^{i\lambda}} f(zue^{i\lambda})\, du,$$

which is valid if $-\frac{1}{2}\pi < \lambda < \frac{1}{2}\pi$ and

$$A|x\cos\lambda - y\sin\lambda| + K|x\sin\lambda + y\cos\lambda| < \cos\lambda.$$

This region also is a rhombus; and $\phi(u)$ is regular inside the region **D** formed by all points inside one rhombus or another. The

region **D** is bounded by four circles, described on the lines joining the origin to the corners of **R**; and is the inverse of the rectangle of the lemma with respect to the origin. Hence $\phi(1/u)$ is regular outside this rectangle. If now C is the contour of the lemma, we have

$$\frac{1}{2\pi i}\int_C e^{zu}\,\phi\left(\frac{1}{u}\right)\frac{du}{u} = \frac{1}{2\pi i}\int_C e^{zu}\left(\sum_{n=0}^{\infty}\frac{n!\,a_n}{u^n}\right)\frac{du}{u} = \sum_{n=0}^{\infty} a_n z^n = f(z);$$

which proves (3.241), with

$$\chi(u) = \frac{1}{u}\,\phi\left(\frac{1}{u}\right).$$

3.25. *Proof of the main theorem.*—We have, by (3.241),

$$f(x+t)-f(x-t) = \frac{1}{2\pi i}\int_C e^{xu}(e^{tu}-e^{-tu})\,\chi(u)\,du,$$

and hence

$$\begin{aligned}\tfrac{1}{2}\int_0^\infty \frac{e^{-t}}{t}\,\{f(x+t)-f(x-t)\}\,dt &= \frac{1}{2\pi i}\int_C e^{xu}\,\chi(u)\,du\int_0^\infty \frac{e^{-t}}{2t}\,(e^{tu}-e^{-tu})\,dt\\ &= \frac{1}{2\pi i}\int_C e^{xu}\,\chi(u)\,\tfrac{1}{2}\log\frac{1+u}{1-u}\,du,\end{aligned}$$

that value of the logarithm being taken which vanishes for $u = 0$. Also

$$f'(x) = \frac{1}{2\pi i}\int_C e^{xu}\,u\,\chi(u)\,du;$$

so that equation (M) takes the form

$$\int_C e^{xu}\,\chi(u)\,\psi(u)\,du = 0,$$

where

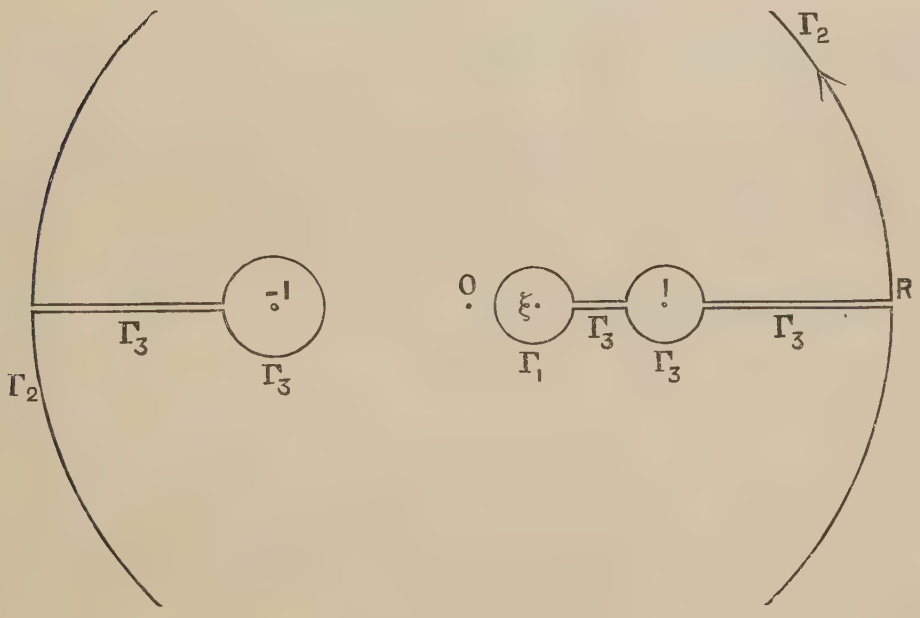
$$\psi(u) = u-\tfrac{1}{2}\log\frac{1+u}{1-u}. \tag{3.251}$$

Let $\mu<1$ be the maximum of $\Re(u)$ on C; multiply by $e^{-\xi x}$, where $\mu<\xi<1$, and integrate over $(0,\infty)$. We obtain

$$\int_C \chi(u)\,\psi(u)\,\frac{du}{\xi-u} = 0. \tag{3.252}$$

We may deform C into the contour $\Gamma = \Gamma_1+\Gamma_2+\Gamma_3$ shown in the figure.* The radii of the small circles are all equal to ρ, and Γ_3 comprises the whole of Γ except the large circle Γ_2 and the small circle Γ_1 round ξ. We have

$$\int_C = \int_\Gamma = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} = 0.$$



It is easily proved that

$$\lim_{\rho\to 0} \frac{1}{2\pi i}\int_{\Gamma_1} = \chi(\xi)\,\psi(\xi), \qquad \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Gamma_2} = -b_1,$$

$$\lim \frac{1}{2\pi i}\int_{\Gamma_3} = -\int_1^\infty \frac{\chi(u)}{\xi-u}\,du - \int_{-\infty}^{-1} \frac{\chi(u)}{\xi-u}\,du.$$

Hence
$$\chi(\xi)\,\psi(\xi) = b_1 + \int_1^\infty \frac{\chi(u)}{\xi-u}\,du + \int_{-\infty}^{-1} \frac{\chi(u)}{\xi-u}\,du.$$

The expression on the right is an analytic function of ξ, regular for all values of ξ except real values such that $\xi \geqslant 1$ or $\xi \leqslant -1$; and so the same is true for $\chi(\xi)\,\psi(\xi)$.

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* Most of the large circle Γ_2 is omitted in the figure. The point $u = \mu$ lies between O and the point nearest to O of the circle Γ_1, and $u = A$ lies between O and μ, so that the rectangle of Lemma ϵ lies entirely inside Γ.

We know already that $\chi(\xi)$ is regular outside the rectangle of Lemma ϵ; and it is easily verified that $\psi(\xi)$ has no zeros inside the rectangle, except its obvious zero, of the third order, for $\xi = 0$. Hence the only possible singularity of $\chi(u)$ is a pole of the third order at $u = 0$. Since $\chi(u)$ is regular at infinity, (3 . 242) reduces to

$$\chi(u) = \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_3}{u^3};$$

and

$$f(x) = \frac{1}{2\pi i}\int_C e^{xu}\chi(u)\,du = b_1 + b_2 x + \tfrac{1}{2}b_3 x^2.$$

3 . 26. We have still to prove the theorem without assuming the existence of all the derivatives of $f(x)$. Suppose now simply that $f(x)$ satisfies (1 . 43), and let

$$\phi(x) = \int_{-\infty}^{\infty} f(y)\,\mathrm{sech}\,(y-x)\,dy = \int_{-\infty}^{\infty} f(x+u)\,\mathrm{sech}\,u\,du.$$

Then if $f(x)$ satisfies (M), so does $\phi(x)$; for

$$\begin{aligned}\tfrac{1}{2}\int_0^{\infty} \frac{e^{-t}}{t}\,\{\phi(x+t)-\phi(x-t)\}\,dt &= \tfrac{1}{2}\int_0^{\infty}\frac{e^{-t}}{t}\,dt\int_{-\infty}^{\infty}\{f(x+t+u)-f(x-t+u)\}\,\mathrm{sech}\,u\,du\\ &= \int_{-\infty}^{\infty} f'(x+u)\,\mathrm{sech}\,u\,du = \phi'(x).\end{aligned}$$

Also

$$\phi(x) = O\left(\int_{-\infty}^{\infty} e^{A|x|+A|u|}\,\mathrm{sech}\,u\,du\right) = O(e^{A|x|}),$$

and plainly all derivatives of $\phi(x)$ exist; so that by what has been proved already

$$\int_{-\infty}^{\infty} f(y)\,\mathrm{sech}\,(y-x)\,dy = \phi(x) = a+bx+cx^2.$$

We can determine constants α, β, γ, such that

$$\int_{-\infty}^{\infty} (\alpha+\beta y+\gamma y^2)\,\mathrm{sech}\,(y-x)\,dy = a+bx+cx^2.$$

Hence

$$\int_{-\infty}^{\infty} F(y)\,\mathrm{sech}\,(y-x)\,dy = 0$$

if

$$F(y) = f(y)-\alpha-\beta y-\gamma y^2.$$

Since $F(y) = O(e^{A|y|})$, we can differentiate repeatedly with respect to x, and then put $x = 0$; thus

$$(3.261) \qquad \int_{-\infty}^{\infty} \frac{F(y)}{(\cosh y)^{2p+1}}\, dy = 0, \qquad \int_{-\infty}^{\infty} \frac{\sinh y\, F(y)}{(\cosh y)^{2p}}\, dy = 0,$$

for all positive integral values of p. The first of these equations can be written

$$\int_0^{\infty} \frac{F(y)+F(-y)}{\cosh y} (\operatorname{sech}^2 y)^p\, dy = 0,$$

or, putting $\operatorname{sech}^2 y = e^{-u}$,

$$\int_0^{\infty} G(u)\, e^{-pu}\, du = 0,$$

where $$G(u) = \tfrac{1}{2} \operatorname{cosech} y \left(F(y)+F(-y)\right).$$

Hence* $G(u) = 0$, *i.e.* $F(y)$ is an odd function of y. It follows in the same way from the second equation (3.261) that $F(y)$ is an even function of y. Hence

$$F(y) = 0, \qquad f(y) = \alpha+\beta y+\gamma y^2,$$

which completes the proof.

3.3. *Proof of Theorem* 8.

It follows without difficulty from our hypotheses that

$$f(x) = O(|x|^{-\frac{1}{2}} e^{|x|})$$

for real x, so that the integral (M) is absolutely convergent. But we shall not use this fact explicitly.

We have, for all values of x and t,

$$\frac{1}{2t}\{f(x+t)-f(x-t)\} = f'(x)+\frac{t^2}{3!} f^{(3)}(x)+\frac{t^4}{5!} f^{(5)}(x)+\ldots.$$

Multiplying by e^{-t}, and integrating term by term,

$$(3.31) \qquad \tfrac{1}{2}\int_0^{\infty} \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\, dt = f'(x)+\tfrac{1}{3}f^{(3)}(x)+\tfrac{1}{5}f^{(5)}(x)+\ldots.$$

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* By the well-known theorem of Lerch: see M. Lerch, "Sur un point de la théorie des fonctions génératrices d'Abel", *Acta Math.*, 27 (1903), 339–351. For references to the many later proofs of the theorem, see G. H. Hardy, "Notes on some points in the integral calculus (42)", *Messenger of Math.*, 46 (1917), 43–48.

This step is justified provided the integrated series is convergent.* Now

$$\sum_{p=1}^{n} f^{(2p-1)}(x) = \sum_{p=1}^{n} a_{2p-1} + x \sum_{p=1}^{n} a_{2p} + \frac{x^2}{2!} \sum_{p=1}^{n} a_{2p+1} + \ldots$$

and hence

$$\left| \sum_{p=1}^{n} f^{(2p-1)}(x) \right| < 2M + |x| 2M + \frac{|x|^2}{2!} 2M + \ldots < 2Me^{|x|},$$

M being the upper bound of the moduli of the sums of alternate coefficients. Hence the series (3.31) is convergent for all values of x, and indeed uniformly for $|x| < K$. The term by term integration is thus justified; and (M) and (3.31) give

$$\tfrac{1}{3} f^{(3)}(x) + \tfrac{1}{5} f^{(3)}(x) + \ldots = 0. \tag{3.32}$$

Since this series is uniformly convergent throughout any finite domain of values of x, we may multiply by x^{-p} and integrate term by term round a contour enclosing the origin. Thus

$$\tfrac{1}{3} a_p + \tfrac{1}{5} a_{p+2} + \tfrac{1}{7} a_{p+4} + \ldots = 0 \quad (p = 3, 4, \ldots), \tag{3.33}$$

and so

$$\tfrac{1}{3} a_p = (\tfrac{1}{3} - \tfrac{1}{5}) a_{p+2} + (\tfrac{1}{5} - \tfrac{1}{7}) a_{p+4} + \ldots. \tag{3.34}$$

Let $\mu_p = \text{Max}(|a_p|, |a_{p+2}|, \ldots)$. Then $\mu_{p+2} \leqslant \epsilon$ for $p \geqslant p_0(\epsilon)$, since $a_p \to 0$. It follows from (3.34) that $|a_p| \leqslant \mu_{p+2} \leqslant \epsilon$, and so

$$\mu_p = \text{Max}(|a_p|, \mu_{p+2}) = \mu_{p+2} \leqslant \epsilon.$$

Repeating the argument, we find that $|\mu_3| \leqslant \epsilon$, $|\mu_4| \leqslant \epsilon$, which is plainly only possible if $\mu_3 = 0$, $\mu_4 = 0$, *i.e.* if $a_3 = a_4 = \ldots = 0$. This proves the theorem.

* See G. H. Hardy, "Further researches in the theory of divergent series and integrals", *Trans. Camb. Phil. Soc.*, 21 (1908), 1–48 (5).

ON THE SOLUTION OF A PAIR OF SIMULTANEOUS DIOPHANTINE EQUATIONS CONNECTED WITH THE NUPTIAL NUMBER OF PLATO

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1. In Adam's edition of Plato's *Republic*,* he writes: "The famous Number of Plato is notoriously the most difficult passage in his writings." Cousin,† in his French translation, omits the passage altogether, leaving a blank, and explaining in a note his inability to make sense of it. The difficulty, no doubt, lies to a great extent in the Greek; but not entirely. Plato is confessedly using cryptic language, he is as they say, "Pythagorising." The Muses are supposed to be speaking, in answer to a question as to the causes of the degeneracy which must have taken place, supposing the ideal Republic really to have existed in the remote past. In accordance with Pythagorean tradition, the Muses indicate a certain number,

ἀριθμὸς γεωμετρικὸς τοιούτου κύριος,

obtained by certain geometrical processes, as underlying the solution of the problem. Such numbers, or periods, they explain, are characteristic of the various species of living creatures, including the universe itself, the Makrokosmos, which Plato regarded as a living creature, created indeed, but endowed, unlike the human creature, the Mikrokosmos, with divine attributes.

The passage (Republic VIII, 546 B, C) may be rendered into English as follows:—

"For that which, though created, is divine, a recurring period exists,

* J. Adam, *The Republic of Plato*, Cambridge University Press, 2 (1907), 265 *seq.* In these *Proceedings* full references to the numerous authors, classical and modern, I have consulted, would be out of place. I have accordingly, as far as possible, referred only to the English standard books, more especially in view of the fact that Dr. Adam's discussion of Plato's number is quoted as the last word on the subject by the most recent continental writers. I should like here to thank Prof. Rose of Aberystwyth for his encouragement and for revision of the classical part of my work.

† V. Cousin, *Œuvres de Platon*, 10 (Paris, Rey et Gravier, 1834), 130. Note 322–341.

which is embraced by a perfect number. For that which is human, however, by that one for which it first occurs that the increasings of the dominants and the dominated,* when they take three spaces† and four boundaries‡ making§ similar and dissimilar and increasing and decreasing, cause all to appear familiar and expressible.

"Whose base, modified as four to three, and married to five, three times increased, yields two harmonies: one equal multiplied by equal,‖ a hundred times the same: the other, equal in length to the former, but oblong,¶ a hundred of the numbers upon** expressible diameters of five, each diminished by one, or by two if inexpressible, and a hundred cubes of three. This sum now, a geometrical number, is lord over all these affairs, over better and worse births; and when in ignorance of them, the guardians

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* *I.e.* the hypotenuse and the sides of a right-angled triangle: Alexander Aphrodisiensis (end of the second century A.D.) explains that the Pythagoreans used these terms in this sense. The passage is quoted in Adam, 269.

† Ἀποστάσεις: the word has been translated "distances"; it has been rendered "dimensions," which is nearer to διαστάσεις, and is identified with this by Adam, 271. Both these seem to me too special in the present mystic phrase: it has been rendered "intervals," and would appear to mean *that which is contained between the boundaries*, that in virtue of which one of these boundaries *stands off* or *stands away* from the others, becomes, shall we say, an *apostate*. In another passage (*Timaeus*, 43 D), Plato uses *the same word* in a sense which is absolutely the same as in my reading of the present passage. He is considering the double τετρακτύς

1
2 3
4 9
8 27

and he speaks of the ἀποστάσεις being "three of each," that is three in each arm of the τετρακτύς, namely those things—to use an indefinite term—which constitute at once the connection and the differentiation of the individuals—the binary and ternary numbers—here grouped to make the arms of the figure. These three things may be taken, in the figure, to be represented by the spaces between the numbers, referred to the direction of the arm in question, but not essentially without a certain width, perfectly indefinite in size, for the rest. The word *space* has here exactly the same sense as in musical notation, "lines and spaces."

‡ Ὅρους, the same word is used in Euclid, Bk. 1, Def. 13,

Ὅρος ἐστὶν ὅ τινός ἐστι πέρας.

"A boundary is that which is the extremity of anything." See Heath's *Euclid*, 1, 143, for an interesting note on the use of this word: ὁ ὅρος is used by Aristotle (Eth. Nic. v, 6, 7, 1131^b, 59), Euclid, Bk. 5, Def. 8, Nicomachus, Theon of Smyrna, in the sense of "term" in a proportion.

§ Literally "the dominating and dominated increasings of the things (*i.e.* the lines composing a right-angled triangle) which make similar . . ."

‖ Recognized expression for "the square of a number." Euclid, Bk. 2, Def. 18.

¶ *I.e.* not square, but the product of two unequal factors; cp. Plato's *Theaetetus*, 148 A, B.

** Recognized expression for "the square on," or, as Heath translates it, "produced by", a number. Euclid, Bk. 7, Def. 19, Heath, 2, 320.

unite the brides and bridegrooms wrongly, the children will not be well-endowed, either in their constitutions or in their fates."

Long intercourse with mathematicians has taught me that their covert allusions to mathematics, as throwing light on philosophical or other matters, are usually as profound as their own mathematical knowledge: and some years ago, when I first found myself face to face with the question what mathematical truths Plato was referring to in these oracular utterances, I felt that Plato was here epitomising some part of his own mathematical cogitations, and that, in unravelling this mystery, we should be gaining a clue by which we may be able to trace somewhat of the intellectual biography of one of the most ancient and eminent of Greek mathematicians. Tradition tells us but little about this. We know that Plato was particularly interested in the theory of proportion (a theory which was in his time somewhat in the stage that the theory of integration has been in ours) and we have more than tradition* in support of the belief that Propositions 11 and 12 of the Eighth Book of Euclid† are due to Plato himself: there can be little doubt that they are enunciated in the *Timaeus*,‡ and, unless I am mistaken, they recur in the present connexion. Plato is credited§ with a partial solution‖ of the equation of Pythagoras

$$x^2+y^2 = z^2, \qquad \text{(I)}$$

namely $$n^2+\{\tfrac{1}{4}(n^2-4)\}^2 = \{\tfrac{1}{4}(n^2+4)\}^2$$

where n is any even integer, Pythagoras himself being said to have given the partial solution

$$\{\tfrac{1}{2}(n^2-1)\}^2+n^2 = \{\tfrac{1}{2}(n^2+1)\}^2,$$

where n is odd. It will be seen that these considerations also reappear in the present discussion.

Tradition¶ tells us that Plato occupied himself with the problem of the duplication of the cube and that he invented a mechanical instrument

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* Nicomachus, about 100 A.D. See Heath, 2, 363.

† "Between two square numbers there is one mean proportional . . ." "Between two cube numbers there are two mean proportionals . . ."

‡ *Timaeus*, 32 A sqq.

§ Proclus, 410–485 A.D. See Heath, 1, 355.

‖ The general solution is given in Euclid, Bk. 10, 28, Lemma 1, Heath, 3, 63.

¶ Eutocius, 6th century. See Gow's *History of Greek Mathematics*, 180, where also the reasons are given why discredit has been thrown on this story, reasons so plausible that I had taken no interest in this instrument, which accounts for my not having seen or noticed the figure. Even if the story of the invention of the machine by Plato himself is not authentic, and the invention be due to one of his pupils, this only corroborates my contention, that the figure itself is Plato's.

for the solution. Now it is a curious fact that it was only after the present paper was completely finished, that I found* a description of this instrument, and it exactly corresponds to the figure which I had myself drawn as illustrating the present passage, as well as the passage in the *Timaeus*, a figure which I had been convinced, from internal evidence, had been in Plato's mind at the time of writing. Problems of three dimensions had undoubtedly occupied Plato's attention: he refers in his writings to the regular solids, some if not all of which were already known to the Pythagoreans before his day. They were traditionally known as the "Platonic figures" and the fact that Plato is stated by Proclus to have initiated the solution of the problem of "golden section,"† agrees with the tradition that he threw light on the theory, in particular, of the dodecahedron. This subject is closely connected with the theory of irrationals,‡ and we know from Plato's own writings that he was enamoured of the pioneer work of Theodorus and Theaetetus in this subject; there can be no doubt whatever about the references to it in the present passage.

From Aristotle§ we learn that, in the speech before us—which seems, by the way, to have presented no serious obscurities for the early commentators—Plato is referring to the Cosmic Triangle, namely that right-angled triangle whose sides are 3 and 4, the former a "male" and the latter a "female" number according to the Pythagoreans. The hypotenuse, which at the same time results from and unites them,‖ is accordingly 5. This triangle, which to the Egyptians symbolized the universe, was connected by the Pythagoreans with marriage. The area, 6, being at once the first of the perfect numbers¶ and the product of the first "male" and "female" numbers, was symbolic of the union of the sexes, as the other Pythagorean marriage number 5 may be held to be symbolic of its fruit.

As Plato evidently regarded the problem of the degeneracy of the human race as belonging to what we now call Eugenics, it was only to be expected that he would make the Muses begin from the Cosmic Triangle.

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* See Heath, 3, 438 and 525.

† Cutting a line in extreme and mean ratio. See Heath, 2, 99.

‡ One wonders whether Plato knew the elegant construction of Euclid, Bk. 13, Prop. 18 (Heath, 3, 503), "to set out the sides of the five figures, and to compare them with one another."

§ Polit. (E. 12, 1316^a, 1–17). The passage is very sketchy. Aristotle uses "diagram" quite vaguely. Heath translates the term "geometrical proposition," 1, 252.

‖ Cp. Plato's *Timaeus* 31 C. and 32 A.B.

¶ $6 = 1 + 2 + 3$. Euclid, Bk. 7, Def. 22. Heath, 2, 293.

Aristotle tells us plainly that Plato is considering the case where "the number of the diagram is made solid," or, as we should say, when it assumes three dimensions.* By "the number of the diagram" we may fairly assume that Aristotle means the marriage number 6, as representing the area of the Cosmic Triangle† and an early commentator who lived between Cicero and Ptolemy‡ connects the passage with the equation

$$3^3+4^3+5^3 = 6^3.$$

2. My own idea, when I first read the passage, was that Plato was essentially alluding to the more general equation

$$x^3+y^3+z^3 = t^3, \qquad \text{(II)}$$

where x, y, and z are the sides and the hypotenuse of a right-angled triangle, so that these quantities are connected by the equation of Pythagoras

$$x^2+y^2 = z^2, \qquad \text{(I)}$$

and that the phrase about *making similar and dissimilar, and increasing and decreasing*, refers to what we modern mathematicians would express by saying that x, y, z and t are *variables*.§ I fancied that Plato knew, or at least, with the intuition of the mathematician, had jumped to the conclusion, that *all the solutions possible of these two simultaneous equations in integers were of the form*

$$3m, \quad 4m, \quad 5m, \quad 6m.$$

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* Euclid, Bk. 7, Def. 17. Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν. "And, when three numbers having multiplied one another make some number, the number so produced is solid." Heath, 2, 278, 290.

† This was the opinion of Cousin (*loc. cit.*), and is supported by Adam. See Adam's Ed. of Plato's *Republic*, 2, 307, footnote [3].

‡ Aristides Quintilianus, quoted by Adam, *loc. cit.*, 374. The reasons deduced by Adam for accepting the number 216, or 6^3, and the above equation, are numerous and convincing, but there is no reason for quoting them further here. It will be seen that I venture to agree with Adam in connecting the equation with the first part of the passage. Cousin connected it also with the latter part, but his remarks are entirely tentative. Paul Tannery also introduces 6^3, but, though I have gained much from his works as to general knowledge, it would be difficult to quote them, more especially as his memoir on the *Nombre nuptial dans Platon* (1882) did not represent his final views. His untimely death prevented his rewriting this article.

§ [Note added Feb. 1924. This contention is supported by the argumentation of R. Vogt in his article "Die Entdeckungsgeschichte des Irrationalen nach Plato", *Bibliotheca Math.* (3), 10 (1909), 103–5, based on the *Theaetetus* (*loc. cit.*) and the *Hippias major*, 303 B and C.]

The fact that I have *a posteriori* proved this to be true is, to say the least, suggestive, more especially since Plato's formula for the solution of the equation of Pythagoras would show that he had fully grasped the idea of a variable integer.

In form the solution which I have given is modern; essentially, however, it does not involve any of those new methods and ideas that are foreign to ancient Greek mathematics. It depends on the consideration of quadratic remainders, or "squares," in the scales of 8 and 3, namely on the two congruence identities,

$$n^2 \equiv 1 \pmod 8$$

and

$$n^2 \equiv 1 \pmod 3,$$

which express the fact that the square of an odd integer, when divided by 8 leaves the remainder 1, and that of a number not divisible by 3, when divided by 3, leaves the same remainder 1.

Beside these, I use, intrinsically, a very simple Lemma, namely that *if* a^3+b^3 *is divisible by* 3, *it is divisible by* 9. The whole question is shown to depend on the solution of the equation

$$\zeta^2 = 12\eta^4+6\alpha^2\eta^2+\alpha^4.$$

The impossibility of this may be proved in different ways.* My method has been to solve on Euclidean lines the Diophantine equation

$$\phi^2+3\psi^2 = 4\chi^4$$

and to show that this equation is incompatible with the relation

$$\psi-\chi^2 = 2^{2\lambda-1}3^{2\mu}\omega^2 \quad (\lambda > 0, \quad \mu \geqslant 0).$$

This result serves to prove the non-existence of solutions of (I) and (II) other than the one above referred, which, for definiteness, I call *the solution of Plato-Pythagoras or P.-P. solution.*

3. In speaking of x and y (the *dominated*) and z (the *dominant*) as becoming three-dimensional—for so, in accordance with Aristotle and Adam, though going a step further than either of them, I interpret the phrase about the three spaces and the four boundaries—I thought I perceived that Plato had in his mind *all possible* rectangular blocks, or, as we

* Mr. Littlewood, who kindly looked through my calculations, has given me an alternative proof, following Fermat's method of "infinite descent", see L. E. Dickson, *History of the Theory of Numbers*, Washington, 2 (1920), 616 *seq.*

should say, cubic combinations formed from these quantities, and that he was recurring to his favourite theme of proportionals. It is, to say the least, interesting that the equation

$$(\tfrac{1}{2}t)^3 = \left\{\frac{x+y+z}{4}\right\}^2 (2z-x-y), \qquad \text{(III)}$$

which results from (I) and (II), and can be very simply proved by Euclidean methods, is precisely of the same form as

$$b^3 = a^2 d,$$

which is obtained from the continued proportion

$$a:b = b:c = c:d,$$

when we recall what we remarked already about Plato's interest in four quantities which are connected by this latter formula. An interesting connection is thus set up between the Nuptial Number of Plato and the celebrated Delian problem of doubling a cube.

To explain the phrase just referred to, Adam, who gives very interesting quotations from the classics in support of the interpretation he adopts, connects it with the configuration of three consecutive edges of a rectangular block parallel to the three edges meeting at a vertex of the block. He interprets the four boundaries to be the four vertices which are the extremities of these edges, and the three spaces between these boundaries as the three edges in question.

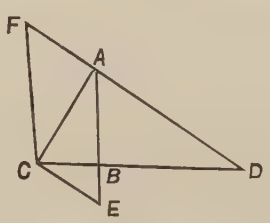
A different figure suggested itself to my mind, it is the figure of Euclid, Bk. VI, 8, used in Bk. VIII, 13, *to find a mean proportional to two given straight lines*, a figure which was certainly habitual with Plato and his contemporaries. We have here a right-angled triangle ACD, with the perpendicular AB drawn from the right angle A to the hypotenuse CD. If we regard this figure* as attached to, or developed from, the central line AB, we have precisely four points as boundaries, A, B, C, and D, and three linear spaces or intervals, namely the surrounding sides AC, CD, and DA.

It was in attempting to understand the passage of the *Timaeus*, above referred to, in which Plato says that, between two "solid" numbers you have to insert two mean proportionals, and not only one, that I began using this well known figure. This part of the *Timaeus* has been always

* This is certainly what the early Greeks would have regarded as a figure. I must confess, much as I was attracted by Adam's picture (272), I am doubtful whether his three lines would have been regarded by Plato as constituting a figure.

supposed to have some connection with Plato's number, and at least two distinguished commentators* and probably many more† have interpreted the four boundaries, or terms, as being the four quantities in a continued proportion, and the three spaces as being the ratios between them: but neither in the one connection nor in the other have the explanations and justifications been convincing.

Having attached such a figure to one of the sides of our right-angled triangle ABC, the text before us naturally suggests performing the same operation on the other side, and on the hypotenuse, this gives us the accompanying figure, in which there are three triangles DBA, CEB, and CFA outside the original triangle ABC, and similar to it.



We only need to compare this figure with that given in Gow's *History of Mathematics*, p. 180, to perceive that leaving out the last point F, and the lines FA, FC, as being equal and parallel to CE, AE, that is suppressing the last triangle FAC, as being completely represented by the triangle ECA, the two figures are identical. My contention that the figure may well have been in Plato's mind, as well as the following reasoning, is therefore only supported by this identity.

In order that the lines in this figure may be what Plato calls *expressible*, that is commensurable with the unit of length, it is easily perceived that we have to express them as the product of three factors, that is by "solid" numbers, and that, apart from a common factor μ, we get in this way *every cubic combination of three quantities a, b, c, without common factor, connected by the equation of Pythagoras* represented by the lines of our figure:—

$$BE = \mu a^3, \quad BC = \mu a^2 b, \quad CE = \mu a^2 c, \quad AB = \mu a b^2, \quad BD = \mu b^3,$$
$$AC = \mu abc, \quad AD = \mu b^2 c, \quad DF = \mu c^3, \quad CF = \mu a c^2.$$

It is not impossible that Plato may have remarked that the length of hypotenuse AC of the triangle ABC is then necessarily a multiple of the eminently Pythagorean number 60.

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* Proclus and Hermann. See Adam, 275.

† Schneider, Cousin.

4. The figure of the right-angled triangle ACD with the perpendicular AB seems to have remained present in Plato's mind during the conclusion of the passage. The fact that the perpendicular is the mean between the segments of the base, combined with the theorem of Pythagoras, gives us, in our figure,

$$AC^2 = CB^2 + CB.BD,$$

and this seems to be what Plato is referring to in speaking of the two harmonies. One of these CB^2 is "equal multiplied by equal," and the other has one side CB equal to the side of the former, but the other side BD unequal, so that $CB.BD$ is "oblong."

The word "harmony," essentially a musical term,* serves in Plato's mathematics to give a certain concrete form to the unification of two numbers, which are to one another as the lengths of two vibrating strings, giving one of the habitual chords, or, more generally are connected by a law, independent of physical conditions, but including the former as a special case. In an earlier passage of the *Republic* (531 C) Plato had already set the problem of the determination of such a law as a subject for pure mathematical research, and in the *Timaeus* (35 B *seq.*) he gives a solution of his own, depending only on the binary and ternary nnmbers of the double τετρακτύς and the principles of division by arithmetic and harmonic means. CB^2 is therefore a "harmony," corresponding, as it does, to two strings in unison, provided CB is an integer. In order that $CB.BD$ may also be a harmony, CB and BD must be integers, and their ratio must be of one of the actual or ideal musical types. Both these things will be true if ABC is similar to the Cosmic Triangle.

The determination of the common factor μ must be made from the context. The oblong is clearly explained by Plato: it is

$$2700 \times 4800 = 3^4 4^4 5^4.$$

Indeed the nearest square integer to twice 5^2 being $7^2 = 49$, this is what was called *the number on the rational diameter* 7 *of* 5, *the number on the irrational diameter being* 50.† This harmony is therefore musically *the flat seventh*, being represented with our fractional notation, by

$$\frac{4800}{2700} = \frac{16}{9} = \left(\frac{4}{3}\right)^2,$$

4/3 giving *a fourth*.

---

* Paul Tannery, *Mémoires scientifiques*, 2 (1912), *Le nombre nuptial dans Platon* (1882) specifies three possible forms of harmonies in the case of "plane" numbers, n^2, $n(n+1)$, and $n \times mn$. See also his article on Pythagoras in the *Grande Encyclopédie*.

† A complete discussion with very interesting quotations is to be found in Adam's Plato's *Republic*, 2, 285, together with Heath's *Euclid*, 1, 399.

This makes $$CD = 7500 = 2700 + 4800,$$

a number specified by Proclus* as making the cycle from birth to birth.

If we identify CB with 2700 and BD with 4800, the common factor of the sides of the triangle ACD is

$$3^2 4 . 5^2,$$

and therefore

$$\mu = 4 \cdot 5^2 = 100,$$

and we have $$AC = 3^2 4\, 5^3 = \tfrac{4}{3} 3^3 . 5^3 = 4500.$$

This seems admirably to fit the description given by Plato: "the base (3^3) of these numbers (*i.e.* the cubic combinations of 3, 4, 5) modified as 4 to 3 and married to 5, three times increased." The base ($\pi\upsilon\theta\mu\acute{\eta}\nu$) of a system of numbers was a recognized term for the *least* of these numbers.†

5. To speak of the number 4500 as "yielding" the two harmonies 2700^2 and 2700×4800 by means of the equation of Pythagoras

$$(4500)^2 = (2700)^2 + 2700 \times 4800$$

is quite in the spirit of these utterances.

The number referred to in the first part of the passage appears then to be $6^3 = 216$. In this conclusion I agree with Adam, though not in the remaining, and I arrive at it by a different process. *The number referred to in the second part of the passage, the Nuptial Number of Plato, appears to be*

$$4500^2 = 20250000.$$

These numbers, as well as the equations

$$3^3 + 4^3 + 5^3 = 6^3,$$

$$\begin{aligned}(4500)^2 &= \{3^3 \times 100\}^2 + \{3^2 \times 4 \times 100\}^2 \\ &= (270 \times 10\}^2 + \{270 \times 10\}\,\{(270 + 210) \times 10\}^2,\end{aligned}$$

by which they were obtained, can be interpreted in the light of what is well known as to the esoteric meaning of these numbers.

6. We proceed to prove that *the only solution,‡ free of a common factor, of the simultaneous Diophantine equations:*

$$x^2 + y^2 = z^2 \qquad (1)$$

$$x^3 + y^3 + z^3 = t^3 \qquad (2)$$

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* See Adam, 277, foot-note.

† Speusippus, trans. by Paul Tannery, *Pour la Science Hellène*, 386.

‡ That is, in positive integers.

is the P.-P. solution (solution of Plato-Pythagoras),

$$3,\ 4,\ 5,\ 6.$$

We take for this purpose the habitual form of the general solution of the equation of Pythagoras (1),

$$\left.\begin{aligned} x &= m.n \\ y &= \tfrac{1}{2}(m^2-n^2) \\ z &= \tfrac{1}{2}(m^2+n^2) \end{aligned}\right\} \qquad (3)$$

where m and n are odd, and we obtain by simple algebra the equations

$$t^3 = \tfrac{1}{2}(x+y+z)^2(2z-x-y) \qquad (4)$$

where

$$x+y+z = m(m+n), \qquad (5)$$

$$2z-x-y = \tfrac{1}{2}(m^2-2mn+3n^2) = \tfrac{1}{2}\{3(m+n)^2-8m(m+n)+6m^2\}. \qquad (6)$$

7. Since x and z are odd and y is even, $2z-x-y$ is odd, and, therefore, by (4), $x+y+z$, *and therefore also* $m+n$, *must be divisible by* 4.*

By (4), the quantities

$$\tfrac{1}{4}(x+y+z) \quad \text{and} \quad 2z-x-y = 3z-(x+y+z)$$

apart from a possible common factor, are cubes. Moreover they cannot have any common factor, save 3,† since z and $x+y$ can have no common factor, in virtue of the equation of Pythagoras, without such a factor dividing both z and xy, which is excluded by the conditions of our problem. Moreover, if one of them is divisible by 3 it is apparent that the other is also.

We shall now show that 3 *is actually a common factor of* $\tfrac{1}{4}(x+y+z)$ *and* $2z-x-y$.

Indeed by (1) we see at once that one or other of x and y is divisible by 3, but not both, and z is not divisible by 3. Hence, by (2),

$$t^3 \equiv 0 \text{ or } \pm 2 \pmod 9.$$

But a cube can only be congruent to 0 or ± 1 (mod 9), therefore

$$t \equiv 0 \pmod 3.$$

Hence, by (4), our result at once appears.

We have proved then that ‡

$$x+y+z = m(m+n) \equiv 0 \pmod{12},$$

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* In the Pythagorean triangle (5, 12, 13), $x+y+z = 30$ is already not divisible by 4.

† 3 only as an unrepeated common factor. The argument shows that no power of 3 higher than the first can be such a factor.

‡ In the Pythagorean triangle (7, 24, 25), $x+y+z = 56$ is divisible by 4 but not by 3; the same is true of (15, 8, 17), where $x+y+z = 40$; for (21, 20, 29), $x+y+z = 70$ is divisible neither by 3 nor by 4.

and thus there are two cases to discuss:

Case A. $\left.\begin{aligned} m &= 3\alpha^3 \\ m+n &= 4\beta^3 \end{aligned}\right\}$ $\beta \equiv \pm 1 \pmod 3, \quad \alpha \equiv 1 \pmod 2.$

Case B. $\left.\begin{aligned} m &= \alpha^3 \\ m+n &= 12\beta^3 \end{aligned}\right\}$ $\alpha \equiv \pm 1 \pmod 3, \quad \alpha \equiv 1 \pmod 2.$

8. We take first Case A. Here

$$\tfrac{1}{3}(2z-x-y) = \tfrac{1}{6}\{3(m+n)^2-8m(m+n)+6m^2\} = 8\beta^6-16\alpha^3\beta^3+9\alpha^6 = \tau^3, \tag{6; A}$$

where α is odd, β is not divisible by 3, *and no two of α, β, τ have a common factor.*

The P.-P. solution is at once apparent; it is given by

$$\alpha = \beta = \tau = 1,$$

and is the *only* solution, of course, in which $\alpha-\beta = 0$, since α and β have no common factor.

We pass on then to the case when $\alpha-\beta$ is not zero.

We may write (6; A) in the form

$$\tau^3-\alpha^6 = 8(\alpha^3-\beta^3)^2, \tag{7; A}$$

that is $$(\tau-\alpha^2)\{(\tau-\alpha^2)^2+3\tau\alpha^2\} = 8(\alpha-\beta)^2\{(\alpha-\beta)^2+3\alpha\beta\}^2,$$

or, since the second factor on the left is clearly odd,

$$\frac{\tau-\alpha^2}{8}\left\{64\left(\frac{\tau-\alpha^2}{8}\right)^2+3\tau\alpha^2\right\} = (\alpha-\beta)^2\{(\alpha-\beta)^2+3\alpha\beta\}^2. \tag{8; A}$$

By the same reasoning as before, *the two factors on the right can have no common factor, save possibly* 3, and we have therefore two cases again to be considered, according as $\alpha-\beta$ is, or is not, divisible by 3.

9. *Case* A 1. $\alpha-\beta \equiv \pm 1 \pmod 3.$

Hence, by (7; A),

$$\left.\begin{aligned} \tau-\alpha^2 &= 8\eta^2 \\ (\tau-\alpha^2)^2+3\tau\alpha^2 &= \xi^2 \end{aligned}\right\}. \tag{9; A 1}$$

But this is impossible, for it gives

$$\xi^2 \equiv 3\tau\alpha^2 \equiv 3\alpha^4 \equiv 3 \pmod 8.$$

10. *Case* A 2. $$\alpha-\beta \equiv 0 \pmod 3,$$

which shows that α, *like* β, *is not divisible by* 3.

We then have, by (8 ; A),

$$\left.\begin{aligned} \tau-a^2 &= 6\eta^2 \\ (\tau-a^2)^2+3\tau a^2 &= 3\zeta^2 \end{aligned}\right\}, \qquad (9\ ;\ \text{A}\,2)$$

where η *is even and* ζ *is odd.* Hence

$$\zeta^2 = 12\eta^4+6a^2\eta^2+a^4 = 3\eta^4+(3\eta^2+a^2)^2, \qquad (10\ ;\ \text{A}\,2)$$

which shows that $$(\zeta-a^2-3\eta^2)(\zeta+a^2+3\eta^2) = 3\eta^4.$$

Let us then write $$\zeta-a^2-3\eta^2 = \theta,$$

and we have $$\theta(\theta+2a^2+6\eta^2) = 3\eta^4,$$

whence $$2a^2 = 3\eta^4/\theta-\theta-6\eta^2. \qquad (11\ ;\ \text{A}\,2)$$

11. This shows that

(i) θ *is even, since* η *is even ;*

(ii) θ *must be a factor of* $3\eta^4$.

Now θ *and* $3\eta^4$ *cannot have a common odd factor, without its cancelling out when we divide one by the other.* For, if they had a common odd prime factor π, which did not cancel out, it would, by (11 ; A 2), be a factor of $2a^2$; but we know that this is impossible, for, on the one hand, a is not divisible by 3, and, on the other, a and η have no common factor. Thus no such odd factor can exist, which proves the above statement.

We must therefore write

$$\left.\begin{aligned} \eta &= 2^\lambda 3^\mu \eta_1\eta_2 \\ \theta &= 2^\kappa 3^\nu \eta_1^4 \end{aligned}\right\} \qquad (12\ ;\ \text{A}\,2)$$

where $\lambda > 0$, $\mu \geqslant 0$. And, in order that the right-hand side of (11 ; A 2) may be divisible by 2, but not by 4, nor by 3,

$$\left.\begin{aligned} \kappa &= 1 \text{ or } 4\lambda-1 \\ \nu &= 0 \text{ or } 4\mu+1 \end{aligned}\right\} \qquad (13\ ;\ \text{A}\,2)$$

Thus our equation (11 ; A 2) becomes

$$a^2 = 2^{4\lambda-1-\kappa}3^{4\mu+1-\nu}\eta_2^4-2^{\kappa-1}3^\nu\eta_1^4-2^{2\lambda}3^{2\mu+1}\eta_1^2\eta_2^2. \qquad (14\ ;\ \text{A}\,2)$$

12. We easily see that $\kappa = 1$ is impossible. For the last equation becomes

$$a^2 = 2^{4\lambda-2}3^{4\mu+1-\nu}\eta_2^4 - 3^\nu\eta_1^4 - 2^{2\lambda}3^{2\mu+1}\eta_1^2\eta_2^2.$$

The left-hand side is $\equiv 1 \pmod 8$ or (mod. 3), but, if $\nu = 0$, the right-hand side is $\equiv -\eta_1^4 \equiv -1 \pmod 3$, while, if $\nu = 2\mu+1$, it is

$$\equiv -3^\nu\eta_1^4 \equiv -3 \pmod 8,$$

whether $\lambda > 1$ or $\lambda = 1$.

We also see at once that, when $\kappa = 4\lambda-1$, ν cannot be zero. For then we have, by (14 ; A 2)

$$a^2 = 3^{4\mu+1}\eta_2^4 - 2^{4\lambda-2}\eta_1^4 - 2^{2\lambda}3^{2\mu+1}\eta_1^2\eta_2^2,$$

and the right-hand side is $\equiv 3 \pmod 8$, whether $\lambda > 1$ or $\lambda = 1$.

13. Thus we have to put

$$\left.\begin{aligned}\kappa &= 4\lambda-1\\ \nu &= 4\mu+1\end{aligned}\right\}$$

and our equation (14 ; A 2) becomes

$$\begin{aligned}a^2 &= \eta_2^4 - 2^{4\lambda-2}3^{4\mu+1}\eta_1^4 - 2^{2\lambda}3^{2\mu+1}\eta_1^2\eta_2^2\\ &= 4\eta_2^4 - 3\{2^{2\lambda-1}3^{2\mu}\eta_1^2+\eta_2^2\}^2. \qquad (15\,;\ \text{A 2})\end{aligned}$$

We shall write this in the following form,

$$\phi^2+3\psi^2 = 4\chi^4 \qquad (16)$$

where

$$\psi-\chi^2 = 2^{2\lambda-1}3^{2\mu}\omega^2. \qquad (17)$$

Here $\lambda > 0$, $\mu \geqslant 0$, and ϕ, ψ, χ, and ω are odd numbers, not divisible by 3, and having no factor common to two of the four.

14. We can easily obtain the complete solution of (16), when we disregard the condition (17), which, as will be seen later, is incompatible with (16). There is a double infinity of solutions including the obvious one where ϕ, ψ, and χ are each unity. The work required is exactly parallel to that used by Euclid in the solution of the equation of Pythagoras, and is only more complicated.

We have

$$(2\chi^2-\phi)(2\chi^2+\phi) = 3\psi^2,$$

and therefore, since the factors on the left have no common factor, we must write

$$\psi = \psi_1\psi_2\,;\quad 2\chi^2 \mp \phi = 3\psi_1^2\,;\quad 2\chi^2 \pm \phi = \psi_2^2 \qquad (18)$$

where the upper signs have to be taken together, or the lower signs together. Hence

$$4\chi^2 = 3\psi_1^2+\psi_2^2, \tag{19}$$

$$\mp 2\phi = 3\psi_1^2-\psi_2^2. \tag{20}$$

Writing (19) in the form

$$(2\chi-\psi_2)(2\chi+\psi_2) = 3\psi_1^2,$$

we get, as before,

$$\psi_1 = \psi_{11}\psi_{12}; \quad 2\chi\mp\psi_2 = 3\psi_{11}^2; \quad 2\chi\pm\psi_2 = \psi_{12}^2. \tag{21}$$

This completes the solution of (16), which, writing for convenience γ and δ for ψ_{11} and ψ_{12}, is as follows :—

$$\left.\begin{aligned} \mp 2\psi_2 &= 3\gamma^2-\delta^2 \\ \psi_1 &= \gamma\delta \end{aligned}\right\},$$

$$\left.\begin{aligned} \mp 3\phi &= 3\psi_1^2-\psi_2^2 \\ \psi &= \psi_1\psi_2 \\ 4\chi &= 3\gamma^2+\delta^2 \end{aligned}\right\},$$

the signs being so chosen that ψ_2 and ϕ are positive; γ and δ being any odd numbers not divisible by 3, *and having no common factor.*

15. We now have to consider the effect of (17). We have

$$4(\psi-\chi^2) = 4\psi_1\psi_2-3\psi_1^2-\psi_2^2 = (\psi_2-\psi_1)(3\psi_1-\psi_2)$$

and $$\mp 2(\psi_2-\psi_1) = 3\gamma^2-\delta^2\pm 2\gamma\delta = (\gamma\pm\delta)(3\gamma\mp\delta)$$

and $$\pm 2(3\psi_1-\psi_2) = \pm 6\gamma\delta+3\gamma^2-\delta^2.$$

Thus $$-2^4(\psi-\chi^2) = (\gamma+\delta)(3\gamma-\delta)(3\gamma^2-\delta^2+6\gamma\delta),$$

or we must, on the right, change δ into $-\delta$. We shall see that the argument is unaffected by this change of sign. We write then in accordance with (17),

$$(\gamma+\delta)(\delta-3\gamma)(3\gamma^2-\delta^2+6\gamma\delta) = 2^{2\lambda+3}3^{2\mu}\omega^2. \tag{22}$$

Now each of the three factors on the left is even, no two of them have a common divisor, and the second and third are certainly not divisible by 3.

Thus we must write

$$\left.\begin{aligned} \gamma+\delta &= 2^i 3^{2\mu}\omega_1^2, \\ -3\gamma+\delta &= 2^j\omega_2^2, \\ 3\gamma^2-\delta^2+6\gamma\delta &= 2^k\omega_3^2, \\ \omega &= \omega_1\omega_2\omega_3, \\ i\geqslant 0,\quad j\geqslant 0,\quad k\geqslant 0,\quad & i+j+k = 2\lambda+3\geqslant 3, \end{aligned}\right\} \tag{23}$$

or else we must change the signs of the second and third factors on the left of (22), *which is equivalent to writing* $-\omega_2^2$ *for* ω_2^2 *and* $-\omega_3^2$ *for* ω_3^2.

We shall discuss the formula (22) completely. We then have

$$\left.\begin{aligned} 4\gamma &= 2^i 3^{2\mu}\quad \omega_1^2-2^j\omega_2^2 \\ 4\delta &= 2^i 3^{2\mu+1}\omega_1^2+2^j\omega_2^2 \end{aligned}\right\}. \tag{24}$$

16. Now, if both i and j were >2, this would be impossible, since γ and δ being odd, 4γ and 4δ are $\equiv 4 \pmod 8$, whereas the expressions equated to them are $\equiv 0 \pmod 8$. *Thus either* i *or* j *must be* $\leqslant 2$.

But we easily see that *one of the pair* i, j *must be* >2, for (24) gives

for $i=j$, $\quad 4\gamma = 2^i\{3^{2\mu}\omega_1^2-\omega_2^2\} \equiv 0 \pmod 8$,

for $i=1,\ j=2$, $\quad 4\gamma = 2\{3^{2\mu}\omega_1^2-2\omega_2^2\} \equiv -2 \pmod 8$,

for $i=2,\ j=1$, $\quad 4\gamma = 2\{3^{2\mu}2.\omega_1^2-\omega_2^2\} \equiv 2 \pmod 8$,

all of which are impossible.

Moreover it is evident that the same argument leads to the same result when we replace ω_2^2 by $-\omega_2^2$, or change the sign of γ. Thus we have the following necessary condition: *One of the indices* (i, j) *must be* $\leqslant 2$ *and the other* >2.

17. We now use the identities

$$3\gamma^2-\delta^2+6\gamma\delta = 3(\gamma+\delta)^2-4\delta^2 = 3.4.\gamma^2-(\delta-3\gamma)^2,$$

and we get by (23)

$$\left.\begin{aligned} 2^{2i}3^{4\mu+1}\omega_1^4-4\delta^2 &= 2^k\omega_3^2 \\ 2^2 3\gamma^2-2^{2j}\omega_2^4 &= 2^k\omega_3^2 \end{aligned}\right\}. \tag{25}$$

Now if $i>2$, or if $j>2$, these equations show that $k=2$, since the

left-hand side of one of them is then divisible by 4, but not by 3. Thus we necessarily have

$$k = 2,$$

in virtue of the conclusion of § 16.

Therefore (25) gives

$$\left.\begin{aligned} 2^{2i}3^{4\mu+1}\omega_1^4 &= 2^2(\delta^2+\omega_3^2) \\ 2^{2j}\omega_2^4 &= 2^2(3\gamma^2-\omega_3^2) \end{aligned}\right\}. \tag{26}$$

But the right-hand side of both these equations is divisible by 2^3 but not by 2^4. Therefore they are impossible.

Moreover the result is the same if we change ω_3^2 into $-\omega_3^2$, and the sign of δ does not come into the argument.

Thus we are led to an absurdity, which shows that the equation (17) is incompatible with (16). Hence Case A leads to no solution, except the P.-P. solution.

18. We next consider Case B (§ 7). Here

$$\tfrac{1}{3}(2z-x-y) = \tfrac{1}{6}\{3(m+n)^2-8m(m+n)+6m^2\} = 72\beta^6-16\alpha^3\beta^3+\alpha^6 = \tau^3 \tag{6 ; B}$$

where α is odd and not divisible by 3, *and no two of α, β, τ have a common factor.*

We write (6 ; B) in the form

$$\tau^3-8\beta^6 = (\alpha^3-8\beta^3)^2 \tag{7 ; B}$$

that is

$$(\tau-2\beta^2)\{(\tau-2\beta^2)^2+6\tau\beta^2\} = (\alpha-2\beta)^2\{(\alpha-2\beta)^2+6\alpha\beta\}^2 \tag{8 ; B}$$

where, since α is odd, $\alpha-2\beta$ is certainly not zero, so that we have no isolated solution corresponding to the P.-P. solution in Case A.

We have again two cases, according as $(\alpha-2\beta)$ is, or is not, divisible by 3.

19. Case (B 1) $\alpha-2\beta \equiv 0 \pmod 3$,

and we must write by (8 ; B),

$$\tau-2\beta^2 = 3\eta^2 \tag{9 ; B 1}$$

But, by (7 ; B) $\tau^3 \equiv \alpha^6 \equiv 1 \pmod 8$,

and therefore $\tau \equiv 1 \pmod 8$.

Hence η is odd and $2\beta^2 \equiv \tau - 3\eta^2 \equiv 1-3 \equiv -2 \pmod 8$,

which is impossible.

20. We are left with Case (B 2)

$$\alpha - 2\beta \equiv \pm 1 \pmod 3;$$

and therefore, by (8; B),

$$\left.\begin{aligned} \tau - 2\beta^2 &= \eta^2 \\ \tau^2 + 2\beta^2\tau + 4\beta^4 &= \zeta^2 \end{aligned}\right\} \qquad (9;\ \text{B 2})$$

where η is odd and not divisible by 3, *like α in Case A* 2.

Also, as in § 19, $\tau \equiv 1 \pmod 8$,

and therefore $\eta^2 + 2\beta^2 \equiv 1 \pmod 8$;

but η is odd, *therefore β must be even, like η in Case A* 2.

Moreover, eliminating τ in (9; B 2), we get

$$\zeta^2 = 12\beta^4 + 6\eta^2\beta^2 + \eta^4, \qquad (10;\ \text{B 2})$$

which is the same as (10; A 2), α being replaced by η, and η by β.

Thus the argument already given in Case A 2 tells us that Case B leads to no solution of our problem.

This completes the proof of the uniqueness of the P.-P. solution.

EUCLIDEAN GEOMETRY OF ANGLE

By D. K. PICKEN.

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1.1. A preliminary account of the work with which this paper is concerned has appeared in the *Math. Gazette*, 11, No. 161 (December 1922), 188. The most elementary parts of the work are there set out, and will be referred to where necessary; § 9, § 11 (v), § 12 (iii), and § 13 of the article give an abstract of this paper.

The subject is the elementary geometrical figure consisting of an *ordered* pair of unbounded intersecting straight lines—a figure which may be called "the line-pair," or (preferably for this paper, as will appear) "the Complete Angle," and denoted thus: (l_1, l_2), or (AB, CD), or (again preferably here) AOB, equivalent to (OA, OB).* All three notations have their uses. For other details of the notation, see the *Math. Gazette* article, § 1.

The purpose of the paper is generality of treatment. Any or every figure may be drawn for each proposition.

1.2. The congruence $(l_1, l_2) \equiv (l_1', l_2')$, or $AOB \equiv A'O'B'$, is shown to be the characteristic Angle-relation of plane geometry, fundamental cases being—

1.2.1. If O, A, A' are collinear and O, B, B' collinear, $AOB \equiv A'OB'$; and, conversely, if $ABC \equiv ABD$, the points B, C, D are collinear.

* The ideas here developed were first suggested to the writer by Townsend (*Modern Geometry*, 1, 18, § 24). It has been pointed out to me that Laguerre (*Œuvres*, 2, 64) has indicated the significance of the line-pair. Mathews (*Projective Geometry*, 14, § 18) uses the term "complete angle," in the same connection but not quite in the same sense. Hilbert (*The Foundations of Geometry*, trans. 14, § 7) introduces the congruence $(h, k) \equiv (h', k')$, but so that $(h, k) \equiv (k, h)$—which makes all the difference, as will be apparent in the present development.

1.2.2. The condition of parallelism of l_1, l_2 is $(l, l_1) \equiv (l, l_2)$, if l be any transversal; or $ABC \equiv DCB$ is the condition that AB, CD be parallel.

1.2.3. The cyclic condition for four points A, B, C, D is* $ABC \equiv ADC$.

2. Every congruence represented by $PQR \equiv UVW$, in which P, Q, R and U, V, W are 3-permutations of four given symbols A, B, C, D, can be given simple geometrical interpretation.

2.1. Collinearity, of the three or four points involved, is always one case.

2.2.1. If $ABC \equiv ACB$, A is at infinity.

2.2.2. If $ABC \equiv BCA$, the triangle ABC is isosceles; apex A.

2.2.3. If $ABC \equiv CBA$, the Complete Angle ABC is rectangular $(AB \perp BC)$.

All other congruences involving only three of the symbols are reducible to these types.

2.3.1.1. If $ABC \equiv ABD$, the points B, C, D are collinear.

2.3.1.2. If $ABC \equiv DBA$, AB is a bisector of the Complete Angle CBD.

2.3.2.1. If $ABC \equiv ADC$, the points A, B, C, D are concyclic.

2.3.2.2. If $ABC \equiv CDA$, the circles ABC, ACD are equal.

2.3.3.1. If $ABC \equiv ADB$, the line BC touches the circle A, B, D.

2.3.3.2. If $ABC \equiv BDA$, the line BC touches the circle through A, B equal to the circle ABD.

2.3.4.1. If $ABC \equiv BAD$, BC and AD are parallel lines.

2.3.4.2. If $ABC \equiv DAB$, the triangle ABE is isosceles (apex E) if E be the intersection of lines BC, AD.

All other congruences involving all four of the symbols A, B, C, D are educible to these.

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* This is the central proposition which has been expressed in some form by several writers. Laguerre *loc. cit.*) also states the companion theorem—§ 2.3.2.2 below.

2 . 4. These results have a striking application to the elementary geometry of Orthocentre and Simson's Line. See *Math. Gazette* article, §§ 10 and 11. See also § 4 of this paper.

3 . 1 1. If $(l_1, l_2) \equiv (l_3, l_4)$ it is useful to name the relation between (l_1, l_2) and (l_4, l_3) "contra-congruence,"* because this gives the means of distinguishing between the two ways in which triangles (or polygons) in a given plane may be quantitatively equal; the relation between corresponding Angles is either *congruence* for all or *contra-congruence* for all. And, as this distinction is purely angular, its most useful expression is by the terms *similar* and *contra-similar*—used of triangles (or polygons).

3 . 1 . 2. Congruence and contra-congruence of Complete Angles are inconsistent with one another except in the case of Straight Angles (*i.e.* the limiting case of coincident lines) and the case of Right Angles. For the latter case (Cf. § 2 . 2 . 3) this fact is of fundamental importance.

3 . 2 . 1 . 1. If $(l_1, l_2) \equiv (l'_1, l'_2)$ and $(l_2, l_3) \equiv (l'_2, l'_3)$, then† $(l_1, l_3) \equiv (l'_1, l'_3)$;

3 . 2 . 1 . 2. *And the same relation holds for contra-congruences.*

3 . 2 . 2. From these it follows that if $(l_1, l_2) \equiv (l_3, l_4)$, then $(l_1, l_3) \equiv (l_2, l_4)$.

3 . 2 . 3. An illustration of the use of the "*ex aequali*" proposition of § 3 . 2 . 1 is given by *direct* proof of the proposition (converse to § 10 (ii) of the *Math. Gazette* article):—

If three equal circles with a common point of intersection A intersect again at B, C, D, the circle BCD is also equal to them.

We have $BCA \equiv ADB$, from the circles of which AB is common chord; and $ACD \equiv DBA$. Hence $BCD \equiv (BC, CD) \equiv (AD, AB) \equiv DAB$.

4 . 1. The proposition linking Orthocentre with Simson's Line has the double complication of "case figures." This gives added interest to the general argument used to prove it.

O being the orthocentre of triangle ABC and P any point of circle ABC, we have to prove that Simson's Line for P bisects OP.

Let AQ, PR be chords of the circle, both perpendicular to the line BC (at L and X respectively), therefore parallel; and let the line PQ intersect the line AR at U, the line BC at T, and the Simson's Line at S.

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* This is the distinctive relationship which Hilbert appears to ignore.

† Hilbert gives a form of this, but through ignoring the distinction between congruence and contra-congruence does not give the important corollary § 3 . 2 . 2. [See *Math. Gazette* article, § 5 (iii), 189.]

The triangles UQA, UPR are *similar* isosceles triangles; for

$$UQA \equiv PQA \equiv PRA \ (\equiv PRU) \equiv QAR \ (\equiv QAU) \equiv QPR \equiv UPR.$$

The triangle TQO, which is isosceles because $OL = L\,Q$ and $TL \perp OQ$, is also *similar* to these triangles, since $TQO \equiv UQA$; hence

$$TOA \equiv TOQ \equiv UAQ \equiv RAO,$$

and

$$OT \parallel AR \parallel XS.$$

The triangle SPX is also *similar* to these, for $SPX \equiv UPR$ and $SXP \equiv SXR \equiv URX \equiv URP$; hence $SP = SX$, and S is the circum-centre of the right-angled triangle PXT.

Thus the Simson's Line bisects TP and OP.

4.2. The discussion of the more general question suggested by the elementary Simson's Line theory—when the point P is not restricted to the circle ABC—may be expressed in the following purely geometrical general form.*

4.2.1.1. If PX, PY, PZ are the perpendiculars from P to the lines BC, CA, AB (or, more generally, isoclinals†), and if the lines AP, BP, CP intersect the circle ABC again at U, V, W,

$$YXP \equiv YCP \equiv ACW \equiv AUW \quad \text{and} \quad PXZ \equiv VUA,$$

i.e. YXP, PXZ are contra-congruent with WUA, AUV; hence (§ 3.2.1) YXZ is contra-congruent with WUV, *i.e.* $YXZ \equiv VUW$.

Thus triangles XYZ, UVW are *similar*.

4.2.1.2. Different sets of isoclinals give triangles which are all *similar*.

4.2.2.1. The ratio $YZ : BC$ is obtainable as that of the diameters of circles AYZ, ABC, since $YAZ \equiv CAB$.

When PX is perpendicular to BC, etc., the former diameter is PA. For isoclinals it is $\operatorname{cosec} i \,.\, PA$, if i denote the acute angle of inclination. It is sufficient to discuss the particular case of perpendicularity.

4.2.2.2. The ratio $BC : VW$ is obtainable from the fact that triangles PBC, PWV are *contra-similar*, since

$$BPC \equiv VPW \quad \text{and} \quad PBC \equiv VBC \equiv VWC \equiv VWP.$$

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* Cp. Townsend, *loc. cit.*, § 64, Cor. 3, 77.

† In the *Math. Gazette* article, owing to a slip, the term "isogonal" was used.

Hence

$$BC : VW = PB : PW = PB \,.\, PC : PC \,.\, PW = PB \,.\, PC : \Pi$$

if Π denote the "power" area of chords divided at P.

4.2.2.3. Thus $YZ : VW = PA \,.\, PB \,.\, PC : \mathsf{D} \,.\, \Pi$,

if D denote the diameter of circle ABC.

4.2.3. The diameter δ of circle XYZ is, therefore, given by

$$\delta \,.\, \Pi = PA \,.\, PB \,.\, PC.$$

4.2.4. Using the proposition*

$$2 \,.\, \mathsf{D} \,.\, \triangle ABC = BC \,.\, CA \,.\, AB,$$

we have

$$\begin{aligned} (\delta : \mathsf{D}) \,.\, (\triangle XYZ : \triangle ABC) &= YZ \,.\, ZX \,.\, XY : BC \,.\, CA \,.\, AB \\ &= PA \,.\, PB \,.\, PC : \mathsf{D}^3 \qquad (\S\, 4.2.2.1) \\ &= (\delta : \mathsf{D}) \,.\, (\Pi : \mathsf{D}^2). \qquad (\S\, 4.2.3) \end{aligned}$$

Hence $\triangle XYZ : \triangle ABC = \Pi : \mathsf{D}^2$; or $\triangle XYZ \propto \Pi$, when P varies.

4.2.5. For isoclinals, $\triangle XYZ : \triangle ABC = \operatorname{cosec}^2 i \,.\, (\Pi : \mathsf{D}^2)$.

4.2.6. When P comes on to the circle ABC, $\Pi \to 0$, $\delta \to \infty$, and $\triangle XYZ \to 0$.

The three points U, V, W all come into coincidence with P. But the point-triangle UVW has clearly all three of its side-lines coincident with the tangent to the circle.

5. The generalized Ptolemy's Theorem — regarded as expressing a characteristic property of the Plane four-point system† — provides an interesting illustration of the Complete Angle methods.

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* A form of Euc. VI, C.

† The old name, Complete Quadrangle, is clearly unsatisfactory. It is obviously important to distinguish between an Angle and its vertex.

5 . 1 . 1. Setting out from four given coplanar points P_1, P_2, P_3, P_4, determine Q so that triangles $P_1P_2P_3$, P_1P_4Q are *similar*; then

$$(P_1P_2, P_1P_3) \equiv (P_1P_4, P_1Q) \quad \text{and} \quad P_1P_2 : P_1P_3 = P_1P_4 : P_1Q ;$$

therefore, alternating (§ 3 . 2 . 2),

$$(P_1P_2, P_1P_4) \equiv (P_1P_3, P_1Q) \quad \text{and} \quad P_1P_2 : P_1P_4 = P_1P_3 : P_1Q$$

and triangles $P_1P_2P_4$, P_1P_3Q are *similar*.

5 . 1 . 2. From these two pairs of similar triangles

$$P_2P_3 . P_1P_4 = P_1P_2 . P_4Q \quad \text{and} \quad P_2P_4 . P_1P_3 = P_1P_2 . P_3Q ;$$

therefore

$$P_4Q : P_3Q : P_3P_4 = P_1P_4 . P_2P_3 : P_2P_4 . P_1P_3 : P_3P_4 . P_1P_2$$
$$= A_1 : A_2 : A_3$$

if A_1, A_2, A_3 denote the rectangle areas determined by opposite joins of the four given points. This determines the *shape* of triangle P_3P_4Q.

5 . 2. The determination of Q in § 5 . 1 . 1 depends on the initial choice of the *ordered* point-pair P_1, P_2. There are thus 12 triangles which can be determined on the same principle, such that each has sides in the proportion $A_1 : A_2 : A_3$.

We proceed to determine the precise relations of these twelve triangles.

5 . 3. The point-pair P_2, P_1 gives Q' such that triangles $P_2P_1P_3$, P_2P_4Q' are *similar*—or triangles $P_1P_2P_3$, P_4P_2Q' *similar*; and triangles $P_1P_2P_4$, P_3P_2Q' *similar*; so that

$$P_1P_3 . P_2P_4 = P_1P_2 . P_4Q' \quad \text{and} \quad P_1P_4 . P_2P_3 = P_1P_2 . P_3Q'.$$

Hence $P_3Q' : P_4Q' : P_3P_4 = A_1 : A_2 : A_3 = P_4Q : P_3Q : P_3P_4$,

and, therefore, $P_3Q' = P_4Q$ and $P_4Q' = P_3Q$.

Further, $(P_3Q, P_1P_3) \equiv (P_2P_4, P_1P_2)$;

therefore $(P_3Q, P_2P_4) \equiv (P_1P_3, P_1P_2) \equiv (P_4Q', P_2P_4)$;

therefore $P_3Q \parallel P_4Q'$; and $P_4Q \parallel P_3Q'$.

Thus P_3QP_4Q' is a parallelogram and triangles P_3P_4Q, P_4P_3Q' are *congruent*.

5.4.1. The six cases that remain to be considered may be tabulated thus :—

	Initial *similar* triangles	Equal areas	Resultant triangle
5.4.1.1.	P_1P_4P, $P_1P'Q$*	$P_1P_4 . P'Q = P_1P' . P_4P$	$Q_{14}P_2P_3$
5.4.1.2.	P_2P_3P, $P_2P'Q$	$P_2P_3 . P'Q = P_2P' . P_3P$	$Q_{23}P_1P_4$
5.4.1.3.	P_2P_4P, $P_2P'Q$	$P_2P_4 . P'Q = P_2P' . P_4P$	$P_1Q_{24}P_3$
5.4.1.4.	P_1P_3P, $P_1P'Q$	$P_1P_3 . P'Q = P_1P' . P_3P$	$P_2Q_{13}P_4$
5.4.1.5.	P_3P_4P, $P_3P'Q$	$P_3P_4 . P'Q = P_3P' . P_4P$	$P_1P_2Q_{34}$
5.4.1.6.	P_1P_2P, $P_1P'Q$	$P_1P_2 . P'Q = P_1P' . P_2P$	$P_3P_4Q_{12}$

Q_{12} being the Q of § 5.1.

The Q' of § 5.3 being Q_{21}, we have

$$\begin{aligned} &\text{triangle } Q_{14}P_2P_3 \equiv \text{triangle } Q_{41}P_3P_2 ; \\ &\text{,, } \quad Q_{23}P_1P_4 \equiv \text{ ,, } \quad Q_{32}P_4P_1 ; \\ &\quad \dots \qquad \dots \qquad \dots \qquad \dots \\ &\text{,, } \quad P_3P_4Q_{12} \equiv \text{ ,, } \quad P_4P_3Q_{21}. \end{aligned}$$

5.4.2. The twelve triangles named in § 5.4.1 are all *similar*. For $Q_{14}P_2P_1 \equiv P_3P_4P_1$ (§ 5.4.1.1) and $P_1P_2P_3 \equiv P_1P_4Q_{12}$ (§ 5.4.1.6), therefore (§ 3.2.1) $Q_{14}P_2P_3 \equiv P_3P_4Q_{12}$;
and, likewise,

$$Q_{14}P_3P_1 \equiv P_2P_4P_1 \quad \text{and} \quad P_1P_3P_2 \equiv P_1P_4Q_{13} ;$$

therefore $$Q_{14}P_3P_2 \equiv P_2P_4Q_{13}.$$

Hence triangles $Q_{14}P_2P_3$, $P_3P_4Q_{12}$, $P_2Q_{13}P_4$ are *similar*.

So triangles $Q_{23}P_1P_4$, $P_1Q_{24}P_3$, $P_4P_3Q_{21}$ are *similar*; &c.; and any one of the twelve triangles, *as named* in § 5.4.1, is *similar* to any other.

6. The applications to elementary geometry of the conic appear to be specially important, as giving quite general expression to the angle theorems.

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* P, P' in each case representing the *other* two of the initial four points, in either order.

A familiar notation will be used, namely: S, S' for foci; C for centre; A, A' for vertices; X, X' for axis-directrix points; PN, PM for perpendiculars from P to axis and directrix.

6.1.1.1. If the secant line P_1P_2 intersect directrix at D,

$$DP_1 : DP_2 = MP_1 : MP_2 = XN_1 : XN_2$$

and is negative or positive according as the directrix is or is not between P_1 and P_2.

Thus* $$P_1D : DP_2 = \pm(SP_1 : SP_2)$$

according as (1) P_1, P_2 are on different branches of a hyperbola, or (2) the case is otherwise. Hence SD is, in case (1), the bisector of the *Simple* Angle P_1SP_2,† in all other cases the *other* bisector of the *Complete* Angle P_1SP_2.

6.1.1.2. From the limiting case of coincidence of P_1, P_2—which is *not* case (1)—if the tangent at P intersect the directrix at Z, PSZ is a Right Angle.

And for the Parabola, triangles SPZ, MPZ are *contra*-congruent (since they are obviously not coincident); hence the tangent (t) to the parabola at P is the bisector of the *Simple* Angle SPM.

6.1.2.1. Adams' property may be stated thus:—

If T be any point of the tangent t, and TJ, TK the perpendiculars from T to the line SP and to the directrix,

$$SJ : SP = ZT : ZP = KT : MP,$$

the ratios having sign, and the proposition being therefore general.

6.1.2.2. The alternated form $SJ : TK = SP : PM$ gives the equality of SJ_1, SJ_2, if T be the intersection of tangents t_1, t_2; and from this follows *contra*-congruence of triangles SJ_1T, SJ_2T, and the fact that the line ST is the bisector of the *Simple* Angle J_1SJ_2.

Now, by § 6.1.2.1, $SJ_1 : SP_1$ and $SJ_2 : SP_2$ are of opposite signs, or of the same sign, according as the directrix is, or is not, between P_1 and

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* $SP_1 : SP_2$ being a signless ratio.

† See *Math. Gazette* article, § 1 and § 12.

P_2. Hence the line ST is the bisector of the *Simple* Angle P_1SP_2, *except* when P_1, P_2 are on different branches of a hyperbola (being in that case the *other* bisector of the *Complete* Angle P_1SP_2).

6.1.3. Combining § 6.1.1.1 and § 6.1.2.2, the bisectors of the *Complete* Angle P_1SP_2 are in all cases the two lines SD, ST; and we have the means of discriminating between them by reference to the *Simple* Angle P_1SP_2.*

6.2.1. If the normal (n) at P intersect the axis at G,

$$SG \parallel PM \quad \text{and} \quad SPM \equiv PSG \quad (\S\, 1.2.2);$$

and GP touches the circle SPM (on diameter PZ), therefore

$$PMS \equiv GPS \quad (\S\, 2.3.3.1).$$

Thus SPM, PMS are *contra*-congruent with GSP, SPG; and triangles SPM, GSP are *contra*-similar.

6.2.2.1. It follows that the *directions* PM, SG (of the parallel lines) are *opposite* directions.

Thus $\quad SG : MP = +e^2, \quad$ or $\quad SG : XN = +e^2.$

So, again, from the other focus and directrix,

$$S'G : X'N = +e^2.$$

Hence
$$SG : GS' = XN : NX' = MP : PM'$$
$$= \pm (PS : PS')\dagger$$

according as P is, or is not, between the directrices. And the tangent and normal are the bisectors of the *Complete* Angle SPS'—the bisector of the *Simple* Angle SPS' being tangent or normal according as the conic is an hyperbola or an ellipse.

6.2.2.2. This discrimination fails in the intermediate case of the parabola—because S' is at infinity and the direction PS' ambiguous. But the Simple Angle SPM can then be used to discriminate (§ 6.1.1.2).

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* It is not suggested that this is the most fundamental way of dealing with this proposition. The object is to illustrate the use of the Complete Angle in familiar elementary processes and to show its bearing on the significance of these processes.

† $(PS : PS')$ being a signless ratio.

6.3.1.1. The equality of inclinations of tangents t_1, t_2 to the focal radii ST, $S'T$ of their intersection is deducible (by a familiar proof) from the propositions of § 6.1.3 and § 6.2.2. But that proof, being not simply a matter of Angle-congruences, raises new difficulties of generality; and the other familiar proof, in the following form, is therefore preferable.

6.3.1.2. If SY_1, SY_2 are the perpendiculars from S to t_1, t_2, etc., then S, T, Y_1, Y_2 are concyclic; and S', T, Y_1', Y_2'; also Y_1, Y_2, Y_1', Y_2' are on the "auxiliary circle" (a proposition of which the proof depends in a simple general way on § 6.2.2.1 and the focus-directrix property).

Hence $Y_1TS \equiv Y_1Y_2S$ or $(t_1, ST) \equiv (Y_1Y_2, SY_2)$;

and $(t_2, S'T) \equiv (Y_1'Y_2', S'Y_1')$; etc.

Also $Y_1Y_2Y_2' \equiv Y_1Y_1'Y_2'$ or $(Y_1Y_2, t_2) \equiv (t_1, Y_1'Y_2')$

and (Right Complete Angles) $(t_2, SY_2) \equiv (S'Y_1', t_1)$;

therefore (§ 3.2.1.2, for *contra*-congruences)

$$(Y_1Y_2, SY_2) \equiv (S'Y_1', Y_1Y_2')$$

and $(t_1, ST) \equiv (S'T, t_2)$;

whence also, of course,* $(t_1, S'T) \equiv (ST, t_2)$.

6.3.2.1. In the case of the parabola (axis x, say), the points Y_1, Y_2 are on the tangent-at-the-vertex (y, say); hence

$$\begin{aligned}(t_1, ST) &\equiv (y, SY_2) \\ &\equiv (x, t_2) \text{ since } (x, y) \equiv (t_2, SY_2), \text{ being Right Complete Angles,} \\ &\equiv (t_2, SP_2), \text{ by § 6.1.1.2.}\end{aligned}$$

Thus $P_1TS \equiv TP_2S$; and, since also $P_1ST \equiv STP_2$ [§ 6.1.3], triangles SP_1T, STP_2 are *similar*.

7. This review, in terms of Complete Angle congruences, of the Angle theory of elementary geometry appears to establish the importance of the method, and to ensure that it will meet successfully further demands made upon it.

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* The Complete Angle statement of the relation has the characteristic precision of the method—commonly lacking in statements of this particular proposition.

Note on the proof of the Circle Theorem (§ 1.2.3).

The proof of this important general theorem, from the elementary definition of the circle, is not quite obvious. After it had been written up in a variety of ways—none quite satisfactory—the idea of the following proof was suggested by my friend and former pupil, acting-Professor E. J. G. Pitman (Canterbury Univ. Coll., N.Z.)* :—

Let A, B be given points of a given circle (centre O), P the representative point of the circle. Denote diametral lines, OA, OP, &c., by a, p, etc. Let u, v be the diametral parallels to AP, BP (or, but for slight convenience of the notation used, AP, BP themselves); and let x be determined by the congruence $(a, x) \equiv (u, v)$.

Then, by repeated alternations, we have

$(x, v) \equiv (a, u) \equiv (u, p)$, from the isosceles triangle OAP }
therefore } § 1.2.2 and § 2.2.2.
$(x, u) \equiv (v, p) \equiv (b, v)$, „ „ „ OBP }

Hence $(x, b) \equiv (u, v) \equiv (a, x)$; and x is therefore one of the bisectors of the Complete Angle AOB.

Now, when P is diametrically opposite to A, u is coincident with a ; therefore x is coincident with v, which, being parallel to BP, is the perpendicular bisector of AB and the bisector of Simple Angle AOB. Hence, from the unbroken continuity of the variation of the lines AP, BP, it is clear that x must be that same line for all the positions of P. And it follows that (u, v), and therefore (AP, BP), remains self-congruent.†

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* For a proof on different lines and a criticism of it, see notes by Prof. E. H. Neville and the author in the *Math. Gazette*, 11 (March 1923), 275, and (Oct. 1923), 386.

† If the tangents at A, B intersect the tangent at P at points Q, R, $(OQ, AP) \equiv (OR, BP)$, being Right Angles; hence, alternating, $QOR \equiv APB$ amplifies the Circle theorem.

A LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS

By E. L. Ince.

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1. *The equation.*

This paper is devoted to a study of the linear differential equation*

$$\frac{d^2y}{dx^2}+\xi \sin 2x \frac{dy}{dx}+(\eta-p\xi \cos 2x)\, y = 0, \tag{1.1}$$

in which p, ξ, and η are constants. When p is a positive integer there exist $p+1$ distinct values of η (dependent upon ξ) for which the equation admits a solution which is a polynomial in $\sin x$ or $\cos x$ of degree p. The $p+1$ functions thus defined satisfy also the homogeneous integral equation*

$$y(x) = \lambda \int_{-\pi}^{\pi} \cos^p(x-s)\, e^{-\frac{1}{2}\xi \cos 2s}\, y(s)\, ds, \tag{1.2}$$

each for an appropriate value of λ (also dependent upon ξ). It will be shown that the properties of these functions depend largely upon the properties of a certain family of plane curves.

For special values of η other than those above mentioned, solutions of period 2π occur, but except in the above cases the solution is of a non-terminating form.

If p and ξ be caused to tend simultaneously to infinity and to zero respectively, in such manner that $p\xi$ remains finite and ultimately becomes 2θ, equation (1.1) becomes Mathieu's equation†

$$\frac{d^2y}{dx^2}+(\eta-2\theta \cos 2x)\, y = 0, \tag{1.3}$$

* Whittaker, *Proc. Edinburgh Math. Soc.*, 33 (1914–15), 22.

† Mathieu, *Journal de Math.* (2), 13 (1868), 137–203. Whittaker and Watson, *Modern Analysis* (1920), Ch. 19.

and the polynomials in $\sin x$ and $\cos x$ become the series for the Mathieu functions.

The normal form of the equation,* viz.,

$$\frac{d^2u}{dx^2}+\{\eta-\tfrac{1}{8}\xi^2-(p+1)\,\xi\cos 2x+\tfrac{1}{8}\xi^2\cos 4x\}\,u=0, \tag{1.4}$$

is obtained by writing $y=e^{\frac{1}{4}\xi\cos 2x}u$; the substitution $z=a\cos^2 x$ then gives the algebraic form

$$4z(z-a)\frac{d^2u}{dz^2}+2(2z-a)\frac{du}{dz} \tag{1.5}$$
$$-\left\{\eta-(p+1)\,\xi\left(\frac{2}{a}\,z-1\right)+\frac{\xi^2}{a^2}\,z(z-a)\right\}u=0.$$

Now let a and ξ tend to zero in such manner that $\lim \xi/a=1$; then the equation becomes

$$4z^2\frac{d^2u}{dz^2}+4z\frac{du}{dz}-\{\eta-2(p+1)\,z+z^2\}\,u=0.$$

Let the second term be removed by writing $u=z^{-\frac{1}{2}}w$, and let $p+1=2k$, $\eta=4m^2$; then the equation assumes the form

$$\frac{d^2w}{dz^2}+\left\{-\tfrac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right\}w=0, \tag{1.6}$$

and thus becomes the confluent hypergeometric equation.†

The equation studied therefore bears the same relation to the confluent hypergeometric equation as Mathieu's equation bears to Bessel's.

2. *The solutions in finite form when p is odd.*

The coefficients in the even solution

$$C_o=\sum_{r=0}^{n}a_r\cos(2r+1)\,x, \tag{2.1}$$

* This is a particular form of Hill's Equation, *Acta Math.*, 8 (1886), 1–36; *Collected Works*, 243–270.

† Whittaker and Watson, *Modern Analysis*, Ch. 16.

where $2n+1=p$, are determined by the set of recurrence relations

$$(n+2)\xi a_1 = \{\eta-1-(n+1)\xi\}\, a_0,$$

$$(2.2)\qquad (n+r+2)\xi a_{r+1} = \{\eta-(2r+1)^2\}\, a_r-(n-r+1)\xi a_{r-1},$$

$$(r=1, 2, \ldots, n-1),$$

$$0 = \{\eta-(2n+1)^2\}\, a_n-\xi a_{n-1}.$$

If the coefficients a_r be eliminated, there arises the algebraic relation between ξ and η which is necessary for the existence of a periodic solution of the finite type C_o, viz.,

$$\begin{vmatrix} \eta-1-(n+1)\xi, & -(n+2)\xi, & 0, & \ldots, & 0, & 0 \\ -n\xi, & \eta-9, & -(n+3)\xi, & \ldots, & 0, & 0 \\ 0, & -(n-1)\xi, & \eta-25, & \ldots, & 0, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & 0, & \ldots, & -\xi, & \eta-(2n+1)^2 \end{vmatrix} = 0,$$

which is of degree $n+1$; let it be denoted by

$$(2.3)\qquad F_n(\xi, \eta) = 0.$$

The coefficients a_r' in the odd solution

$$(2.4)\qquad S_o = \sum_{r=0}^{n} a_r' \sin(2r+1)x$$

are determined by the same set of recurrence relations (2.2) as the constants a_r in the even solution, except that the first relation of the set is to be replaced by

$$(n+2)\xi a_1' = \{\eta-1+(n+1)\xi\}\, a_0.$$

The algebraic relation necessary for the existence of a solution of finite type S_o is therefore

$$(2.5)\qquad F_n(-\xi, \eta) = 0.$$

3. *The solutions in finite form when p is even.*

The coefficients in the even solution

$$C_e = \sum_{r=0}^{n} a_r \cos 2rx, \tag{3.1}$$

where $2n = p$, are determined by the relations

$$\begin{aligned}
(n+1)\xi a_1 &= \eta a_0, \\
(n+2)\xi a_2 &= (\eta-4)a_1 - 2n\xi a_0, \\
(n+r+1)\xi a_{r+1} &= (\eta-4r^2)a_r - (n-r+1)\xi a_{r-1}, \qquad (r = 2, 3, \ldots, n-1), \\
0 &= (\eta-4n^2)a_n - \xi a_{n-1}.
\end{aligned} \tag{3.2}$$

Consequently ξ and η are connected by the algebraic relation

$$\begin{vmatrix}
\eta, & -(n+1)\xi, & 0, & \ldots, & 0, & 0 \\
-2n\xi, & \eta-4, & -(n+2)\xi, & \ldots, & 0, & 0 \\
0, & -(n-1)\xi, & \eta-16, & \ldots, & 0, & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0, & 0, & 0, & \ldots, & -\xi, & \eta-4n^2
\end{vmatrix} = 0,$$

which is of degree $n+1$ and will be denoted by

$$\Phi_n(\xi, \eta) = 0. \tag{3.3}$$

Lastly, the coefficients in the odd solution

$$S_e = \sum_{r=1}^{n} a'_r \sin 2rx \tag{3.4}$$

are determined by the same set of relations (3.2), with the exception that the first relation is omitted, and the second replaced by

$$(n+2)\xi a'_2 = (\eta-4)a'_1.$$

The algebraic relation between ξ and η is expressed by equating to zero the preceding determinant deprived of its first row and first column. This relation is of degree n and will be denoted by

$$\phi_n(\xi, \eta) = 0. \tag{3.5}$$

It will be observed that

$$\Phi_n(-\xi, \eta) = \Phi_n(\xi, \eta); \quad \phi_n(-\xi, \eta) = \phi_n(\xi, \eta).$$

4. *Simple cases.*

The following cases admit of an elementary treatment, viz.,

(*a*) $p = 0, \quad n = 0,$

$$\Phi_0(\xi, \eta) \equiv \eta = 0, \quad \phi_0(\xi, \eta) \text{ does not exist.}$$

(*b*) $p = 1, \quad n = 0,$

$$F_0(\xi, \eta) \equiv \eta - \xi - 1 = 0.$$

One even and one odd real periodic solution of finite form exists for every value of η.

(*c*) $p = 2, \quad n = 1,$

$$\Phi_1(\xi, \eta) \equiv \eta^2 - 4\eta - 4\xi^2 = 0.$$

Two real even periodic solutions of finite form exist for every value of η except when $0 < \eta < 4$. In this range the solutions are imaginary.

$$\phi_1(\xi, \eta) \equiv \eta - 4 = 0.$$

(*d*) $p = 3, \quad n = 1,$

$$F_1(\xi, \eta) \equiv \eta^2 - 2\eta(\xi + 5) + 9 + 18\xi - 3\xi^2 = 0.$$

Real periodic solutions of finite form, even and odd, exist for all values of η except when $3 < \eta < 9$. To each admissible value of η correspond, in general, two distinct values of ξ.

(*e*) $p = 4, \quad n = 2$ (Fig. 1),

$$\Phi_2(\xi, \eta) \equiv \eta^3 - 20\eta^2 + (64 - 16\xi^2)\eta + 192\xi^2 = 0.$$

Two real even periodic solutions of finite form exist for every value of η except $0 < \eta < 4, \quad 12 < \eta < 16$.

$$\phi_2(\xi, \eta) \equiv \eta^2 - 20\eta + 64 - 4\xi^2 = 0.$$

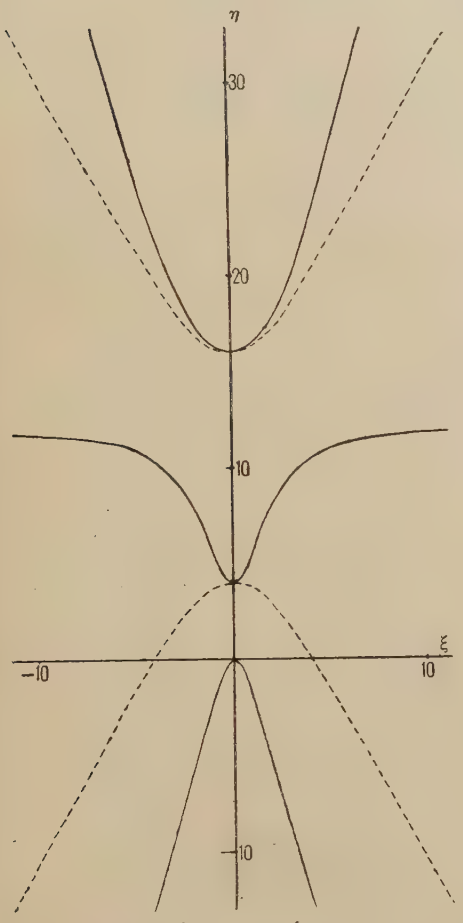

Fig. 1, $p = 4$,
$\Phi_2(\xi, \eta) = 0$ (full line),
$\phi_2(\xi, \eta) = 0$ (broken line).

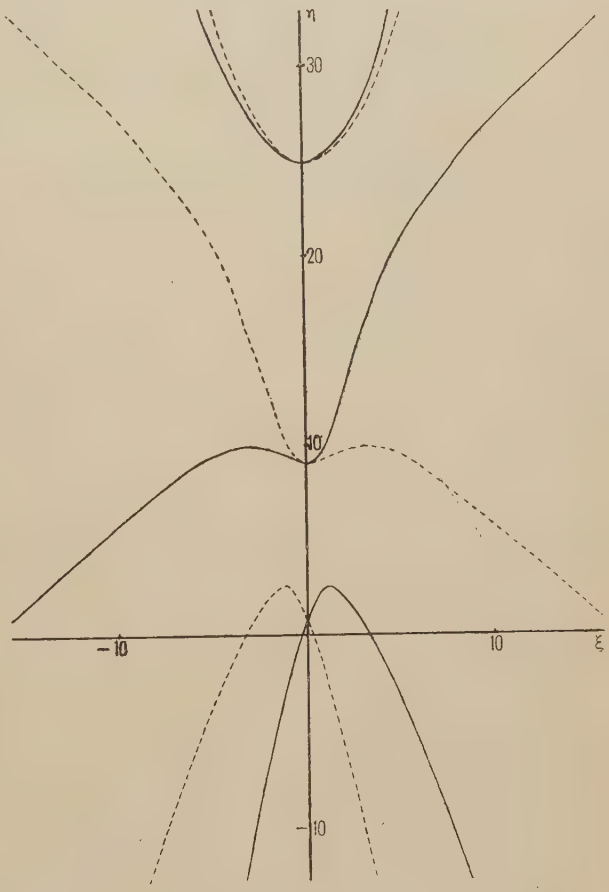

Fig. 2, $p = 5$,
$F_2(\xi, \eta) = 0$ (full line),
$F_2(-\xi, \eta) = 0$ (broken line).

Two real odd periodic solutions exist for all values of η except $4 < \eta < 16$.

(*f*) $p = 5, \quad n = 2$ (Fig. 2),

$$F_2(\xi, \eta) \equiv \eta^3 - (35 + 3\xi)\eta^2 + (259 + 102\xi - 13\xi^2)\eta$$
$$- (225 + 675\xi - 205\xi^2 - 15\xi^3) = 0.$$

Real periodic solutions of finite form, even and odd, exist for all values of η. When $\eta < 2{\cdot}70\ldots$, $9 < \eta < 10$, $\eta > 25$ to each value of η correspond three distinct even and three distinct odd real solutions.

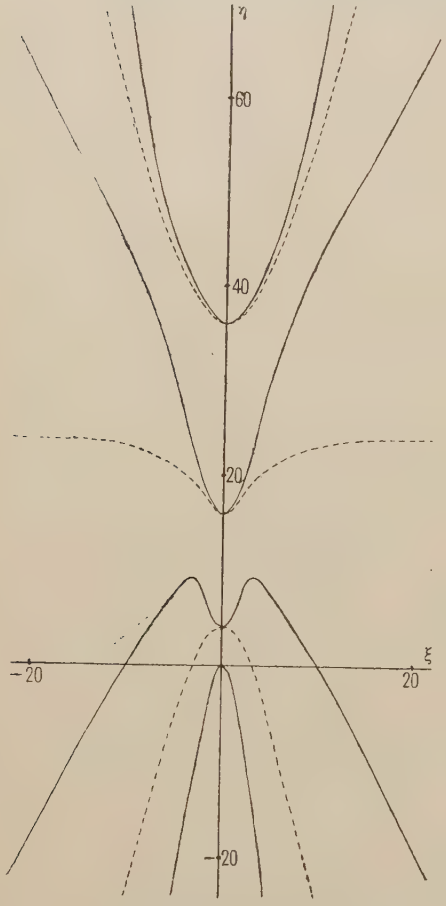

Fig. 3, $p = 6$,
$\Phi_3(\xi, \eta) = 0$ (full line),
$\phi_3(\xi, \eta) = 0$ (broken line).

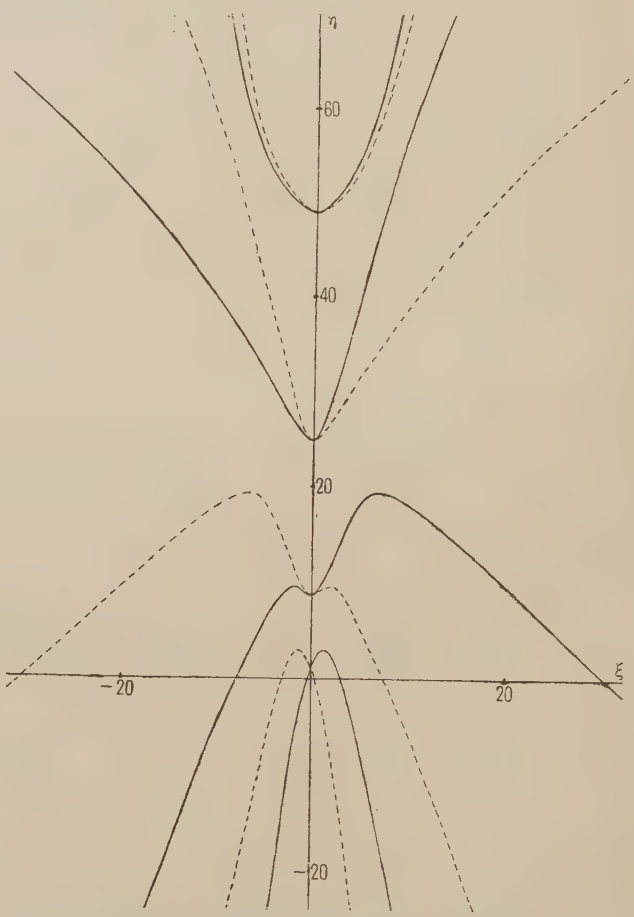

Fig. 4, $p = 7$,
$F_3(\xi, \eta) = 0$ (full line),
$F_3(-\xi, \eta) = 0$ (broken line).

The cases $p = 4$, 5, 6, 7 are illustrated by the diagrams given. (Figs. 1–4).

5. *Determination of η as a series in ξ, p odd.*

It is now necessary to express the periodic solution C_o as a series ordered in powers of ξ. Let the solution be

$$y = \cos rx + v_1\xi + v_2\xi^2 + \ldots + v_s\xi^s + \ldots, \tag{5.1}$$

where r is odd and $r \leqslant p$, and let

$$\eta = r^2+a_1\xi+a_2\xi^2+\ldots+a_s\xi^s+\ldots. \tag{5.2}$$

If y and η be substituted in (1.1) the following general relation appears :—

$$v_s''+\sin 2x\, v_{s-1}'-p\cos 2x\, v_{s-1}+a_s\cos rx$$
$$+a_{s-1}v_1+a_{s-2}v_2+\ldots+a_1v_{s-1}+r^2v_s = 0.$$

The coefficient v_s is to be determined as an even periodic function of x in which $\cos rx$ does not appear. This condition determines a_s, thus

$$\int_0^{2\pi} a_s\cos^2 rx\,dx = p\int_0^{2\pi} v_{s-1}\cos rx\cos 2x\,dx-\int_0^{2\pi} v_{s-1}'\cos rx\sin 2x\,dx$$

whence

$$2\pi a_s = \int_0^{2\pi} v_{s-1}\{(p-r+2)\cos(r-2)x+(p+r+2)\cos(r+2)x\}\,dx.$$

If the coefficients of $\cos(r-2)x$ and $\cos(r+2)x$ in v_{s-1} be A_{s-1} and B_{s-1} respectively, then

$$a_s = \tfrac{1}{2}(p-r+2)A_{s-1}+\tfrac{1}{2}(p+r+2)B_{s-1}.$$

v_s may now be evaluated by quadratures.

It is thus found that

$$a_1 = 0 \qquad (r>1),$$

$$v_1 = \frac{p+r}{8(r-1)}\cos(r-2)x-\frac{p-r}{8(r+1)}\cos(r+2)x,$$

$$a_2 = \frac{p^2+2p+r^2}{8(r^2-1)} \qquad (r>2),$$

$$v_2 = \frac{(p+r)(p+r+2)}{128(r-1)(r-2)}\cos(r-4)x+\frac{(p-r)(p-r-2)}{128(r+1)(r+2)}\cos(r+4)x,$$

$$a_3 = 0 \qquad (r>3),$$

v_3 involves terms in $\cos(r\pm 6)x$ and $\cos(r\pm 2)x$,

$$a_4 = \frac{(5r^2+7)(p^4+4p^3)-6(r^4-7r^2-6)p^2-4(3r^2+1)(r^2-4)p+r^2(r^2-4)(r^2-5)}{512(r^2-1)^3(r^2-4)} \qquad (r>4),$$

v_4 involves terms in $\cos(r\pm 8)x$ and $\cos(r\pm 4)x$,

$$a_5 = 0 \qquad (r > 5),$$

and so on.

In general $a_s = 0$ when s is odd and $s < r$, but since v_{r-1} contains a term in $\cos(r-2)x$, and is the first v of even suffix to contain such a term, a_r will not be zero, and will be the first a of odd suffix which is not zero.

If now the odd solution

$$y = \sin rx + u_1\xi + u_2\xi^2 + \ldots + u_s\xi^s + \ldots \tag{5.3}$$

be taken, together with

$$\eta = r^2 + \beta_1\xi + \beta_2\xi^2 + \ldots + \beta_s\xi^s + \ldots, \tag{5.4}$$

it is found that* $\beta_s = a_s, \quad s \leqslant r-1.$

Hence, since

$$\eta = r^2 + \Sigma a_s\xi^s \quad \text{and} \quad \eta = r^2 + \Sigma\beta_s\xi^s$$

are the approximations to

$$F_n(\xi, \eta) = 0 \quad \text{and} \quad F_n(-\xi, \eta) = 0$$

respectively in the neighbourhood of the point $\xi = 0$, $\eta = r^2$, it follows that the curves have r-point contact at that point.

The total number of intersections at the points

$$\xi = 0, \quad \eta = 1, 9, \ldots, p^2$$

is $$1 + 3 + \ldots + p = \tfrac{1}{4}(p+1)^2 = (n+1)^2.$$

But each curve is of degree $(n+1)$, so that there can be no intersections other than those specified.

Thus $F(\xi, \eta) = 0$ and $F(-\xi, \eta) = 0$ have no common roots except when $\xi = 0$, and therefore the differential equation (1) cannot have at the same time a solution of the type C_o and one of the type S_o except in the trivial case of $\xi = 0$.

---

* The argument of § 8, below, shows that $\beta_r < a_r$, for each branch of $F_n(-\xi, \eta) = 0$ is below or above the corresponding branch of $F_n(\xi, \eta) = 0$ according as $\xi >$ or < 0.

6. *Determination of η as a function of ξ, p even.*

The process employed in the previous section may now be repeated formally when p is even. Let the even solution be

$$y = \cos rx + v_1\xi + v_2\xi^2 + \ldots + v_s\xi^s + \ldots, \tag{6.1}$$

where r is even and $r \leqslant p$, and let

$$\eta = r^2 + a_1\xi + a_2\xi^2 + \ldots a_s\xi^s + \ldots. \tag{6.2}$$

The corresponding odd solutions will be of the form

$$y = \sin rx + u_1\xi + u_2\xi^2 + \ldots + u_s\xi + \ldots, \tag{6.3}$$

with r even, and

$$\eta = r^2 + \beta_1\xi + \beta_2\xi^2 + \ldots + \beta_s\xi^s + \ldots. \tag{6.4}$$

In this case $a_s = \beta_s = 0$ for all odd values of s; a_2 and a_4 have the values given in the preceding section when $r > 2$ and $r > 4$ respectively. In general,

$$\beta_s = a_s \quad \text{if} \quad s \leqslant r-1.$$

Hence the curves $\Phi_n(\xi, \eta) = 0$ and $\phi_n(\xi, \eta) = 0$ have r-point contact at the point $(0, r^2)$, and the total number of intersections at the points $\xi = 0$, $\eta = 0, 4, 16, \ldots, p^2$ is

$$0+2+4+\ldots+p = \tfrac{1}{4}p(p+2) = n(n+1).$$

The curves $\Phi_n(\xi, \eta) = 0$ and $\phi_n(\xi, \eta) = 0$ are of degrees $n+1$ and n respectively, and consequently they can have no points of intersection other than those specified. Consequently equation (1) cannot have at the same time two solutions of the type C_e and S_e, unless $\xi = 0$.

The general conclusion is that the differential equation in question cannot have simultaneous periodic solutions for any value of ξ, real or complex, except $\xi = 0$.

7. *Asymptotes.*

The asymptotes* are given below, the terms in square brackets

* The earlier members in each set were calculated by successive approximation direct from the equation of the curve. These suggested the general form, which was verified. The work is lengthy, and quite without interest. A referee, however, kindly communicates the note: "I have been able to verify (7.1) in part by successive approximation to the solution of (1.1) for great values of ξ, η. The first approximation is

$$y = A\cos^{p-s}x\sin^s x, \quad \eta = \xi(p-2s),$$

and the work does not seem laborious".

indicating the approximations of the curves to the asymptotes.

$F_n(\xi, \eta) = 0$ has $n+1$ asymptotes, viz.,

$$\eta = (2n+1)\xi + (2n+1) \quad [+n(2n+1/\xi],$$
$$\eta = -(2n-1)\xi + (6n+1) \quad [-3n(2n-1)/\xi],$$
$$\ldots \quad \ldots$$

and, in general,

$$\eta = (-1)^s(2n-2s+1)\xi + (2s+1)(2n+1) - 2s^2 \quad (7.1)$$
$$[+(-1)^s(2n-2s+1)\{(2s+1)n - s(s-1)\}/\xi]$$
$$s = 0, 1, 2, \ldots, n.$$

$\Phi_n(\xi, \eta) = 0$ has $n+1$ asymptotes, viz.

$$\eta = 2n\xi + 2n \quad [+n(2n-1)/\xi],$$
$$\eta = (2n-4)\xi + 10n - 8 \quad [+(n-2)(10n-9)/\xi],$$
$$\ldots \quad \ldots$$

and, in general,

$$\eta = (2n-4s)\xi + 2(4s+1)n - 8s^2 \quad (7.2)$$
$$[+(n-2s)\{2(4s+1)n - 8s^2 - 1\}/\xi],$$
$$s = 0, 1, 2, \ldots, \tfrac{1}{2}(n-2) \text{ or } \tfrac{1}{2}(n-1),$$

together with the set obtained by writing $-\xi$ for ξ. When n is even the set ends with the single asymptote

$$\eta = 2n(n+1) \quad [-n(n^2-1)(n+2)/\xi^2]$$

parallel to the ξ-axis.

$\phi_n(\xi, \eta) = 0$ has n asymptotes, viz.,

$$\eta = (2n-2)\xi + 6n - 2 \quad [+(n-1)(6n-3)/\xi],$$
$$\eta = (2n-6)\xi + 14n - 18 \quad [+(n-3)(14n-19)/\xi],$$
$$\ldots \quad \ldots$$

and, in general,

$$\eta = (2n-4s-2)\xi + 2(4s+3)n - 2(2s+1)^2 \quad (7.3)$$
$$[+(n-2s-1)\{2(4s+3)n - 2(2s+1)^2 - 1\}/\xi],$$
$$s = 0, 1, 2, \ldots, \tfrac{1}{2}(n-3) \text{ or } \tfrac{1}{2}(n-2),$$

with a similar set in $-\xi$. When n is odd the set ends with

$$\eta = 2n(n+1) \quad [-n(n^2-1)(n+2)/\xi^2].$$

Each curve, therefore, has its maximum number of linear asymptotes.

8. *The question of double points.*

The curve $F_n(\xi, \eta) = 0$ is composed of $n+1$ branches cutting the η-axis in the points 1, 9, 25, ... $(2n+1)^2$. It will be proved that

(a) no two branches intersect one another,

(b) no branch intersects itself.

Consider large positive values of ξ, and let the curves $F_n(\xi, \eta) = 0$ and $F_n(-\xi, \eta) = 0$ be taken together. The asymptotes to these curves occur in the following order (reckoning from $\eta = -\infty$ to $\eta = \infty$):—

(i) the asymptote of $F_n(-\xi, \eta) = 0$ of gradient $-(2n+1)$,

(ii) ,, ,, $F_n(\xi, \eta) = 0$,, ,, $-(2n-1)$,

(iii) ,, ,, $F_n(-\xi, \eta) = 0$,, ,, $-(2n-3)$,

(iv) ,, ,, $F_n(\xi, \eta) = 0$,, ,, $-(2n-5)$,

and so on alternately. Since the curves $F_n(\xi, \eta) = 0$ and $F_n(-\xi, \eta) = 0$ do not intersect when $\xi > 0$, the branches of these curves separate one another. In particular, when $\xi > 0$, the branch of $F_n(-\xi, \eta)$ which passes through the point $(0, r^2)$ lies below the branch of $F_n(\xi, \eta) = 0$ which passes through $(0, r^2)$ and above the branch of $F_n(\xi, \eta) = 0$ which passes through $\{0, (r-2)^2\}$. These two branches cannot therefore intersect one another. A similar argument applies when $\xi < 0$. Theorem (a) is therefore proved.

Since the branches of $F_n(\xi, \eta) = 0$ are distinct, a double point can only arise by a branch cutting itself. This is impossible, since a straight line through the double point (parallel to the η-axis, for example) would cut the curve in more than $n+1$ points. Theorem (b) is therefore proved.

Thus the curve $F_n(\xi, \eta) = 0$ has no double points. Similarly, by considering the curves $\Phi_n(\xi, \eta) = 0$ and $\phi_n(\xi, \eta) = 0$ together, it is easily proved that these curves have no double points.

9. *Allocation of branches to asymptotes.*

On referring back to the previous section it will be seen that for large positive values of ξ the lowest branch of $F_n(\xi, \eta) = 0$, viz., the branch through (0, 1), approaches the asymptote of gradient $-(2n-1)$; the next lowest branch, viz., that through (0, 9), approaches the asymptote of gradient $-(2n-5)$, and so on. Similarly, for negative values of ξ the branch of $F_n(\xi, \eta) = 0$ through (0, 1) approaches the asymptote of gradient $(2n+1)$, that through (0, 9) approaches the asymptote of gradient $(2n-1)$, and so on.

In general, the branches of the curve $F_n(\xi, \eta) = 0$ which intersect the η-axis in the points r^2 and $(2n-r+2)^2$ are asymptotic to

$$\eta = (2n-2r+3)\xi+(2r-1)(2n+1)-2(r-1)^2$$

and to $$\eta = -(2n-2r+1)\xi+(2r+1)(2n+1)-2r^2.$$

In the curve $\Phi_n(\xi, \eta) = 0$, the branches which intersect the η-axis in the points r^2 and $(2n-r)^2$ are asymptotic to

$$\eta = \pm(2n-2r)\xi+2(2r+1)n-2r^2,$$

and, in particular, when n is even, the branch through $(0, n^2)$ is asymptotic to

$$\eta = 2n(n+1).$$

In the curve $\phi_n(\xi, \eta) = 0$, the branches which intersect the η-axis in the points r^2 and $(2n-r+2)^2$ are asymptotic to

$$\eta = \pm(2n-2r+2)\xi+2(2r-1)n-2(r-1)^2,$$

and, in particular, when n is odd, the branch through $\{0, (n+1)^2\}$ is asymptotic to

$$\eta = 2n(n+1).$$

10. *The number of solutions of finite form which occur for a given value of η.*

Let p be odd, then the problem of determining the number of real and distinct even solutions of finite form which correspond to a given value of η reduces to determining the number of roots of the equation

$$F_n(\xi, \eta) = 0$$

for the given value of η. This problem does not admit of an exact solution, but the results of the preceding section give at once a lower limit to the

number of possible solutions of finite form in any given range of values of η; thus there are at least $n-r$ (r odd) even solutions in each of the ranges $r^2 < \eta < (r+2)^2$ and $(2n-r)^2 < \eta < (2n-r+2)^2$. In particular, there are $n+1$ solutions (the maximum number) when $\eta < 1$ or $\eta > (2n+1)^2$. The fewest possible solutions can occur in the range $(n-2)^2 < \eta < n^2$, when n is odd, where there may be no solution, or in the range

$$(n-1)^2 < \eta < (n+3)^2,$$

when n is even, where there may be only one solution.

Since the odd solutions are obtained by reversing the sign of ξ and replacing x by $\frac{1}{2}\pi - x$, they are equal in number, in any range, to the even solutions.

The corresponding problem for p even is dealt with in the same way by considering the two equations

$$\Phi_n(\xi, \eta) = 0, \qquad \phi_n(\xi, \eta) = 0$$

for given values of η.

The equation $$\Phi_n(\xi, \eta) = 0$$

has at least $n-r-2$ (n, r even) or $n-r-1$ (n odd, r even) real roots for any value of η in the ranges

$$r^2 < \eta < (r+2)^2 \quad \text{and} \quad (2n-r-2)^2 < \eta < (2n-r)^2.$$

In particular, it has n (n even) or $n+1$ (n odd) real roots when $\eta < 0$ or $\eta > 4n^2$. When n is odd there may be no real root for values of η in the range $(n-1)^2 < \eta < (n+1)^2$. When n is even the matter is complicated by the presence of the asymptote $\eta = 2n(n+1)$. In the range $(n-2)^2 < \eta < n^2$ there may be no solution of finite form; when $n = 2$ there is also no such solution in the range $12 < \eta < 16$.

Similarly, the equation

$$\phi_n(\xi, \eta) = 0$$

has at least $n-r$ (n, r even) or $n-r-1$ (n odd, r even) real roots for any value of η in $r^2 < \eta < (r+2)^2$ and in $(2n-r)^2 < \eta < (2n-r+2)^2$. In particular, it has n (n even) or $n-1$ (n odd) real roots when $\eta < 4$ or $\eta > 4n^2$. When n is even there may be no real root for values of η in the range $n^2 < \eta < (n+2)^2$. When n is odd there may be no root for values of

η in the range $(n-1)^2 < \eta < (n+1)^2$. Further, when $n = 3$ there is no real root in the range $24 < \eta < 36$, and when $n = 5$ there is no real root when $60 < \eta < 64$.

11. *The occurrence of periodic solutions in general.*

The solutions which have been considered in the preceding paragraphs are the only solutions which are polynomials in $\sin x$ and $\cos x$, but these are not the only solutions of period 2π. Thus, for instance, when $p = 2n+1$,

$$C_o = \sum_{r=n+1}^{\infty} a_r \cos(2r+1)x \tag{11.1}$$

is a solution of (1.1) provided the coefficients are determined by the recurrence relations

$$(n+r+2)\xi a_{r+1} = \{\eta-(2r+1)^2\}a_r+(r-n-1)\xi a_{r-1} \quad (r \geqslant n+1),$$

and η is determined by means of an infinite determinantal equation. Furthermore,

$$C_e = \sum_{r=0}^{\infty} a_r \cos 2rx \tag{11.3}$$

satisfies (1.1), when $p = 2n+1$, if the coefficients satisfy the relations

$$(n+\tfrac{3}{2})\xi a_1 = \eta a_0,$$

$$(n+\tfrac{5}{2})\xi a_2 = (\eta-4)a-(2n+1)\xi a_0,$$

$$(n+r+\tfrac{3}{2})\xi a_{r+1} = (\eta-4r^2)a_r+(r-n-\tfrac{3}{2})\xi a_{r-1} \quad (r \geqslant 2), \tag{11.4}$$

and η is determined as before. These series for C_o and C_e are convergent for sufficiently small values of ξ, n being finite,* but the region of their convergence, in general, is unknown.

The results of § 8 have shown that the value of η which, for a given positive value of ξ, corresponds to an even finite periodic solution is greater than the value of η which, for the same value of ξ, corresponds to the allied odd periodic solution. The same relation is true, when ξ is negative, in the case of solutions proceeding in even multiples of the

* The proof is the same as that for the series which represent the Mathieu functions. See Whittaker and Watson, *Modern Analysis*, § 19.61.

argument, but is reversed in the case of those proceeding by odd multiples of the argument. Since the process of §§ 5, 6 reveals no formal difference in the expansion of η in terms of ξ whether the solution terminates or not, this conclusion is also true when non-terminating solutions are considered.

There are also solutions which are periodic and of period $2s\pi$ $(s > 1)$, of the form

$$C = \sum_{i=-\infty}^{\infty} a_i \cos\left(2i + \frac{r}{s}\right)x, \tag{11.5}$$

in which the integers r and s are prime to one another, and $r < s$. The coefficients a_i are determined from the recurrence relations

(11.6)

$$\left(i+1+\tfrac{1}{2}p+\frac{r}{2s}\right)\xi a_{i+1} = \left\{\eta - \left(2i+\frac{r}{s}\right)^2\right\}a_i + \left(i-1-\tfrac{1}{2}p+\frac{r}{2s}\right)\xi a_{i-1}$$

$$(-\infty < i < \infty),$$

in which a_0 and a_1 (say) are arbitrary, and η is determined as before. In the corresponding odd solution

$$S = \sum_{i=-\infty}^{\infty} a_i' \sin\left(2i + \frac{r}{s}\right)x, \tag{11.7}$$

the coefficients a_i' are determined by the relations (11.6), a_0' and a_1' (say) being considered arbitrary.* The values of η corresponding to a solution of type C admit also a solution of type S, and thus two such solutions always coexist. Thus, p being fixed, through each point on the η-axis of a (ξ, η)-diagram of ordinate $\{2i \pm (r/s)\}^2$, there will pass a single curve, and to every point on each of these curves there corresponds a pair of simultaneous solutions of the forms (11.5) and (11.7).

12. *The general solution and the question of stability.*

The general solution of an equation with periodic coefficients, such as (1.1), which is unaltered when x is replaced by $-x$, is

$$y = Ae^{\mu x}\psi(x) + Be^{-\mu x}\psi(-x), \tag{12.1}$$

---

* The constants a_0, a_1, a_0', a_1' are not independent; relations between them are necessary for the convergence of the series. For the case of $r/s = \frac{1}{2}$, see *Proc. Camb. Phil. Soc.*, 21 (1922), 120. See also Poole, *Proc. London Math. Soc.* (2), 20 (1922), 378.

in which $\psi(x)$ is purely periodic, with period 2π, and μ depends upon the constants of the equation, viz., p, ξ, and η in the present case. If these constants be such as to make μ a pure imaginary or zero, the solution is said to be stable; otherwise it is unstable.*

In order to obtain a solution which is, in the first place, appropriate to small values of ξ and η, $\psi(x)$ is assumed to have the form

$$\psi(x) = 1+v_1\xi+v_2\xi^2+\ldots, \tag{12.2}$$

and μ is found to be given by†

$$\mu^2 = -\eta+\frac{1}{8}\xi^2-\frac{(p+1)^2\xi^2}{8(1-\eta)}+\ldots. \tag{12.3}$$

When $\mu = 0$ and ξ and η are small, (12.3) gives that branch of $\Phi_n(\xi, \eta) = 0$ which passes through the origin. Thus μ is real, and the solution unstable, for values of (ξ, η) which lie below this particular branch of $\Phi_n(\xi, \eta) = 0$. Above this branch a region of stability exists.

For the purpose of studying the question of stability in the neighbourhood of the branches of $F_n(\xi, \eta) = 0$ and $F_n(-\xi, \eta) = 0$ which cut the η-axis in $\eta = 1$, $\psi(x)$ is now assumed to have the form‡

$$\psi(x) = \sin(x-\sigma)+v_1\xi+v_2\xi^2+\ldots, \tag{12.4}$$

where σ is a parameter depending upon the constants of the equation, then

$$\mu = -\tfrac{1}{4}(p+1)\xi\sin 2\sigma+\ldots, \tag{12.5}$$

$$\eta = 1-\tfrac{1}{2}(p+1)\xi\cos 2\sigma+\ldots. \tag{12.6}$$

When $\sigma = 0$ or $\frac{1}{2}\pi$, $\mu = 0$ and the value of η becomes that which is associated with the branch of $F_n(-\xi, \eta) = 0$ or $F_n(\xi, \eta) = 0$ which passes through the point (0, 1) and $\psi(x)$ becomes the corresponding odd or even periodic solution.

When $0 < \sigma < \frac{1}{2}\pi$, μ is real and the value of η given by (12.6) corresponds to the region which lies between the branches of $F_n(-\xi, \eta) = 0$

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* See A. W. Young, *Proc. Edinburgh Math. Soc.*, 32 (1914), 89–90.

† Tisserand, *Mécanique Céleste*, 3 (1894), 8.

‡ Cf. Whittaker, *Proc. Edinburgh Math. Soc.*, 32 (1914), 76.

and $F_n(\xi, \eta) = 0$ which meet in $(0, 1)$. This region is therefore one of instability.

Let σ be a pure imaginary, then μ is imaginary and η real. For small values of σ and ξ at least, the region now mapped out is that below the branch of $F_n(-\xi, \eta) = 0$ through $(0, 1)$ when $\xi > 0$ and below the branch of $F_n(\xi, \eta) = 0$ through $(0, 1)$ when $\xi < 0$. If σ be a complex quantity of the form $\frac{1}{2}\pi + i\rho$, where ρ is real, μ is imaginary and η real. In this case the region mapped out lies above the branch of $F_n(\xi, \eta) = 0$ through $(0, 1)$ when $\xi > 0$, and above the branch of $F_n(-\xi, \eta) = 0$ through $(0, 1)$ when $\xi < 0$.

In general, through the point $(0, r^2)$ where r is an integer there pass two curves to which correspond periodic solutions. The region between these curves is one of instability, as is also the region below the single curve through $(0, 0)$. There is reason to believe that no other regions of instability exist, for it has been seen that through each point on the η-axis, of ordinate r^2/s^2, where r and s are integers prime to one another and $s > 1$, there passes a single curve which corresponds to two simultaneous solutions of period $2s\pi$. It is probable that such curves form a set everywhere-dense over the (ξ, η)-plane except in the regions of instability. The corresponding periodic solutions are, of course, far from being the only stable solutions which the equation admits.

13. *Mathieu's Equation.*

In equation (1 . 1) let $\xi \to 0$, $p \to \infty$ in such manner that $\lim p\xi = 2\theta$ is finite. The equation then becomes Mathieu's equation

$$\frac{d^2y}{dx^2} + (\eta - 2\theta \cos 2x)\, y = 0. \tag{13 . 1}$$

Many of the results of the previous pages may be expected to remain true in the limiting case, thus :

(*a*) For no value of θ except $\theta = 0$ has Mathieu's equation two periodic solutions of period 2π. This fact has already been proved by the present author by an independent method.*

(*b*) For small values of θ periodic solutions are given by

$$\eta = r^2 + \frac{\theta^2}{2(r^2-1)} + \frac{(5r^2+7)\,\theta^4}{32(r^2-1)^3(r^2-4)} + \ldots \quad (r > 4). \tag{13 . 2}$$

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* *Proc. Camb. Phil. Soc.*, 21 (1922), 117.

(*c*) The form of the asymptotes in §7 shows that, in the limiting case, the (θ, η) curves which correspond to solutions of period 2π have no asymptotes.

(*d*) The results of §10 show that for any given value of θ there exists an infinity of values of η to which correspond periodic solutions, and *vice versa*.

(*e*) For any given value of θ except $\theta = 0$ the value of η which corresponds to the Mathieu function $ce_{2n}(x, \theta)$ is greater than that which corresponds to $se_{2n}(x, \theta)$. The same relation holds with regard to $ce_{2n+1}(x, \theta)$ and $se_{2n+1}(x, \theta)$ when θ is positive, but is reversed when θ is negative.

(*f*) The conclusions drawn in §12 as to the regions of stability hold also for Mathieu's equation.

THE THEORY OF CLOSED REPEATING POLYGONS IN EUCLIDEAN SPACE OF TWO DIMENSIONS.

By W. P. D. MacMahon.

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In the course of the past eighteen months the writer has been closely associated with Major P. A. MacMahon in an investigation of the design of repeating patterns in two and three dimensions, a subject the idea of which was first broached by Major MacMahon in a little book, entitled "New Mathematical Pastimes," recently published.* A paper dealing with the preliminaries of this investigation was communicated to the Royal Society in November last,† and a further development based upon this paper is now communicated to the London Mathematical Society. The author believes that many of the results given in the present paper are new and may be of general interest. The treatment of rectilineal figures and the classification of repeats herein presented appear to be amply justified in that they have led directly to important extensions of theorems of Euler and Cauchy and to results in three dimensions which do not seem to be obtainable so easily in any other way.

Definition.—By a repeating polygon, or "repeat" as it is sometimes called for brevity, is understood a closed rectilineal figure satisfying such conditions of form and construction that any number of other figures directly similar to it may be assembled together so as to cover continuously two-dimensional space.

The word "polygon" is here taken to include the triangle and quadrilateral.

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* Camb. Univ. Press, 1921.

† *Proc. Royal Soc.* (A), 101 (1922), 80–94.

I. *General nature of the repeat assemblage.*

(i) If a number of directly similar polygons be so assembled together as to form a tesselation it follows, from the definitions of the summits and edges (sides) of a plane figure as enunciated by Cayley,* that :—

(*a*) The units or fundamental cells of the assemblage are in *side-to-side* contact with one another, so that a side of one unit coincides with a side of another,† and consequently,

(*b*) Groups of angles are formed at the summits of each unit, the sum of the angles in each group being always 2π.

(ii) Side-to-side contact between repeating polygons is of two kinds :—

(*a*) A side of one repeat coincides with a *corresponding* side of another, in which case the two polygons are in point-symmetry in respect to the mid-point of the common boundary between them, and each is the point-image or symmetric of the other with respect to that point. Contact of this nature is referred to as *point-image* contact.

(*b*) A side of one repeat is in contact with a side of another to which it is equal but does not correspond. Contact of this nature is referred to as *parilateral* contact.

It is convenient to state at once that all figures that occur in the course of the paper are lettered as to their summits or vertices, $A, B, C, \ldots$ in cyclic order, and that the sides are numbered 1, 2, 3, ..., so that the side AB is always referred to as 1, BC as 2, CD as 3, etc.

(iii) By a *system of contact* is to be understood the manner in which a repeat participates in an assemblage with respect to all its numbered sides. We may find, for example, that *every* side of a fundamental cell is in point-image contact with a side of another, or that every side of the fundamental cell is in parilateral contact with a side of another ; or again, as is most usually the case, that both kinds of contact are involved.

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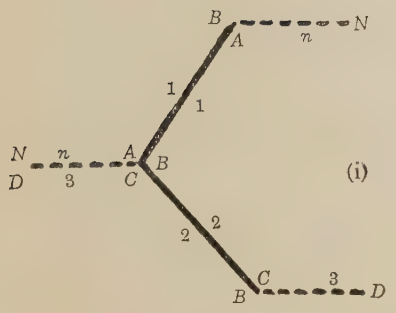
* "*On the partitions of a close.*"—"Any point whatever on a curve may be considered as the point of meeting of two curves, or, in the case of a closed curve, as the point where the curve meets itself. . . . A point where a curve cuts or meets itself or any other curve is a *summit*. . . . The curve which is the path from a summit to itself or to any other summit is an *edge*."—*Phil. Mag.* (4), 21 (1861), 426, and cf. works, *passim*.

† It follows that the "order" of a repeat is determined by assemblage only.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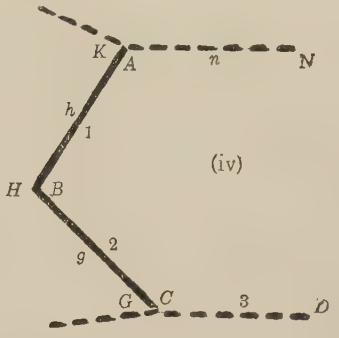
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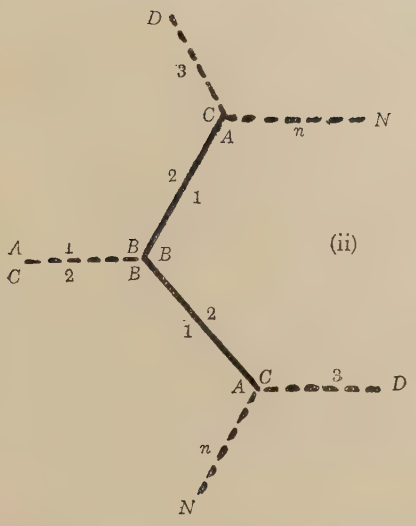
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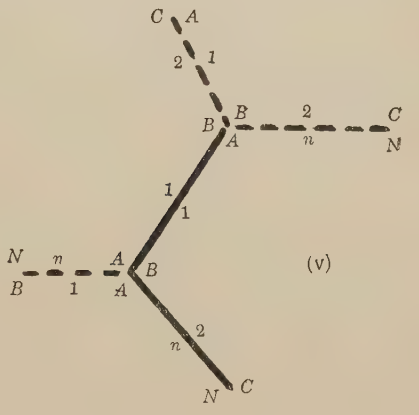
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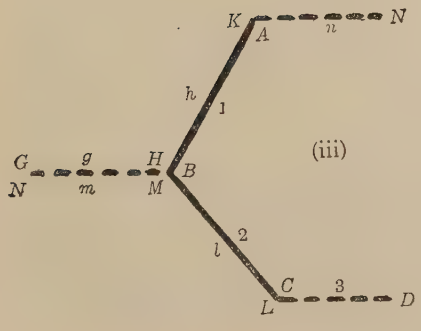
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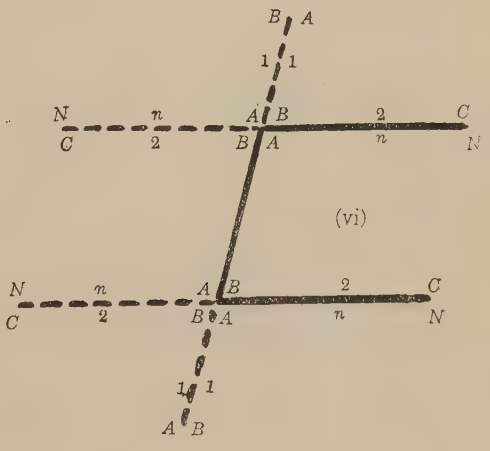
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(iv) *Angle groups.*—These are the groups already referred to as being formed at the vertices of each fundamental cell in an assemblage. It is clear that the angles of these groups are all of them angles of the repeat, and consequently the angles of the repeat must be such that they can be distributed into sets of angles all different or with repetitions, the sum of the angles in each set being equal to 2π. The restrictions thus imposed upon the angles of the repeat will be rendered more definite by a consideration of the manner in which the angles of each fundamental cell are associated together by the contact to which it may be subjected in the assemblage.

These considerations must, however, be prefaced by two observations:—

(*a*) For the purposes of this discussion a repeat assemblage is understood to be *homonomous*, that is, in which a *definite system of side-to-side contact prevails throughout and is the law of growth of the whole.* Heteronomous assemblages—and such are occasionally possible—are of no interest in the present investigation.

(*b*) Since the sum of the interior angles of any polygon of n sides is given by $(n-2)\pi$, and for $n > 4$ the sum of the angles exceeds 2π, it is clear that in an assemblage of repeats whose order is higher than the fourth no angle-group can be composed of *all* the angles or of more than four differently lettered angles of the unit repeat. Now, these angle groups being formed at the vertices of each unit, it is evident that if the constituents of these groups are to be studied the contact to which pairs of *adjacent* sides of the fundamental cell or unit may be subjected must be examined, since it is thus that the angles of the unit are, for the most part, associated in the groups.

(v) *Analysis of contact.* (For figures see accompanying plate.)

(*a*) *Two adjacent sides in point-image contact.* — A, B, C, ..., N (Fig. 1) is a repeat of n sides in the assemblage of which two adjacent sides, say 1 and 2, are in point-image contact. Then the angle-group at B contains the three angles A, B, C. Since the assemblage must be homonomous, either $(A+B+C) = 2\pi$ and the sides 3, n are in parilateral contact, or $(A+B+C) < 2\pi$.

Assuming that the repeat is not a triangle,* the angle of deficiency in the latter case is clearly not A, B, or C, and is therefore *one* of the

* If the figure is a triangle we have the unique case of an assemblage in which *each* completed angle-group contains *all* the angles of the repeat, each occurring twice over.

remaining angles of the repeat, since no angle-group can be composed of more than four differently lettered angles of the fundamental cell. It follows that the sides 3, n must be in point-image contact and the repeat is a quadrilateral. The assemblage being homonomous, it is evident that the angles brought into association by the contact system must continue so associated, and *the group constituted by them is uniquely determined.*

(*b*) *Two adjacent sides in parilateral contact with respect to one another.*—As before, $A, B, C, \ldots, N$ (Fig. 2) is an n-sided repeat, but in the assemblage two sides, say 1 and 2, are in parilateral contact with respect to one another. Since the assemblage is homonomous, the angle-group at B will contain no other angle of the repeat except B. B must therefore be a sub-multiple of 2π, and cannot occur associated in a group with any other angle of the repeat.

If the adjacent sides are in parilateral contact with two other and differently numbered sides, so that the contact is, say, 1 to h, 2 to l (Fig. 3), it is clear that either the angle-group at B will be constituted by the angles B, H, M, in which case g and m will be in parilateral contact, *or* the group will contain a fourth angle which will not be either B, H, or M, *or* the group will be constituted by B and one other angle cognate to it, in which case the two sides with which 1 and 2 are in parilateral contact will themselves be adjacent sides. Let them be g, h. (Fig. 6.)

If this is a possible case, clearly B and H must be cognate angles; it follows, moreover, that if the four sides concerned be written down in cyclic order, 1, 2, g, h, the extremes and the means must be equal sides. Again, the group H, B is uniquely determined.

(*c*) *One side in point-image contact, the two sides adjacent to it at either extremity being in parilateral contact in respect to one another.* (Fig. 5.)—This case is inadmissible *unless* the angle-groups at A and B each contain a fourth angle, which must be A in the group at A and B in that at B. It is clear that if both A and B are less than 180° they must be *supplementary* angles; if one of them is greater than 180° they must be *cognate* angles. (Fig. 6.) In either case the sides 2, n are both equal and parallel.

Further examples might be given, but sufficient cases have been taken to establish a fundamental law of repetition which we call the "Law of Angle Distribution"—namely, that *a closed polygon may be a repeat if, and only if, its angles are such that they may be associated together in sets of two, three, or four differently lettered angles, the sum of the angles in each set being equal to* π *or* 2π.

The case referred to in (*b*), Fig. 2, p. 79, is no exception to this law, as may be demonstrated as follows.

Let $A, B, C, \ldots, N$ be a repeat of any number of sides; suppose that in its assemblage two adjacent sides, say 1 and 2, are in parilateral contact.

It is clear that the angle B contained by these sides must—

(i) be a sub-multiple of 2π (*e.g.* $2\pi/\lambda$) and less than 180°,

(ii) occur in a group, the composition of which is $B, B, \ldots, \lambda$ times.

Then as to the remaining angles of the figure it follows that either (i) one or more than one of them constitutes with B a set the sum of whose angles is π or 2π; or (ii) the sum of the remaining angles is a multiple of π. (ii) is clearly inadmissible, and the former case only remains for consideration.

(*a*) If the set is constituted by B together with *one* of the remaining angles, let this angle be K. Then it must be

(i) a sub-multiple of 2π and less than 180°,

(ii) contained by two sides subject to parilateral contact, other than those containing B.

Further, if $K = 2\pi/\mu$, since both B and K are less than 180°, we have

$$B+K = \pi,$$

that is

$$\frac{2\pi}{\lambda}+\frac{2\pi}{\mu} = \pi,$$

or

$$\frac{1}{\lambda}+\frac{1}{\mu} = \frac{1}{2},$$

whence, since λ and μ are both integral,

(i) $\lambda = 6, \quad \mu = 3;$

(ii) $\lambda = 4, \quad \mu = 4.$

The repeat must therefore have at least two distinct pairs of equal adjacent sides, the contained angles being either 120°, 60°, or 90°, 90°.

(*b*) If the set is constituted by B together with *two* of the remaining angles, let these be K, L. Then $(K+L)$ is either supplementary or cognate to B. In the former case $(K+L)$ is a sub-multiple of 2π, hence K and L each

occur the same number of times in the group in which they are associated.

The latter case is considered later on in § 2.

The case of a set constituted by B together with more than two of the remaining angles of the repeat does not occur.

(vi) The following table gives, in generalized form, the distributions into sets of 2, 3, or 4 of the angles of repeating polygons.

Order	Sum of int. angles	2π	2π	2π		π	π
$2m$ $m>2$	$(2m-2)\pi$	4	2	2	...	0	0
		2	2	2	...	2	2
		3	3	2	...	0	0
$2m+1$ $m>1$	$(2m-1)\pi$	2	2	2	...	3	0
		3	2	2	...	2	0

The 2's under columns headed "2π" are to be continued so far as to make the sum of all the numbers in the row equal to $2m$ or $2m+1$.

For convenience and brevity of reference the table may be written as follows, using the partition notation* :—

Order	Content of Partition	No. of Parts	Double Partition
$2m$	$2m$	m	$(4\,2^{m-2})$ (0) (2^{m-2}) (2^2) $(3^2\,2^{m-3})$ (0)
$2m+1$	$2m+1$	m	(2^{m-1}) (3) $(3\,2^{m-2})$ (2)

* Where $(a^s b^t \ldots)$ means a partition of n into s parts each equal to a, t parts each equal to b, etc., so that $sa+tb+\ldots=n$. The partition in the left-hand bracket refers to the columns in the first table headed 2π; that in the right-hand bracket refers to the columns headed π, *and is fixed in each case.*

The following are immediately deduced from this table :—

(i) No *convex repeat polygon exists of order higher than the sixth.*

(ii) A repeat polygon having k re-entering angles does not exist for an order $< 2k+2$ or $> 2k+6$.

(iii) A repeat polygon of order $2m$ has either $m-3$, $m-2$, or $m-1$ re-entering angles.

(iv) A repeat polygon of order $2m+1$ has either $m-2$ or $m-1$ re-entering angles.

(v) It follows that five consecutive orders of polygon exhibit, as repeats, polygons with any specified order of concavity.

(vii) *Lateral conditions of repetition.*—Although every repeat polygon is subject to the law of angle distribution, it is not necessarily completely determined in this way; conditions of relationship between the sides must, in almost every case, be satisfied in addition.

Turning once more to the table, it is seen that—

(i) As the order of the repeat increases, the number of sets of *two* angles whose sum is 2π increases also, while from the *fifth* order onwards the number of sets of *three* angles whose sum is 2π or π, and of *four* angles whose sum is 2π, remains unchanged.

(ii) The number of sets of angles whose sum is π does not increase with the order of the repeat.

It is evident, therefore, that the relationship existing between sides considered in connection with "π" sets and "2π" sets of *two* angles are of particular importance. These relationships constitute what may be called "π" and "2π" lateral conditions of repetition.

A. "π" *lateral conditions.*—The two angles concerned must be

(i) contained by three adjacent sides of which the first is equal and parallel to the third; or

(ii) each a sub-multiple of 2π, and each contained by a pair of equal adjacent sides.

B. "2π" *lateral conditions.*—The two angles concerned (of which one is $> 180°$) must be

(i) contained by three adjacent sides of which the first is equal and parallel to the third; or

(ii) contained by two distinct pairs of adjacent sides so related that if the four sides be written down in cyclic order the extremes and the means each form a pair of equal sides, the angle of inclination of the extremes to one another being equal to that of the means, and either $0°$ or a sub-multiple of 2π.

These conditions, to which another will be added in a more suitable place, are sufficiently general to cover all cases considered in this paper. It may be remarked, however, that from the 6th order onwards an obvious amplification is required, which nevertheless in no way affects their essential character. In what follows these conditions are referred to as A(i), A(ii), B(i), B(ii).

2. *The triangle.*

The preceding remarks are sufficient to establish *every* triangle as a repeat, since the sum of the interior angles of every triangle is equal to π, and assemblage involving point-image contact only associates all three angles, each occurring twice at the vertices of each unit.

The contact system is written 1 to 1, 2 to 2, 3 to 3, with respect to the numbered sides. We have thus a general case. Moreover, an *isosceles* triangle may be assembled with parilateral contact in respect to its two equal sides, provided

(i) the angle contained by the equal sides is a sub-multiple of 2π;

(ii) each of the remaining angles is a sub-multiple of 2π.

We have thus three cases only:—

(i) an equilateral triangle;

(ii) a right-angled isosceles triangle;

(iii) an isosceles triangle, the angle contained by the equal sides being $120°$, *i.e.* a $\pi/3$-isosceles triangle.

If the equal sides are 1, 3, the contact system is 1 to 3, 2 to 2.

It is to be noted that the number of different ways in which a repeating polygon may be assembled depends, in general, upon the number of different conditions of repetition it satisfies. The terms *simplex*, *duplex*, etc., will be used to denote this characteristic ; thus, while the triangle in general is a simplex repeat, the three particular cases mentioned above are each of them *duplex* repeats, the assemblage resulting from the contact system 1 to 3, 2 to 2 being considered *proper* to them.

The quadrilateral.

The fact that the sum of the interior angles of any quadrilateral is equal to 360° is, in view of what has been already stated, sufficient to establish *every* quadrilateral figure as a repeat assembling according to the contact system, 1 to 1, 2 to 2, 3 to 3, 4 to 4.*

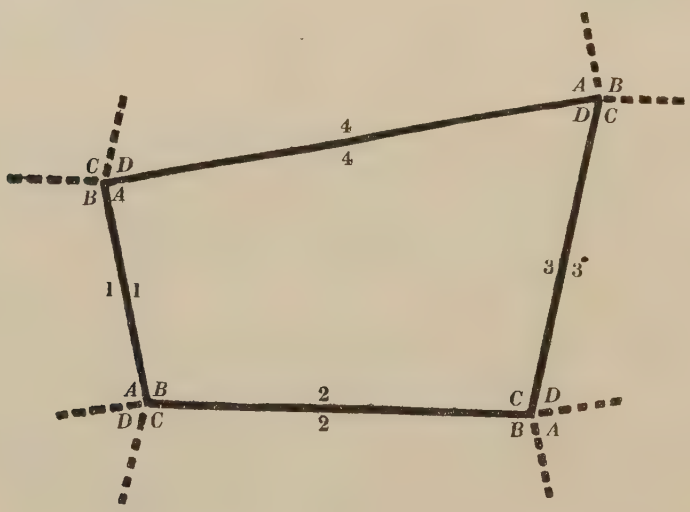


FIG. 1.

(i) Convex.

This repeat, which may be termed the *general quadrilateral*, is clearly

* In Sommerville's "Non-Euclidean Geometry," p. 86, reference is made to the quadrilateral as a repeat, but the author appears to restrict the property of repetition to the *convex* forms. The present writer, in agreement with Dr. G. T. Bennett, regards this restriction as inadmissible.

of type $\{(4\,2^{m-2})(0)\}$, $m = 2$, and is simplex in character, since the property of repetition is in no wise dependent on lateral conditions.

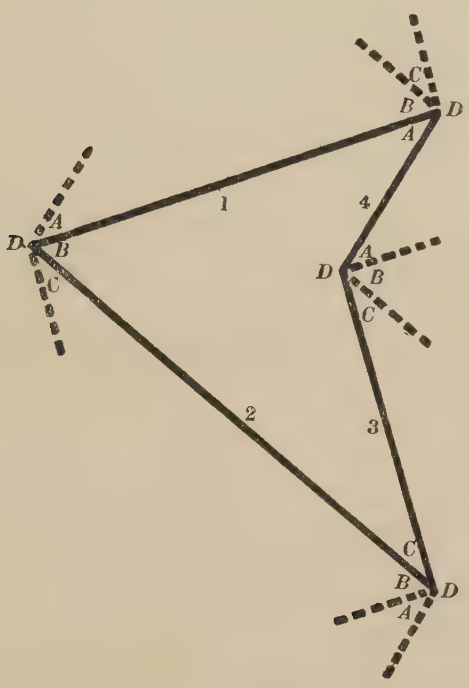


FIG. 2. (ii) Concave.

Quadrilateral repeats of the second type $\{(2^{m-2})(2^2)\}$ are those whose angles may be distributed into two sets of two angles, the sum of the angles in each set being 180°. It follows that parilateral contact will be involved in the assemblage and lateral conditions imposed, viz. (i) two alternate sides equal and parallel, or (ii) two pairs of equal adjacent sides, each pair containing a sub-multiple of 2π, the two angles being supplementary.

It is clear that if either of these conditions is satisfied with regard to one of the "π" sets essential to the type the condition respecting the other "π" set is satisfied also. In this type, therefore, the property of repetition is dependent on the satisfying of the lateral conditions in regard to one "π" set only.

Quadrilaterals may be found to satisfy either of the given conditions which leads to a subdivision of the type into two sub-types, $\{(2^{m-2})(2^2)\}_\alpha$ and $\{(2^{m-2})(2^2)\}_\beta$.

I

$\{(2^{m-2})(2^2)\}_\alpha$.

Quadrilateral repeats satisfying the first of the lateral conditions. A repeat of this sub type is clearly any parallelogram ; it is quadruplex in character, the four contact systems being :—

(i) 1 to 3, 2 to 2, 4 to 4 ;
(ii) 1 to 1, 2 to 4, 3 to 3 ;
(iii) 1 to 3, 2 to 4 ;
(iv) 1 to 1, 2 to 2, 3 to 3, 4 to 4.

The assemblages resulting from (i), (ii), (iii) are proper to the sub-type.

II

$\{(2^{m-2})(2^2)\}_\beta$.

This sub-type contains three members only, and here the pairs of supplementary angles must be either (i) 90°, 90° or (ii) 120°. 60°. If (i) we have the *square ;* if (ii) we have the $\pi/3$-*Rhombus and the Kite.* Each of these is, of course, uniquely determined as to shape. The *square* and $\pi/3$-*Rhombus* as repeats are quintuplex, the five contact systems being :—(i) 1 to 2, 3 to 4 ; (ii) 1 to 3, 2 to 2, 4 to 4 ; (iii) 1 to 1, 2 to 4, 3 to 3 ; (iv) 1 to 3, 2 to 4 ; (v) 1 to 1, 2 to 2, 3 to 3, 4 to 4.

The assemblage resulting from (i) is proper to the square, and will be always that referred to as the " square assemblage."

The *kite* as a repeat is duplex, the two assemblages being :—(i) 1 to 2, 3 to 4 ; (ii) 1 to 1, 2 to 2, 3 to 3, 4 to 4.

The assemblage resulting from (i) is proper to the kite and also to the $\pi/3$-Rhombus.

The pentagon.

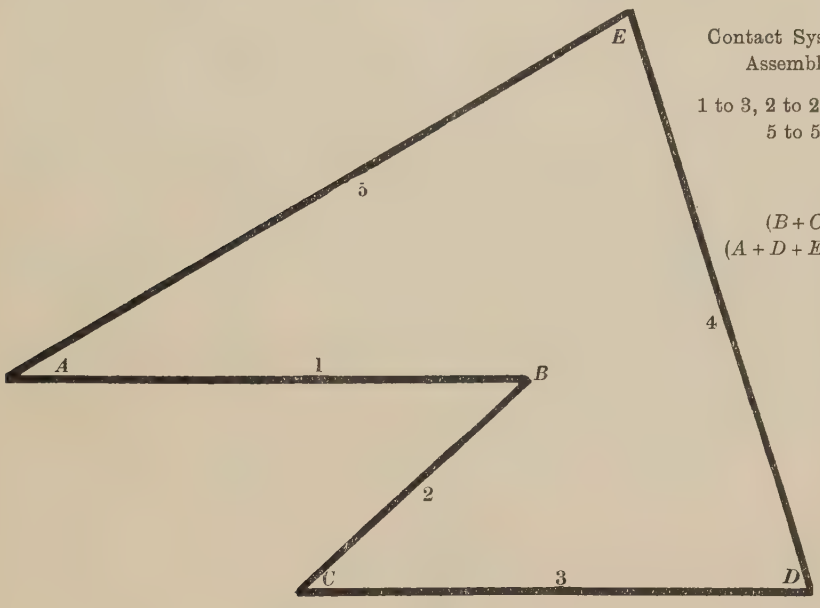
There are two distinct types to be considered—namely $\{(2^{m-1})(3)\}$ and $\{(3\ 2^{m-2})(2)\}$. In each case lateral conditions are imposed, and there is consequently no general case.

I

$\{(2^{m-1})(3)\}$ $(m = 2)$.

The lateral conditions are those regarding " 2π " sets, *i.e.* B(i) and (ii) previously given (p. 83), and we have accordingly a subdivision into two sub-types $\{(2^{m-1})(3)\}_\alpha$ and $\{(2^{m-1})(3)\}_\beta$ according as the first or the second of the conditions is satisfied by the repeat. Evidently every repeat of the type is concave.

(*a*) $\{(2^{m-1})(3)\}_a$. The construction is seen at once from the adjoining figure, the restricted sides being those numbered 1 and 3.



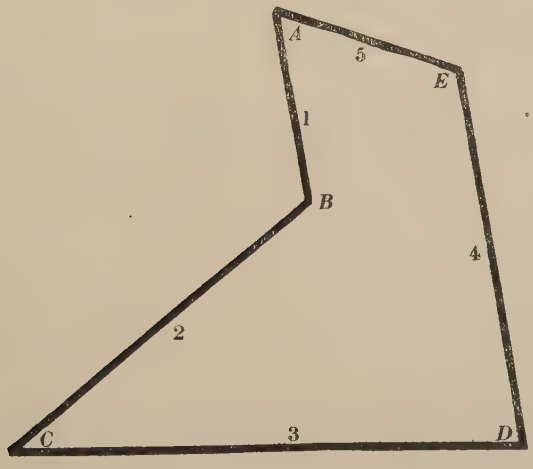
Contact System of Assemblage

1 to 3, 2 to 2, 4 to 4, 5 to 5

$(B + C) = 2\pi$
$(A + D + E) = \pi$

FIG. 3.
Pentagon Repeat, Type $\{(2^{m-1})(3)\}_a$ $(m = 2)$.

It is clear that an infinite variety of construction is possible. The repeat is simplex, the contact system being 1 to 3, 2 to 2, 4 to 4, 5 to 5.



Contact System of Assemblage

1 to 5, 2 to 4, 3 to 3

$A = 60°$
$1 = 5$
$2 = 4$
$(B + E) = 2\pi$
$(A + C + D) = \pi$

FIG. 4.
Repeat Pentagon Sub-Type $\{(2^{m-1})(3)\}_\beta$ $(m = 2)$.

(*b*) $\{(2^{m-1})(3)\}_\beta$. Satisfying lateral condition B (ii). It is to be noted that the inclination of the extremes (and the means) of the restricted sides to one another must in this case be either 120°, 90°, or 60°. This further restriction arises from the intimate connection of all repeats of this type with the triangle; in fact, the general triangle together with the three isosceles repeat triangles, the general quadrilateral, the $\pi/3$-Rhombus constitute six primitive or base repeats from which *every other repeat whatever* may be derived by a process of transformation or system of finite displacements, the properties essential to the base *per se* remaining invariant in all derivatives of the same type or sub-type.

The angular restriction in the present instance suggests at once that the repeat under consideration is a derivative of one of the isosceles triangle bases—as is, in fact, the case.

The repeat is simplex, the contact system being 1 to 5, 2 to 4, 3 to 3, as shown in the assemblage given (Fig. 5).

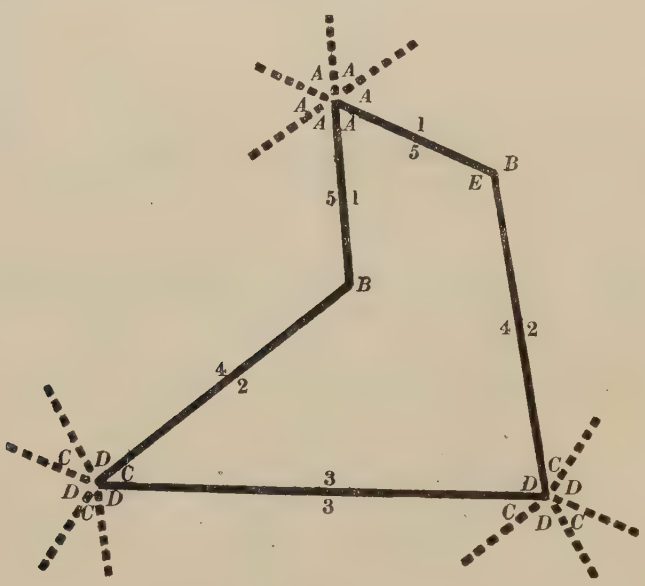


FIG. 5.

Owing to the absence of " 2π " sets, the lateral conditions are A (i) and (ii), and we have again a sub-division into two sub-types. The repeats are both convex and concave.

(*a*) $\{(3^{m-2})(2)\}_\alpha$ satisfying condition A (i). The construction presents no difficulties. With one exception, which will be mentioned further on, all repeats of this sub-type are simplex, the contact system being 1 to 3, 2 to 2, 4 to 4, 5 to 5.

II

$\{(3\ 2^{m-2})(2)\}$ $(m = 2)$.

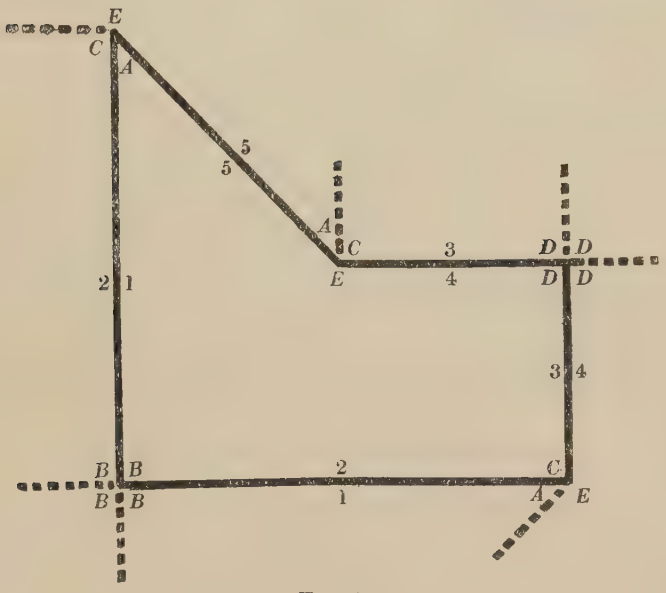


FIG. 6.

(*b*) $\{(3^{m-2})(2)\}_\beta$ satisfying condition A (ii). In this case the pairs of supplementary angles must be either (i) 90°, 90° or (ii) 120°, 60°. We observe, however, that the repeats are not in this case uniquely determined in shape, but allow of an infinite variety of construction. Two examples are of some particular interest, the first being a convex equilateral pentagon, the supplementary angles being each 90°. This is known as the "haystack" repeat, and is of special interest in the construction of symmetrical repeating patterns.*

The second is that of the repeat obtained by constructing a right-angled isosceles triangle on one side of a square; the resulting pentagon is a repeat satisfying the conditions of *both* sub-types α and β. It is the only pentagon repeat that is duplex, the contact systems being:—

(i) $\{(3\ 2^{m-2})(2)\}_\alpha$ 1 to 3, 2 to 2, 4 to 4, 5 to 5;

(ii) $\{(3\ 2^{m-2})(2)\}_\beta$ 1 to 2, 3 to 3, 4 to 5.

This repeat is fundamentally derivable from two distinct bases, (i) a particular case of general triangle, and (ii) an isosceles right-angled triangle.

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* Cf. *Proc. Royal Soc.* (A), 101 (1922), 80–94 (85–88).

The hexagon.

There are three types to be considered :—

(i) $\{(4\,2^{m-2})(0)\}$,

(ii) $\{(2^{m-2})(2^2)\}$,

(iii) $\{(3^2\,2^{m-3})(0)\}$.

It is to be noted that the presence of a "2π" set excludes a general case in (i) and (ii), and that for the same reason all repeats of these two types are concave.

I

$\{(4\,2^{m-2})(0)\}$ $(m = 3)$.

This type, as will be observed, is essentially the same as that of the general quadrilateral, and we have merely to allow for the lateral conditions imposed by the altered value of m, in this case B(i), and complete the hexagon as we please. The repeat is simplex, the contact system being 1 to 3, 2 to 2, 4 to 4, 5 to 5, 6 to 6.

II

$\{(2^{m-2})(2^2)\}$ $(m = 3)$.

As in the case of the quadrilateral, we have again a subdivision into two sub-types, $\{(2^{m-2})(2^2)\}_\alpha$ and $\{(2^{m-2})(2^2)\}_\beta$, according as the first or the second of the "π" lateral conditions is satisfied; moreover, since the "2π" lateral conditions may be satisfied in either of two ways, we have a further subdivision into $\{(2^{m-2})(2^2)\}_{\alpha_1}$ and $\{(2^{m-2})(2^2)\}_{\alpha_2}$.

(*a*) $\{2^{m-2})(2^2)\}_{\alpha_1}$. Satisfying conditions A(i) and B(i). Taking 1 equal and parallel to 3 $(B+C) = 2\pi$ and 4 equal and parallel to 6 $(E+F) = \pi$, the assemblage may be made according to either of the two contact systems (i) 1 to 3, 2 to 2, 4 to 6, 5 to 5; (ii) 1 to 3, 2 to 2, 4 to 4, 5 to 5, 6 to 6, of which (i) is that proper to the repeat.

$\{(2^{m-2})(2^2)\}_{\alpha_2}$. Satisfying conditions A(i) and B(ii). The repeat is triplex. Taking 1 . 4, 2 . 6, 3 . 5, as pairs of equal and parallel sides $(A+B) = \pi$, $(C+F) = 2\pi$, the three contact systems are (i) 1 to 4, 2 to 6,

3 to 5; (ii) 1 to 1, 2 to 6, 3 to 5, 4 to 4; (iii) 1 to 4, 2 to 2, 3 to 3, 5 to 5, 6 to 6; of which (i) and (ii) are proper to the repeat.

FIG. 7.

(*b*) $\{2^{m-2})(2^2)\}_\beta$. In this case the "π" lateral condition is A(ii) and the angles contained by the two pairs of equal adjacent sides must be either (i) 90°, 90°, or (ii) 120°, 60°. We take the former for purposes of illustration. The "2π" lateral condition is B(ii) and is the only condition applicable in this case, the reason for which is to be found in the close connexion between this repeat and the *square* repeat of the same type. A partially assembled example of this repeat is given (Fig. 7). The repeat is simplex, the contact system being 1 to 2, 3 to 6, 4 to 5. The assemblage is of a kind which may conveniently be termed *rotational*.

III

$$\{(3^2\,2^{m-3})(0)\} \quad (m = 3).$$

Since any closed polygon of order $2m$ contains m pairs of opposite sides, it is easily shown that, where any two such sides are parallel, the sum of all the angles lying adjacently at either extremity of these sides is $(m-1)\pi$. In the present case, in which $m = 3$, we have, accordingly, where any two opposite sides of a hexagon are parallel, the six angles distributed into two sets each containing three angles whose sum is 360°.

These angles will be associated together in groups in the assemblage, only if the two parallel and opposite sides are in contact, whence we deduce that *any hexagon is a repeat* of the type we are considering, *if it has one pair of opposite sides equal and parallel.* We will call this lateral condition C(i). Moreover, it is found that if a hexagon has *three* pairs of equal adjacent sides, the angle contained by each pair a sub-multiple of 2π, and the sum of the three angles equal to 2π, a repeat of this type is given. We will call this lateral condition C(ii).

We have here the case referred to (p. 80) in which one angle, being a sub-multiple of 2π, is associated with two of the remaining angles of the repeat in a set, the sum of the three angles being equal to 2π. This case appears to exist only when *each* of the three angles is a sub-multiple of 2π, so that λ, μ, ν being integral and > 2, we have the equation

$$\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{\nu}=1$$

with the unique solution $\lambda = \mu = \nu = 3$.

There are clearly, therefore, two sub-types,

$$\{(3^2\,2^{m-3})(0)\}_\alpha \quad \text{and} \quad \{(3^2\,2^{m-3})(0)\}_\beta$$

according as condition C(i) or C(ii) is satisfied.

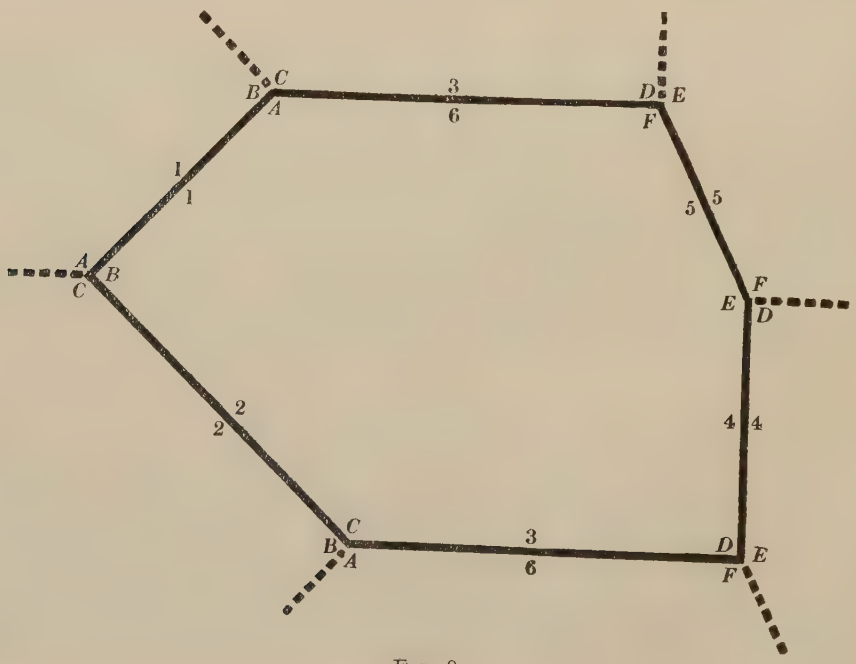

FIG. 8.

(*a*) $\{(3^2\,2^{m-3})(0)\}_\alpha$. Satisfying condition C(i). The repeat may be convex, or concave with one or two re-entrant angles. If only one pair of opposite sides is restricted the repeat is simplex, the contact system being 1 to 1, 2 to 2, 3 to 6, 4 to 4, 5 to 5. An example is given (Fig. 8). If, however, all three pairs of opposite sides are restricted, the repeat is duplex assembling *either* as above or according to the contact system 1 to 4, 2 to 5, 3 to 6. No assemblage is possible in which only two pairs of opposite sides are in parilateral contact.

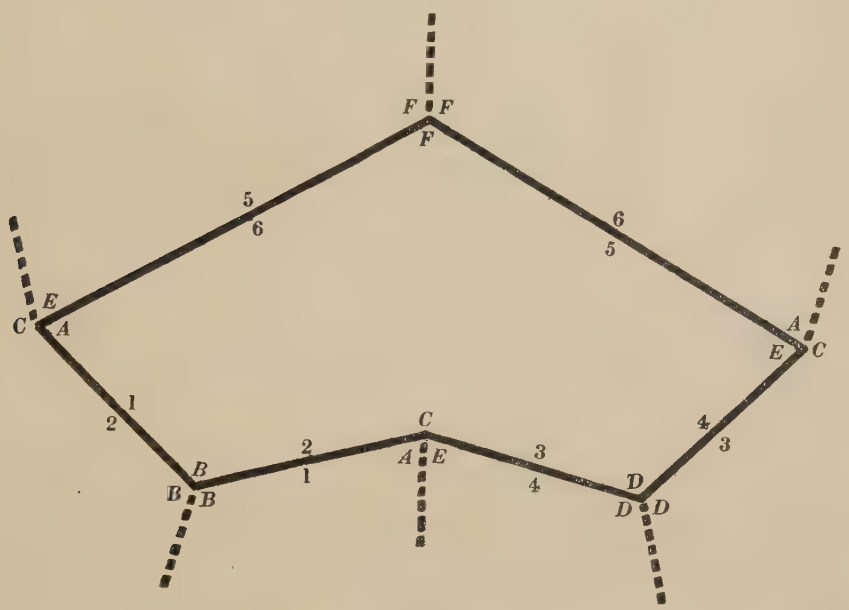

FIG. 9.

(*b*) $\{(3^2\,2^{m-3})(0)\}_\beta$. Satisfying condition C(ii). The repeat may be convex or concave, but in the latter case one re-entrant angle only is possible. A partial assemblage of a concave example is given (Fig. 9). The repeat is simplex, the contact system being 1 to 2, 3 to 4, 5 to 6.

The regular hexagon is a member of this sub-type and the contact system just given is proper to it as such; since, however, it may also be regarded as a particular case of $\{(3^2\,2^{m-3})(0)\}_\alpha$, its character is accordingly triplex. Hexagons of this sub-type are derived fundamentally from the $\pi/3$-Rhombus.

A THEOREM ON POWER SERIES

By J. E. LITTLEWOOD.

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1. Let $f(z) = a_0 + a_1 z + \ldots$ be regular for $|z| = r < 1$,

$$F(r) = \sum_{n=0}^{\infty} |a_n| r^n,$$

$$M(r) = \max_{|z|=r} |f(z)|,$$

$$M_2(r) = \sqrt{\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right)} = \sqrt{\left(\sum_{n=0}^{\infty} |a_n|^2 r^{2n}\right)}.$$

$M_2(r)$ is the mean square modulus of f on the circle $|z| = r$; $F(r)$ is the majorant of f. M_2 and F depend only on the *moduli* of the coefficients a_n. Obviously

$$M_2(r) \leqslant M(r) \leqslant F(r).$$

We have, by Schwarz's inequality,

$$(1.1) \quad M(r) \leqslant F(r) = \Sigma |a_n| r^{\frac{1}{2}n} r^{\frac{1}{2}n} \leqslant \sqrt{(\Sigma |a_n|^2 r^n \Sigma r^n)} = M_2(r^{\frac{1}{2}})(1-r)^{-\frac{1}{2}}.$$

We shall confine ourselves always to functions for which

$$(1.2) \qquad M_2(r) < \gamma(1-r)^{-\beta}, \quad \beta \geqslant 0,$$

i.e. to functions for which $M_2(r)$ is of order not exceeding β. Then

$$M_2(r) \leqslant M_2(r^{\frac{1}{2}}) < 2^{\beta}\gamma(1-r)^{-\beta},$$

and the functions $M_2(r)$ and $M_2(r^{\frac{1}{2}})$ are of the same order.* (1.1) now shows that the order of $M(r)$ or $F(r)$ cannot exceed that of their lower bound $M_2(r)$ by more than $\frac{1}{2}$. This result (and the proof) is due to Hardy.†

Let us, for illustration, consider the special case in which $|a_n| = 1$. It has been shown by Hardy and the present writer jointly that various sets of real

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* That is $\overline{\lim}\, \log M_2(r^{\frac{1}{2}}) \Big/ \log \frac{1}{1-r} = \overline{\lim}\, \log M_2(r) \Big/ \log \frac{1}{1-r}$.

† G. H. Hardy, *Quart. J. of Math.*, 44 (1913), 147.

numbers $a_0, a_1, \ldots$ exist, such that the function $f(z) = \Sigma e^{2\pi i a_n} z^n$ has, for some r arbitrarily near unity, the property

$$A'(1-r)^{-\frac{1}{2}} < |f| < A(1-r)^{-\frac{1}{2}},$$

so that $$M(r) < AM_2(r), \qquad F(r) > A' M_2(r)(1-r)^{-\frac{1}{2}}.$$

For example we may take $a_n = \vartheta n^2$, where ϑ is any irrational constant of a certain arithmetical type,* or $a_n = \phi n \log n$, where ϕ is any real constant other than zero.†

The trivial case $f(z) = \Sigma z^n$ provides the relations

$$M(r) = F(r) = M_2(r)(1-r)^{-\frac{1}{2}}.$$

Finally, to complete our collection of possibilities, it is easily shown that for $f = \Sigma n!\, e^{2\pi i a_n} z^{n!}$ we have, uniformly in the set of α's,

(1.3) $$F(r) \sim M(r) \leqslant (\sqrt{2}+\epsilon) M_2(r) \quad (r \to 1),$$

and $$F(r) \sim M(r) \sim M_2(r)$$

for *some* values of r tending to unity.

The relations between F and M_2 can be summed up in the statements:

(i) The difference in order of F and M_2 is at most $\frac{1}{2}$.

(ii) In special cases the difference is actually $\frac{1}{2}$.

(iii) In very special cases the order of F and M_2 is the same.

There is nothing more of interest to be said about F.

The problem of the connection of M and M_2 is more difficult, since M, unlike M_2 and F, depends on the *arguments* of the a_n. We regard the function $M_2(r)$, or, what is the same thing, the sequence of moduli $|a_0|, |a_1|, \ldots$, as given, and our problem is the dependence of $M(r)$ on the arguments. Observing that the choice $\arg a_n = 0$ makes $M(r) = F(r)$, we can infer at once from our existing information:

(i) The difference in order of $M(r)$ and $M_2(r)$ is at most $\frac{1}{2}$.

(ii) For special $|a_n|$ and selected $\arg a_n$ the difference is actually $\frac{1}{2}$.

(iii) For very special $|a_n|$ [(1.3)] but *arbitrary* $\arg a_n$ the order of $M(r)$ and $M_2(r)$ is the same.

(iv) For $|a_n| = 1$ and selected $\arg a_n$ the order of $M(r)$ and $M_2(r)$ is the same.

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* G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation II", *Acta Math.*, 37 (1914), 193–238.

† "Some Problems of Diophantine Approximation", *Proc. Nat. Sci. Acad.*, 2 (1916), 583–586.

It is instructive also to observe, in the case $|a_n| = 1$, how the integral

$$\int_0^{2\pi} |f|^2 \, d\theta$$

acquires its "order 1 as $r \to 1$", by contributions from different parts of the circle $|z| = r$. Our facts indicate that, to put the matter roughly, $M_2(r)$ acquires, for some sets of a's, its order $\frac{1}{2}$ from the presence of a single point on $|z| = 1$ in the neighbourhood of which f is of order 1, and for other sets from a more or less uniform distribution of order $\frac{1}{2}$ over the circle $|z| = 1$.

The facts we have put forward suggest that for *every* given set of moduli $|a_0|, |a_1|, \ldots$, there exists a set $a_0, a_1, \ldots$, such that, for $f = \Sigma |a_n| e^{2\pi i a_n} z^n$, $M(r)$ does not greatly exceed its lower bound $M_2(r)$. Theorem 2 shows that this is true for functions satisfying (1.2): the theorem is capable of extension, but at the price of added complications, which make an exhaustive discussion hardly worth while. The argument has some affinity, on the one hand to Bohr and Landau's recent paper* on the behaviour of $1/\zeta(1+it)$, and on the other to the proof of one of the results concerning "Lindelöf's hypothesis" recently published jointly by Hardy and the present writer.† The proof of the general theorem is necessarily a little involved, and I have thought it best to begin with a proof of a simpler theorem, of some independent interest, in which the main ideas of the paper are clearly displayed.

2. THEOREM 1.—*Let* $a_n \geqslant 0$ $(0 \leqslant n \leqslant m)$. *Then there exists a set of real numbers* $\alpha_0, \alpha_1, \ldots, \alpha_m$ *for which*

$$\operatorname*{Max}_{|z|=1} \left| \sum_0^m a_n e^{2\pi i \alpha_n} z^n \right| \leqslant A \sqrt{\left(\sum_0^m a_n^2 \right)} \sqrt{\left(\log (m+2) \right)},$$

where A *is an absolute constant.*

A's will always denote positive absolute constants. We may suppose

$$m \geqslant 1, \quad \sum_0^m a_n^2 > 0, \quad \alpha_0 = 0, \quad 0 \leqslant \alpha_n \leqslant 1.$$

Let $$z = e^{i\theta}, \quad f(z) = \sum_0^m a_n e^{2\pi i \alpha_n} z^n.$$

Let k be a positive integer, and

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* H. Bohr and E. Landau, *Göttinger Nachrichten* (1923), 303–330.

† G. H. Hardy and J. E. Littlewood, *Proc. Royal Soc.* (A), 103 (1923), 403–412 (405–406).

$$\{f(z)\}^k = \sum_0^{km} b_n z^n, \tag{2.1}$$

so that

$$J = J(k) = \frac{1}{2\pi}\int_0^{2\pi} |f|^{2k}\, d\theta = \sum_0^{km} |b_n|^2. \tag{2.2}$$

Now

$$b_n = \Sigma c(\nu_1, \nu_2, \ldots)\, a_{n_1}^{\nu_1} a_{n_2}^{\nu_2} \ldots \exp\{2\pi i(\nu_1 \alpha_{n_1} + \nu_2 \alpha_{n_2} + \ldots)\} \tag{2.3}$$

where

$$c(\nu_1, \nu_2, \ldots) = \frac{k!}{\nu_1!\, \nu_2! \ldots} \leqslant k^k, \tag{2.4}$$

the summation is taken over all $n_1, n_2, \ldots, \nu_1, \nu_2, \ldots$, subject to

$$\begin{cases} \nu_r > 0, \quad 0 \leqslant n_1 < n_2 < n_3 < \ldots \\ \nu_1 + \nu_2 + \ldots \quad = k \\ n_1\nu_1 + n_2\nu_2 + \ldots = n, \end{cases} \tag{2.5}$$

and we interpret a_n to be zero for $n > m$.

Let us write $$I_m(\phi) = \int_0^1 da_1 \int_0^1 da_2 \ldots \int_0^1 \phi\, da_m.$$

Then

$$I_m(J) = \sum_{n=0}^{km} I_m(|b_n|^2). \tag{2.6}$$

Now for $n \leqslant km$

$$|b_n|^2 = \Sigma_1 c^2(\nu_1, \nu_2, \ldots)\, a_{n_1}^{2\nu_1} a_{n_2}^{2\nu_2} \ldots + \Sigma_2 c(\nu_1 \ldots)\, c(\nu_1' \ldots)\, a_{n_1}^{\nu_1} \ldots a_{n_1'}^{\nu_1'} \ldots \exp(2\pi i L),$$

where $$L = \nu_1\alpha_{n_1} + \nu_2\alpha_{n_2} + \ldots - \nu_1'\alpha_{n'_1} - \nu_2'\alpha_{n'_2} - \ldots,$$

Σ_1 is taken over all ν_r and n_r subject to (2.5), and Σ_2 is taken over a certain set of ν_r, n_r, ν_r' and n_r', for which we do not have simultaneously $n_1 = n_1'$ and $\nu_1 = \nu_1'$, $n_2 = n_2'$ and $\nu_2 = \nu_2'$, ... For each term of Σ_2 there exists an α_p such that

$$L = \nu\alpha_p + (a\ \textit{linear function of}\ \alpha\textit{'s other than}\ \alpha_p), \quad \nu \neq 0.$$

Hence
$$\int_0^1 \exp(2\pi iL)\, da_p = 0,$$

$$I_m\Big(\exp(2\pi iL)\Big) = 0,$$

$$I_m(J) = \sum_{n=0}^{km} \Sigma_1 c^2(\nu_1, \nu_2, \ldots)(a_{n_1}^2)^{\nu_1}(a_{n_2}^2)^{\nu_2}\ldots$$

$$\leqslant k^k \sum_{n=0}^{km} \Sigma_1 c(\nu_1, \nu_2, \ldots)(a_{n_1}^2)^{\nu_1}(a_{n_2}^2)^{\nu_2}\ldots$$

$$= k^k(a_0^2+a_1^2+\ldots+a_m^2)^k = (\sqrt{k}T)^{2k},$$

where $T = \sqrt{\Big(\sum_0^m a_n^2\Big)}$. Hence *there exists a set* $a_1, a_2, \ldots, a_m$ *for which*

(2.7) $$J \leqslant (\sqrt{k}T)^{2k}.$$

Supposing now the a's to have this set of values, and so fixing the meaning of $f(z)$, let $|f|$ attain its maximum M for $z = e^{i\theta_0}$. We have

(2.8) $$|f'(z)| \leqslant \sum_0^m n a_n \leqslant \sqrt{\Big(\sum_0^m n^2 \sum_0^m a_n^2\Big)} \leqslant m^2 T.$$

Hence, if $0 \leqslant \phi \leqslant \frac{1}{2}m^{-2}$,

$$|f(e^{i\theta_0+i\phi})| \geqslant M - |f(e^{i\theta_0+i\phi}) - f(e^{i\theta_0})| = M - \left|\int_0^\phi f'(e^{i\theta_0+i\phi})\, de^{i\theta_0+i\phi}\right|$$

$$\geqslant M - \phi m^2 T \geqslant M - \tfrac{1}{2}T$$

$$\geqslant \tfrac{1}{2}M,$$

since $M \geqslant T$. We therefore have

$$\frac{1}{2m^2}(\tfrac{1}{2}M)^{2k} \leqslant \int_0^{\frac{1}{2}m^{-2}} |f(e^{i\theta_0+i\phi})|^{2k}\, d\phi \leqslant J \leqslant (\sqrt{k}T)^{2k}.$$

$$M \leqslant 2\sqrt{k}(2m^2)^{1/(2k)}.\, T.$$

The integer k is at our disposal; if we choose to be $[\log(m+2)]$ we obtain

$$M \leqslant A\sqrt{k}T \leqslant A\sqrt{\{\log(m+2)\}}\, T,$$

the desired result.

3. THEOREM 2.—*Let*

$$\beta \geqslant 0, \quad a_n \geqslant 0, \quad M_2(r) = \sqrt{\Big(\sum_0^\infty a_n^2 r^{2n}\Big)}, \quad M_2(r) < \gamma(1-r)^{-\beta} \quad (r<1).$$

Then there exists a set of real numbers $a_0, a_1, \ldots$ *such that*

$$f(z) = \sum_0^\infty a_n e^{2\pi i a_n} z^n \tag{3.1}$$

satisfies

$$M(r) = \operatorname*{Max}_{|z|=r} |f(z)| \leqslant B\mathfrak{M}(r) + C = BM_2(r)\sqrt{\left(\log \frac{2}{1-r}\right)} + C, \tag{3.2}$$

for every $r < 1$, *where* B *depends only on* β, *and* C *only on* β *and* γ.

For a *given* r the argument of § 2 yields fairly easily an f (or a set $a_0, a_1, \ldots$) satisfying (3.2). To make f independent of r, however, is more difficult, and Theorem 2 is more involved than Theorem 1.

We have $$f^k = \Sigma b_n z^n,$$

$$J = J(k, r) = \frac{1}{2\pi}\int_0^{2\pi} |f|^{2k}\, d\theta = \sum_0^\infty |b_n|^2 r^{2n} \quad (r < 1), \tag{3.3}$$

where

$$b_n = \Sigma c(\nu_1, \nu_2, \ldots)\, a_{n_1}^{\nu_1} a_{n_2}^{\nu_2} \ldots \exp\{2\pi i(\nu_1 a_{n_1} + \nu_2 a_{n_2} + \ldots)\}, \tag{3.4}$$

and the summation is taken over all $n_1, n_2, \ldots, \nu_1, \nu_2, \ldots$ subject to (2.5).

The proof is best developed in a series of Lemmas. B's and C's denote throughout positive constants depending only on β, β and γ respectively. A's denote positive absolute constants.

4. LEMMA 1.—*For* $\rho < 1$

$$\operatorname*{Max}_{|\xi| \leqslant \rho} |f'(\xi)| \leqslant B\gamma\rho^{-1}(1-\rho)^{-B_1},$$

where we may suppose $B_1 > 2$.

We have in fact

$$|f'(\xi)| \leqslant \rho^{-1}\Sigma n a_n \rho^n \leqslant \rho^{-1}\sqrt{(\Sigma a_n^2 \rho^n \,.\, \Sigma n^2 \rho^n)} \leqslant A\rho^{-1} M_2(\rho^{\frac{1}{2}})(1-\rho)^{-2}$$
$$< A\rho^{-1}\gamma(1-\rho^{\frac{1}{2}})^{-\beta}(1-\rho)^{-2} < B\gamma\rho^{-1}(1-\rho)^{-\beta-2}.$$

5. LEMMA 2.—*There exists an increasing sequence*

$$r_1 = \tfrac{1}{2},\ r_2,\ r_3,\ \ldots$$

such that $r_m \to 1$ *as* $m \to \infty$,

$$r_m - r_{m-1} = (1-r_m)^{B_1}, \tag{5.1}$$

$$r_m - r_{m-1} > Bm^{-B}, \tag{5.2}$$

and

$$\log(m+2) < B \log\left(\frac{2}{1-r_{m-1}}\right). \tag{5.3}$$

Take $r_1 = \frac{1}{2}$, and let us write $u_m = 1-r_m$. The equation (5.1) becomes $u_{m-1} = u_m + u_m^{B_1}$. If $u_{m-1} > 0$ there is a single positive solution for u_m, and $u_m < u_{m-1}$. It follows that an increasing sequence $\frac{1}{2}, r_2, r_3, \ldots$ exists, satisfying (5.1), with $r_m < 1$. (5.1) shows further that $r_m \to 1$.

We write next $v_m = mu_m$. Then (5.1) gives

$$v_m - v_{m-1} = u_{m-1} - u_m^{B_1} v_m \geqslant u_{m-1} - u_m v_m.$$

It follows that either $v_m \geqslant 1$ or else $v_m \geqslant v_{m-1}$, whence $v_m > B$ for all m. Thus $1-r_m > Bm^{-1}$, and (5.2) follows from (5.1).

Finally, in proving (5.3) we may suppose $m > 3$, in which case

$$1 > r_m = r_1 + \sum_2^m (r_n - r_{n-1}) > \sum_2^{m-1} (r_n - r_{n-1}) = \sum_2^{m-1} (1-r_n)^{B_1} > (m-3)(1-r_{m-1})^{B_1},$$

$$m+2 \leqslant 6(m-3) < 6(1-r_{m-1})^{-B_1},$$

whence we obtain (5.3).

6. After these preliminaries we can proceed with the main argument. Let μ be a positive integer.

LEMMA 3.—*There exists an* $N = N(\mu)$ (*depending only on the* a_n *and* μ) *such that for* $r = r_m$, $m \leqslant \mu$, $k \leqslant \mu$, *and every set* (α_n),

$$0 \leqslant J - \sum_0^N |b_n|^2 r^{2n} < 1.$$

We may suppose the α_n to be zero, evidently the most unfavourable case. Then for each k there exists a $\nu(k)$ such that

$$\sum_{n+1}^{\infty} |b_n|^2 r_\mu^{2n} < 1 \quad (n \geqslant \nu),$$

and we have only to take $N = \operatorname{Max}_{k \leqslant \mu} \nu(k)$.

We write $$I_N(\phi) = \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 \phi \, d\alpha_N.$$

LEMMA 4.—*For* $r = r_m$, $m \leqslant \mu$, $k \leqslant \mu$, *we have*

$$I_N(J) \leqslant T^{2k} = \left(\sqrt{k}\, M_2(r) + 1\right)^{2k}.$$

By Lemma 3 $$I_N(J) \leqslant I_N\left(\sum_0^N |b_n|^2 r^{2n}\right) + 1.$$

If $n \leqslant N$, none but a's of suffixes $\leqslant N$ can occur in b_n. The argument

leading to (2 . 7) therefore gives

$$I_N\left(\sum_0^N\right) \leqslant k^k \left(\sum_0^N |a_n|^2 r^{2n}\right)^k$$

$$I_N(J) \leqslant k^k M_2^{2k}(r)+1 \leqslant T^{2k}.$$

7. LEMMA 5.—*There exists a set* $a_1, a_2, \ldots$ *(depending on* μ*) such that*

(7 . 1) $$J \leqslant 2m^3 T^{2k} \quad (r = r_m,\ k \leqslant m,\ m \leqslant \mu).$$

We show in fact that we can choose $a_1, a_2, \ldots, a_N$ in such a way that (7 . 1) is true when $a_n = 0$ $(n > N)$. Consider the "unit cube" in N dimensions in the variables $a_1, a_2, \ldots, a_N$. Let $V(k, m)$ be the measure of the set of points $(a_1, \ldots, a_N)$ in the cube for which

$$J(k, r_m) > 2m^3 T^{2k}.$$

Then $$2m^3 T^{2k} V(k, m) \leqslant I_N(J) \leqslant T^{2k},$$

$$V(k, m) \leqslant \tfrac{1}{2} m^{-3}.$$

(7 . 2) $$V(\mu) = \sum_{m=1}^{\mu} \sum_{k=1}^{m} V(k, m) < \tfrac{1}{2} \sum_1^{\infty} m^{-2} < 1.$$

Let S be the set of points $(a_1, \ldots, a_N)$ in the unit cube for which it is not true that (7 . 1) holds for all $k \leqslant m$, and all $m \leqslant \mu$.* Evidently $mS \leqslant V(\mu)$. It follows from (7 . 2) that the complementary set of S is not null, and this proves the lemma.

8. Let us denote by $f_\mu(z)$ the function f for which the a's are determined by Lemma 5.

LEMMA 6.—*For* $r \leqslant r_\mu$

(8 . 1) $$M_\mu(r) = \operatorname*{Max}_{|z|=r} |f_\mu| \leqslant B\mathfrak{M}(r)+C.$$

Suppose first $r > \frac{1}{2}$. Let m be the integer determined by

$$r_{m-1} \leqslant r < r_m,$$

so that $2 \leqslant m \leqslant \mu$, and let f_μ attain its maximum $M = M_\mu(r)$ on $|z| = r$ at $z_0 = re^{i\psi}$. Let ϕ satisfy $0 \leqslant \phi \leqslant r_m - r_{m-1}$. Then

$$|f_\mu(r_{m-1} e^{i\psi+i\phi})| \geqslant M - \left| \int_{z_0}^{r_{m-1} e^{i\psi+i\phi}} f'_\mu(\xi)\, d\xi \right|$$

the path of integration being a straight line, and so lying in $|\xi| \leqslant r$.

* When $a_n = 0$ $(n > N)$.

By Lemma 1

$$|f_\mu(r_{m-1}e^{i\psi+i\phi})| \geqslant M - Cr^{-1}(1-r)^{-B_1}s, \tag{8.2}$$

where $s = |r_{m-1}e^{i\psi+i\phi} - z_0| \leqslant \phi r_{m-1} + (r_m - r_{m-1}) < 2(r_m - r_{m-1})$.

Since $r_{m-1} \geqslant \frac{1}{2}$ (8.2) gives

$$|f_\mu(r_{m-1}e^{i\psi+i\phi})| > M - C(r_m - r_{m-1})(1-r_m)^{-B_1} = M - C$$

by Lemma 2. Hence, for (each) $k \leqslant m$, either $M < C$, or else

$$J(k, r_m) \geqslant J(k, r_{m-1}) \geqslant \frac{1}{2\pi}\int_0^{r_m - r_{m-1}} |f_\mu(r_{m-1}e^{i\psi+i\phi})|^{2k}\, d\phi$$

$$\geqslant \frac{1}{2\pi}(r_m - r_{m-1})(M-C)^{2k} > Bm^{-B}(M-C)^{2k}$$

by Lemma 2, and so, by Lemma 5,

$$(M-C)^{2k} < Bm^B T^{2k}$$

$$M < Bm^{B/k}\left(\sqrt{k}\, M_2(r) + 1\right) + C. \tag{8.3}$$

The inequality (8.3) is therefore true in any case in which $r > \frac{1}{2}$.

We now take $k = [\log(m+2)] \leqslant m$, when (8.3) gives

$$M < B\left(\sqrt{k}\, M_2(r) + 1\right) + C < B\sqrt{\{\log(m+2)\}}\, M_2(r) + C \tag{8.4}$$

$$< BM_2(r)\sqrt{\left(\log\frac{2}{1-r_{m-1}}\right)} + C < BM_2(r)\sqrt{\left(\log\frac{2}{1-r}\right)} + C.$$

The desired inequality is therefore proved when $r > \frac{1}{2}$. But when $r \leqslant \frac{1}{2}$ we have, by (1.1),

$$M(r) \leqslant M_2\left((\tfrac{1}{2})^{\frac{1}{2}}\right)(1-\tfrac{1}{2})^{-\frac{1}{2}} < C \leqslant B\mathfrak{M}(r) + C.$$

This completes the proof of the lemma.

9. A well known argument* now suffices to deduce Theorem 2. Denote the a's of f_μ by $a_1(\mu)$, $a_2(\mu)$, ... Then there exists a set $(a_1, a_2, \ldots)$ and an increasing sequence $(\mu_1, \mu_2, \ldots)$ such that as $r \to \infty$

$$a_1(\mu_r) \to a_1, \quad a_2(\mu_r) \to a_2, \quad \ldots.$$

In fact let a_1 be a limit point of the set $a_1(\mu)$ $(\mu = 1, 2, \ldots)$, (μ_{1r}) an increasing sequence of μ's for which $a_1(\mu_{1r}) \to a_1$. Now let a_2 be a limit

* It is part of the argument in Montel's Theorem. See, *e.g.*, the note by P. Montel at the end of Goursat's *Cours d'Analyse*, 3.

point of $\alpha_2(\mu_{1r})$ $(r = 1, 2, \ldots)$ and let

$$\mu_{21} = \mu_{11},\ \mu_{22},\ \mu_{23},\ \ldots$$

be an increasing sequence for which $\alpha_2(\mu_{2r}) \to \alpha_2$. And so on,* the n-th sequence being

$$\mu_{n1} = \mu_{11},\ \mu_{n2} = \mu_{22},\ \ldots,\ \mu_{n,n-1} = \mu_{n-1,n-1},\ \mu_{nn},\ \mu_{n,n+1},\ \ldots.$$

Now let $\mu_r = \mu_{rr}$. Clearly $\alpha_n(\mu_r) \to \alpha_n$, for each n, as $r \to \infty$.

The set $\alpha_1, \alpha_2, \ldots$ satisfies the requirements of Theorem 2. For clearly $f_{\mu_n}(z) \to f(z)$ as $n \to \infty$, uniformly in $|z| \leqslant r < 1$. But if $n > \nu$ (and so $\mu_n > \nu$), where $r_\nu > r$, we have, by Lemma 6, $M_{\mu_n}(r) \leqslant B\mathfrak{M}(r) + C$. Hence, making $n \to \infty$,

$$M(r) \leqslant B\mathfrak{M}(r) + C.$$

* It is not worth while to avoid the Multiplicative Axiom explicitly; the problem is of a familiar type.

A GROUP OF GENERALIZED HYPERGEOMETRIC SERIES: RELATIONS BETWEEN 120 ALLIED SERIES OF THE TYPE $F\left[\begin{matrix}a, b, c \\ e, f\end{matrix}\right]$

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1. The object of this note is to provide an index to the relations between certain series all of the type defined by the equations

$$F\left[\begin{matrix}a, b, c \\ e, f\end{matrix}\right] \equiv 1+\frac{a\,.\,b\,.\,c}{1\,.\,e\,.\,f}+\frac{a(a+1)\,b(b+1)\,c(c+1)}{1\,.\,2\,.\,e(e+1)\,.\,f(f+1)}+\cdots$$

$$\equiv \frac{\Gamma(e)\,\Gamma(f)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}\sum_{r=0}^{\infty}\frac{\Gamma(a+r)\,\Gamma(b+r)\,\Gamma(c+r)}{\Gamma(1+r)\,\Gamma(e+r)\,\Gamma(f+r)}. \qquad (1)$$

An elaborate discussion of this group of series was published* in 1879 by J. Thomae, who approached the subject through the calculus of finite differences. Many of the theorems set out by Thomae have been rediscovered in recent years. References will be found in a paper† by Prof. G. H. Hardy.

Some guide to facilitate the comparison of the formulæ which connect the 120 series in question is necessary. The notation adopted by Thomae is not convenient for this purpose, and, in fact, it masks the symmetry of the group; it is not easy to discover in his memoir such formulæ as may be required in particular applications.

The notation which is now introduced will, it is hoped, provide a simple clue to the labyrinth.

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* *Journal für Math.*, 87 (1879), 26.

† A Chapter from Ramanujan's Note-Book. *Proc. Camb. Phil. Soc.*, 21 (1923), 492–503.

2. The fundamental relations are two. The first* may be written

$$F\left[\begin{matrix}a, b, c\\ e, f\end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\,F\left[\begin{matrix}e-a, f-a, s\\ s+b,\ s+c\end{matrix}\right] \qquad (2)$$

where s is equivalent to $\quad e+f-a-b-c.$

It is proved by Thomae by substitution of the expansion

$$\frac{1}{\Gamma(r+e+f-a)}\,F\left[\begin{matrix}e-a, f-a\\ r+e+f-a\end{matrix}\right]$$

for the factor

$$\frac{\Gamma(a+r)}{\Gamma(e+r)\,\Gamma(f+r)},$$

which occurs in the general term in (1), and changing the order of summation.† An alternative proof, by the integration of ordinary hypergeometric series, is given in the same paper.‡

The second fundamental relation is

$$F\left[\begin{matrix}a, b, c\\ e, f\end{matrix}\right] = \frac{\Gamma(1-a)\,\Gamma(e)\,\Gamma(f)\,\Gamma(c-b)}{\Gamma(e-b)\,\Gamma(f-b)\,\Gamma(1+b-a)\,\Gamma(c)}\,F\left[\begin{matrix}b, b-e+1, b-f+1\\ 1+b-c,\ 1+b-a\end{matrix}\right]$$

$$+\text{a similar expression obtained by interchanging } b \text{ and } c. \qquad (3)$$

Thomae's proof§ depends on the general theory of the calculus of finite differences.

A more direct method‖ is to obtain the four-term equation

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(e)\,\Gamma(f)}\,F\left[\begin{matrix}a, b, c\\ e, f\end{matrix}\right] = \sum_{a,\,b,\,c} e^{-i\pi a}\,\frac{\Gamma(a)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(e-a)\,\Gamma(f-a)}\,F\left[\begin{matrix}a, 1+a-e, 1+a-f\\ 1+a-b,\ 1+a-c\end{matrix}\right]$$

from a consideration of the integral

$$\frac{1}{2\pi i}\int e^{-i\pi z}\,\frac{\Gamma(-z)\,\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)}{\Gamma(e+z)\,\Gamma(f+z)}\,dz,$$

and, after multiplying both sides of the equation by $e^{i\pi a}$, to equate imaginary parts.

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* Thomae, *l.c.*, equation 11.
† Cf. Hardy, *l.c.*, 499.
‡ Thomae, *l.c.*, 72.
§ *L.c.*, equation 45.
‖ Cf. Hardy, *l.c.*, 501.

3. We now introduce* the following notation :—

$r_0, r_1, r_2, r_3, r_4, r_5$ are six parameters such that

$$\sum_0^5 r = 0. \tag{4}$$

With the r's are associated α's and β's such that

$$\alpha_{lmn} = \tfrac{1}{2} + r_l + r_m + r_n, \tag{4 a}$$

$$\beta_{mn} = 1 + r_m - r_n. \tag{4 b}$$

Functions Fp and Fn are defined by the equations

$$Fp(0\,;\ 4,\ 5) = \frac{1}{\Gamma\alpha_{123}\,\Gamma\beta_{40}\,\Gamma\beta_{50}}\, F\begin{bmatrix}\alpha_{145} & \alpha_{245} & \alpha_{345} \\ & \beta_{40} & \beta_{50}\end{bmatrix}, \tag{5}$$

$$Fn(0\,;\ 4,\ 5) = \frac{1}{\Gamma\alpha_{045}\,\Gamma\beta_{04}\,\Gamma\beta_{05}}\, F\begin{bmatrix}\alpha_{023} & \alpha_{013} & \alpha_{012} \\ & \beta_{04} & \beta_{05}\end{bmatrix}. \tag{6}$$

The Fn function is derived from the corresponding Fp function by changing the signs of all the r's. By permutation of the suffixes 60 Fp's and 60 Fn's can be found.

If

$$\left.\begin{aligned} \alpha_{145} = a, \quad \alpha_{245} = b, \quad \alpha_{345} = c \\ \beta_{40} = e, \quad \beta_{50} = f \\ \alpha_{123} \equiv e+f-a-b-c = s \end{aligned}\right\}, \tag{7}$$

then all the α's and β's can be expressed in terms of a, b, c, e, and f. They are set out in Table I.

TABLE I.

Expressions for α's and β's in terms of a, b, c, e, f.

$(s = e+f-a-b-c)$.

$\alpha_{012} = 1-c$	$\alpha_{345} = c$	$\beta_{01} = 2-(s+a)$	$\beta_{12} = 1+a-b$	$\beta_{21} = 1-a+b$
$\alpha_{013} = 1-b$	$\alpha_{245} = b$	$\beta_{02} = 2-(s+b)$	$\beta_{23} = 1+b-c$	$\beta_{32} = 1-b+c$
$\alpha_{023} = 1-a$	$\alpha_{145} = a$	$\beta_{03} = 2-(s+c)$	$\beta_{31} = 1+c-a$	$\beta_{13} = 1-c+a$
$\alpha_{014} = 1-f+a$	$\alpha_{235} = f-a$	$\beta_{04} = 2-e$	$\beta_{14} = 1-b-c+f$	$\beta_{41} = 1+b+c-f$
$\alpha_{015} = 1-e+a$	$\alpha_{234} = e-a$	$\beta_{05} = 2-f$	$\beta_{24} = 1-a-c+f$	$\beta_{42} = 1+a+c-f$
$\alpha_{024} = 1-f+b$	$\alpha_{135} = f-b$	$\beta_{10} = s+a$	$\beta_{34} = 1-a-b+f$	$\beta_{43} = 1+a+b-f$
$\alpha_{025} = 1-e+b$	$\alpha_{134} = e-b$	$\beta_{20} = s+b$	$\beta_{15} = 1-b-c+e$	$\beta_{51} = 1+b+c-e$
$\alpha_{034} = 1-f+c$	$\alpha_{125} = f-c$	$\beta_{30} = s+c$	$\beta_{25} = 1-a-c+e$	$\beta_{52} = 1+a+c-e$
$\alpha_{035} = 1-e+c$	$\alpha_{124} = e-c$	$\beta_{40} = e$	$\beta_{35} = 1-a-b+e$	$\beta_{53} = 1+a+b-e$
$\alpha_{045} = 1-s$	$\alpha_{123} = s$	$\beta_{50} = f$	$\beta_{45} = 1+e-f$	$\beta_{54} = 1-e+f$

* Cf. E. W. Barnes, *Quart. J. of Math.*, 41 (1910), 136–140, 139.

In Tables IIA and IIB the parameters of the Fp's and Fn's are given.

TABLE IIA.

$$Fp(u;\ v, w) \equiv \frac{1}{\Gamma\alpha_{xyz}\,\Gamma\beta_{vu}\,\Gamma_{wu}}\, F\begin{bmatrix} \alpha_{vwx}\ \alpha_{vwy}\ \alpha_{vwz} \\ \beta_{vu}\ \beta_{wu} \end{bmatrix}.$$

	v, w	α_{xyz}	Numerator Parameters			Denominator Parameters		No.
$Fp(0)$	4, 5	s	a,	b,	c	e,	f	1
	2, 3	a	s,	$e-a$,	$f-a$	$s+b$,	$s+c$	3
	1, 4	$f-a$	a,	$e-b$,	$e-c$	e,	$s+a$	6
$Fp(1)$ [$Fp(2)$ & $Fp(3)$ are of this type]	0, 2	c	$1-e+b$,	$1-f+b$,	$1-a$	$1+b-a$,	$2-s-a$	6
	0, 4	$f-a$	$1-s$,	$1-f+b$,	$1-f+c$	$1+b+c-f$,	$2-s-a$	3
	2, 3	$1-s$	$e-a$,	$f-a$,	$1-a$	$1-a+b$,	$1-a+c$	3
	2, 4	$1-e+c$	b,	$e-a$,	$1-f+b$	$1+b+c-f$,	$1+b-a$	12
	4, 5	$1-a$	$1-s$,	b,	c	$1+b+c-f$,	$1+b+c-e$	3
$Fp(4)$ [$Fp(5)$ is of this type]	0, 1	$f-a$	$1-e+a$,	$1-b$,	$1-c$	$2-e$,	$1+f-b-c$	6
	0, 5	s	$1-e+a$,	$1-e+b$,	$1-e+c$	$2-e$,	$1-e+f$	2
	1, 2	$1-e+c$	$f-c$,	$1-c$,	s	$1+f-a-c$,	$1+f-b-c$	6
	1, 5	$1-a$	$1-e+a$,	$f-c$,	$f-b$	$1+f-e$,	$1+f-b-c$	6

TABLE IIB.

$$Fn(u;\ v, w) \equiv \frac{1}{\Gamma\alpha_{uvw}\,\Gamma\beta_{uv}\,\Gamma\beta_{uw}}\, F\begin{bmatrix} \alpha_{uyz}\ \alpha_{uzx}\ \alpha_{uxy} \\ \beta_{uv}\ \beta_{uw} \end{bmatrix}.$$

	v, w	α_{uvw}	Numerator Parameters			Denominator Parameters		No.
$Fn(0)$	4, 5	$1-s$	$1-a$,	$1-b$,	$1-c$	$2-e$,	$2-f$	1
	2, 3	$1-a$	$1-s$,	$1-e+a$,	$1-f+a$	$2-s-b$,	$2-s-c$	3
	1, 4	$1-f+a$	$1-a$,	$1-e+b$,	$1-e+c$	$2-e$,	$2-s-a$	6
$Fn(1)$ [$Fn(2)$ & $Fn(3)$ are of this type]	0, 2	$1-c$	$e-b$,	$f-b$,	a	$1+a-b$,	$s+a$	6
	0, 4	$1-f+a$	s,	$f-b$,	$f-c$	$1-b-c+f$,	$s+a$	3
	2, 3	s	$1-e+a$,	$1-f+a$,	a	$1+a-b$,	$1+a-c$	3
	2, 4	$e-c$	$1-b$,	$1-e+a$,	$f-b$	$1-b-c+f$,	$1+a-b$	12
	4, 5	a	s,	$1-b$,	$1-c$	$1-b-c+f$,	$1-b-c+e$	3
$Fn(4)$ [$Fn(5)$ is of this type]	0, 1	$1-f+a$	$e-a$,	b,	c	e,	$1-f+b+c$	6
	0, 5	$1-s$	$e-a$,	$e-b$,	$e-c$	e,	$1+e-f$	2
	1, 2	$e-c$	$1-f+c$,	c,	$1-s$	$1-f+a+c$,	$1-f+b+c$	6
	1, 5	a	$e-a$,	$1-f+c$,	$1-f+b$	$1+e-f$,	$1-f+b+c$	6

4. It will be seen that in the new notation equation (2) may be written

$$Fp(0;\ 4, 5) = Fp(0;\ 2, 3). \tag{8}$$

By interchange of r_4 and r_1 we find that

$$Fp(0\,;\ 1,\ 5) = Fp(0\,;\ 2,\ 3),$$

and hence that

$$Fp(0\,;\ 4,\ 5) = Fp(0\,;\ 1,\ 5). \tag{9}$$

Accordingly all permutations of the indices 1 to 5 are legitimate. Each of the 10 expressions $Fp(0\,;\ v,\ w)$ represents the same* function, which may conveniently be denoted by $Fp(0)$. Similarly with the other Fp's and the Fn's.

The hypergeometric series by which $Fp(u\,;\ v,\ w)$ is defined at the head of Table IIA is divided by Γa_{xyz}. The condition that the series may be convergent is $a_{xyz} > 0$, and therefore a_{xyz} may be referred to as the "convergency indicator."

In this Table only representative forms are given. The permutation of the indices 1, 2, 3 corresponds with the permutation of a, b, c, whilst the permutation of indices 4, 5 corresponds with the permutation of e, f. It is stated that $Fp(2)$ and $Fp(3)$ are of the same "type" as $Fp(1)$, implying that they can be derived from it by the interchange of b or c with a. For example, by comparison with $Fp(1\,;\ 0,\ 2)$ it is seen that $Fp(2\,;\ 0,\ 1)$ must have the "convergency indicator" c and the parameters $1-e+a$, $1-f+a$, $1-b$; $1+a-b$, $2-s-b$. In a similar sense $Fp(4)$ and $Fp(5)$ are of the same type, the interchange of e and f being suggested.

In the last column of the Table is given the number of different series to be obtained by permutation of the letters a, b, and c, and of e and f.

5. Turning to the second fundamental relation (equation 3) and referring to Tables I, IIA, and IIB, we find, by use of the relation

$$\pi \operatorname{cosec} \pi(1+b-c) = \Gamma(1+b-c)\,\Gamma(c-b),$$

that

$$\frac{\sin \pi\beta_{23}}{\pi \Gamma a_{023}} Fp(0) = \frac{Fn(2)}{\Gamma a_{134}\,\Gamma a_{135}\,\Gamma a_{345}} - \frac{Fn(3)}{\Gamma a_{124}\,\Gamma a_{125}\,\Gamma a_{245}}. \tag{10}$$

* *Vide* Hardy, *l.c.*, 499.

Change of the signs of the r's leads to the relation

$$\frac{\sin \pi\beta_{32}}{\pi\Gamma a_{145}} Fn(0) = \frac{Fp(2)}{\Gamma a_{025}\Gamma a_{024}\Gamma a_{012}} - \frac{Fp(3)}{\Gamma a_{035}\Gamma a_{034}\Gamma a_{013}}. \tag{11}$$

By the combination of 3 equations like (10) it is found* that

$$\frac{\sin \pi\beta_{45}}{\Gamma a_{012}\Gamma a_{013}\Gamma a_{023}} Fp(0) + \frac{\sin \pi\beta_{50}}{\Gamma a_{124}\Gamma a_{134}\Gamma a_{234}} Fp(4) + \frac{\sin \pi\beta_{04}}{\Gamma a_{125}\Gamma a_{135}\Gamma a_{235}} Fp(5) = 0. \tag{12}$$

Similarly

$$\frac{\sin \pi\beta_{54}}{\Gamma a_{345}\Gamma a_{245}\Gamma a_{145}} Fn(0) + \frac{\sin \pi\beta_{05}}{\Gamma a_{035}\Gamma a_{025}\Gamma a_{015}} Fn(4) + \frac{\sin \pi\beta_{40}}{\Gamma a_{034}\Gamma a_{024}\Gamma a_{014}} Fn(5) = 0. \tag{13}$$

If $Fn(2)$ be eliminated from the relation of type (10) connecting $Fp(5)$, $Fn(0)$, $Fn(2)$, and the relation of type (11) connecting $Fn(2)$, $Fp(0)$, $Fp(5)$, it follows† that

$$\frac{Fp(0)}{\Gamma a_{120}\Gamma a_{130}\Gamma a_{230}\Gamma a_{240}\Gamma a_{140}\Gamma a_{340}} + \frac{\sin \pi\beta_{05} Fn(0)}{\pi\Gamma a_{123}\Gamma a_{124}\Gamma a_{134}\Gamma a_{234}} = K_0 Fp(5), \tag{14}$$

where K_0 is defined by the equation

$$\pi^3 K_0 = \sin \pi a_{145} \sin \pi a_{245} \sin \pi a_{345} - \sin \pi a_{123} \sin \pi \beta_{40} \sin \pi \beta_{50}. \tag{15}$$

The symmetry of K_0 is shown by the equivalent relation

$$4\pi^3 K_0 = \sum_{u=1}^{u=5} \cos \pi(r_0 + 2r_u) - \cos 3\pi r_0. \tag{16}$$

The analogue of 14 is

$$\frac{Fn(0)}{\Gamma a_{345}\Gamma a_{245}\Gamma a_{145}\Gamma a_{135}\Gamma a_{235}\Gamma a_{125}} + \frac{\sin \pi\beta_{50} Fp(0)}{\pi\Gamma a_{045}\Gamma a_{035}\Gamma a_{025}\Gamma a_{015}} = K_0 Fn(5). \tag{17}$$

All the three-term linear relations between the 120 hypergeometric series are typified by the equations 10–14 and 17.

6. As an example of the use of the tables let us examine the formula of Ramanujan, which is the kernel of Prof. Hardy's paper, "A Chapter

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* Cf. Thomae, *l.c.*, equation 46; Hardy, *l.c.*, equation 7.1.

† Cf. Thomae, *l.c.*, equation 53.

from Ramanujan's Note-Book", to which such frequent reference has been made. This formula* is, in our notation,

$$F\left[\begin{matrix}a, b, c\\ e, f\end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)}\,F\left[\begin{matrix}a, b, f-c\\ a+b-e+1, f\end{matrix}\right]$$

$$+\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(a+b-e)\,\Gamma(e+f-a-b-c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(f-c)\,\Gamma(e+f-a-b)}\,F\left[\begin{matrix}e-a, e-b, e+f-a-b-c\\ e-a-b+1, e+f-a-b\end{matrix}\right]. \tag{18}$$

By the use of Table II we identify the hypergeometric series as multiples of $Fp[0;\ 4,\ 5]$, $Fn[5;\ 0,\ 3]$, and $Fn[3;\ 0,\ 5]$ respectively, so that, after division throughout by $\Gamma(s)\,\Gamma(e)\,\Gamma(f)$, the formula may be rewritten (with the help of Table I) in the form,

$$Fp[0;\ 4,\ 5] = \frac{\pi}{\sin\pi\beta_{53}}\,\frac{\Gamma a_{035}}{\Gamma a_{234}\,\Gamma a_{134}\,\Gamma a_{123}}\,Fn[5;\ 0,\ 3]$$

$$+\frac{\pi}{\sin\pi\beta_{35}}\,\frac{\Gamma a_{035}}{\Gamma a_{145}\,\Gamma a_{245}\,\Gamma a_{125}}\,Fn[3;\ 0,\ 5]. \tag{19}$$

This equation is equivalent to (10) with indices 2 and 5 exchanged. It follows that Ramanujan's formula can be derived from our two fundamental relations (2) and (3).

The logical order is to write down a relation of the type (3) between $Fp[0;\ 1,\ 2]$, $Fn[5;\ 3,\ 4]$, and $Fn[3;\ 5,\ 4]$, and then to transform the terms by equation (2) as in the derivation of equations (9) and (10).

7. As an example of a limiting case we may investigate the relation between the series, obtained by equating the parameter c to unity,

$$F\left[\begin{matrix}a, b, 1\\ e, f\end{matrix}\right] \quad \text{and} \quad F\left[\begin{matrix}2-e, 2-f, 1\\ 2-a, 2-b\end{matrix}\right].$$

These are multiples of $Fp(0;\ 4,\ 5)$ and $Fn(3;\ 1,\ 2)$. Now when $c = 1$ some of the Fp's and Fn's reduce to a single term on account of the occurrence of $1-c$ in the numerator; one of these is $Fp(4;\ 0,\ 1)$.

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* Hardy, *l.c.*, equation 5 . 2.

From the appropriate relation, analogous to (11), between $Fp(0)$, $Fn(3)$, and $Fp(4)$, we deduce the formula *

$$F\begin{bmatrix} a, & b, & 1 \\ e, & f & \end{bmatrix} + \frac{(1-f)(1-e)}{(1-a)(1-b)} F\begin{bmatrix} 2-e, & 2-f, & 1 \\ 2-a, & 2-b & \end{bmatrix}$$

$$= \frac{\Gamma(e+f-a-b-1)\,\Gamma(e)\,\Gamma(f)\,\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(e-a)\,\Gamma(e-b)\,\Gamma(f-c)\,\Gamma(f-b)}, \quad (20)$$

which may be written in the symmetrical form

$$\sum_{-\infty}^{\infty} \frac{1}{\Gamma(e+n)\,\Gamma(f+n)\,\Gamma(1-a-n)\,\Gamma(1-b-n)}$$

$$= \frac{\Gamma(e+f-a-b-1)}{\Gamma(e-a)\,\Gamma(e-b)\,\Gamma(f-a)\,\Gamma(f-b)}. \quad (21)$$

This elegant generalization of Gauss's Theorem can be proved by the application of the method of residues to the integral

$$\int \frac{\Gamma(b+z)}{\Gamma(e+z)\,\Gamma(f+z)\,\Gamma(1-a-z)\sin \pi z}\, dz.$$

8. *Terminating Series.*—When the parameter c is a negative integer, say $-m$, the series $F\begin{bmatrix} a, & b, & c \\ e, & f & \end{bmatrix}$ terminates. Moreover a terminating series can be written in the reverse order. Examination shows in fact that, when $a_{345} \equiv c = -m$,

$$\Gamma a_{124}\,\Gamma a_{125}\,Fp(0) = (-)^m\,\Gamma a_{023}\,\Gamma a_{013}\,Fn(3). \quad (22)$$

This equation is a degenerate form of (10).

There are 18 terminating series altogether; three of these are forms of $Fp(0)$, viz. $Fp[0;\ 4,\ 5]$, $Fp[0;\ 3,\ 5]$, $Fp[0;\ 3,\ 4]$. These three when reversed give $Fn[3;\ 1,\ 2]$, $Fn[4;\ 1,\ 2]$, $Fn[5;\ 1,\ 2]$ respectively.

The relations between the 18 series are shown collectively † in the

* Due to Dougall: *vide* Hardy, *l.c.*, equation 8.1.

† The equation connecting $Fp[0;\ 4,\ 5]$ and $Fp[1;\ 4,\ 5]$ was established by Sheppard, *Proc. Lond. Math. Soc.* (2), 10 (1912), 476 (equation 18).

equations

$$\Gamma a_{123}\,\Gamma a_{124}\,\Gamma a_{125}\,Fp(0) = \Gamma a_{023}\,\Gamma a_{024}\,\Gamma a_{025}\,Fp(1) = \Gamma a_{013}\,\Gamma a_{014}\,\Gamma a_{015}\,Fp(2)$$
$$= (-)^m \Gamma a_{123}\,\Gamma a_{023}\,\Gamma a_{013}\,Fn(3) = (-)^m \Gamma a_{124}\,\Gamma a_{024}\,\Gamma a_{014}\,Fn(4)$$
$$= (-)^m \Gamma a_{125}\,\Gamma a_{025}\,\Gamma a_{015}\,Fn(5). \qquad (23)$$

The other series, such as the expansions of $Fn(0)$ do not show any specially simple relations.

9. *Saalschütz's Case.*—If the parameter c is a negative integer and at the same time $s = 1$, then for $Fp(1)$ a series can be selected which reduces to a single term and it follows from equation (23) that

$$F\left[\begin{matrix} a, b, c \\ e, f \end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(1-f+b)\,\Gamma(1-e+b)}{\Gamma(e-c)\,\Gamma(f-c)\,\Gamma(f-a)\,\Gamma(e-a)}. \qquad (23\,a)$$

Bearing in mind that c is a negative integer, whilst $(e-b)+(f-a) = c+1$, we can rewrite this result in the form*

$$F\left[\begin{matrix} a, b, c \\ e, f \end{matrix}\right] = \frac{\Gamma(e)\,\Gamma(1-f+a)\,\Gamma(1-f+b)\,\Gamma(1-f+c)}{\Gamma(e-a)\,\Gamma(e-b)\,\Gamma(e-c)\,\Gamma(1-f)}. \qquad (23\,b)$$

10. *Dixon's Case.*—The special case in which

$$a+e = b+f = c+1 \qquad (24)$$

so that

$$r_0+r_3 = r_1+r_4 = r_2+r_5 = 0 \qquad (25)$$

is interesting. In this case the hypergeometric series can be summed.

The definitions show that under these conditions

$$\left.\begin{aligned} Fp(0) = Fn(3), \quad Fp(1) = Fn(4), \quad Fp(2) = Fn(5) \\ Fn(0) = Fp(3), \quad Fn(1) = Fp(4), \quad Fn(2) = Fp(5) \end{aligned}\right\}. \qquad (26)$$

On account of these equalities relations such as (10), which in general connect three functions, reduce in this special case to relations between

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* Hardy (*l.c.*, 497) gives references to three papers by L. Saalschütz. A slip of Hardy's is to be noted; on the right-hand side of 5.1, $\Gamma(y+z+m+1)$, etc., should be written in place of $\Gamma(y+z+m)$, etc.

two functions. Thus (10) becomes after some rather heavy algebra*

$$2^{-2r_0}\,\Gamma(\tfrac{1}{2}-r_0)\,\Gamma(1-\tfrac{1}{2}a_{024})\,\Gamma(1-\tfrac{1}{2}a_{012})\,Fp(0)$$
$$=2^{-2r_5}\,\Gamma(\tfrac{1}{2}-r_5)\,\Gamma(1-\tfrac{1}{2}a_{345})\,\Gamma(1-\tfrac{1}{2}a_{135})\,Fp(5) \qquad (27)$$

The occurrence of halves in this equation is perhaps surprising. It is a result of using the addition formula for sines. It follows from (27) that

$$2^{-2r_0}\,\Gamma(\tfrac{1}{2}-r_0)\,\Gamma(1-\tfrac{1}{2}a_{012})\,\Gamma(1-\tfrac{1}{2}a_{024})\,\Gamma(1-\tfrac{1}{2}a_{015})\,\Gamma(1-\tfrac{1}{2}a_{045})\,Fp(0)$$

is invariant for changes of suffixes such that the symmetry with respect to 0 and 3, 1 and 4, 2 and 5, is preserved.

The method of this paper does not show that the invariant is an absolute constant. Comparison with a formula discovered† by A. C. Dixon shows, however, that it is and that the value is π, *i.e.* that, when the conditions (25) hold, then

$$Fp(0)=\frac{2^{2r_0}\pi}{\Gamma(\tfrac{1}{2}-r_0)\,\Gamma(1-\tfrac{1}{2}a_{012})\,\Gamma(1-\tfrac{1}{2}a_{024})\,\Gamma(1-\tfrac{1}{2}a_{015})\,\Gamma(1-\tfrac{1}{2}a_{045})}. \qquad (28)$$

Dixon's formula may be written

$$F\left[\begin{matrix}a, & b, & c\\ 1-a+c, & 1-b+c\end{matrix}\right]$$
$$=\frac{\Gamma\frac{1}{2}}{2^c\,\Gamma(\frac{1}{2}+\frac{1}{2}c)}\,\frac{\Gamma(1-a+c)}{\Gamma(1-a+\frac{1}{2}c)}\,\frac{\Gamma(1-b+c)}{\Gamma(1-b+\frac{1}{2}c)}\,\frac{\Gamma(1-a-b+\frac{1}{2}c)}{\Gamma(1-a-b+c)}. \qquad (29)$$

11. *Watson's Case.*—Interchange of parameters enables us to evaluate also the series determined by $Fp[0;\ 4,\ 5]$, when subject to the conditions

$$r_0+r_5=r_1+r_2=r_3+r_4=0, \qquad (30)$$

namely‡

$$F\left[\begin{matrix}a, & b, & c\\ \frac{1}{2}(a+b+1), & 2c\end{matrix}\right]=\frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b)\,\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\,\Gamma(\frac{1}{2}-\frac{1}{2}b+c)} \qquad (31)$$

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* See page 114.

† *Proc. Lond. Math. Soc.* (1), 35 (Jan. 1903), 284. A new proof of Dixon's formula has been given by Watson. *Proc. London Math. Soc.* (2) 22 (1924), xxxii.

‡ Formula 31 can be derived from 29 directly by using equation (9) and making the obvious substitutions. The special case of (31) in which a is a negative integer was discovered by Watson. His demonstration is published in *Records, etc.* for Feb. 14, 1924. The formula was quoted by Hardy (*l.c.*, equation 8.2) but with two errata. The factor $(-2)^n$ should be deleted and $\Gamma(2\lambda)$ substituted for $\Gamma(2\mu)$.

12. *Another transformation of Dixon's Formula.*—There is a third case of the same type. It is that in which

$$r_0+r_3=r_1+r_2=r_4+r_5=0 \tag{32}$$

so that $a+b=1$

and $e+f=2c+1$ } (33)

When these conditions hold, then the analogue of (28) reduces to

$$F\left[\begin{matrix}a, b, c\\ e, f\end{matrix}\right]=\frac{\pi\Gamma(e)\,\Gamma(f)}{2^{2c-1}\,\Gamma(\frac{1}{2}a+\frac{1}{2}e)\,\Gamma(\frac{1}{2}a+\frac{1}{2}f)\,\Gamma(\frac{1}{2}b+\frac{1}{2}e)\,\Gamma(\frac{1}{2}b+\frac{1}{2}f)} \tag{34}$$

It will be seen that there are no other new results to be obtained by the application of the standard transformations to Dixon's formula.

Postscript (added May 21st, 1924).—I take advantage of a spare half-page to give the proof of (27).

Equation (10) may be written

$$\frac{Fp(5)}{Fp(0)}=\Gamma a_{134}\,\Gamma a_{135}\,\Gamma a_{345}\left[\frac{\sin\pi\beta_{23}}{\pi\Gamma a_{023}}+\frac{1}{\Gamma a_{124}\,\Gamma a_{125}\,\Gamma a_{245}}\right],$$

so that when the conditions (26) are satisfied

$$\frac{Fp(5)}{Fp(0)}=\frac{\Gamma(\frac{1}{2}-r_0)}{\Gamma(\frac{1}{2}-r_5)}\Gamma a_{135}\,\Gamma a_{345}\left[\frac{\sin\pi\beta_{23}}{\pi}+\frac{1}{\Gamma(\frac{1}{2}+r_1)\,\Gamma(\frac{1}{2}-r_1)}\right].$$

The expression in square brackets is

$$\pi^{-1}[\sin\pi(r_3-r_2)+\sin\pi(\tfrac{1}{2}+r_1)]$$

or

$$(2/\pi)\,.\,\sin(\tfrac{1}{2}\pi a_{135})\,\sin(\tfrac{1}{2}\pi a_{345}).$$

Hence

$$\frac{Fp(5)}{Fp(0)}=\frac{2\,.\,\Gamma(\frac{1}{2}-r_0)\,.\,2^{a_{135}-1}\,\Gamma(\frac{1}{2}a_{135})\,\Gamma(\frac{1}{2}+\frac{1}{2}a_{185})\,2^{a_{345}-1}\,\Gamma(\frac{1}{2}a_{345})\,\Gamma(\frac{1}{2}+\frac{1}{2}a_{345})}{\Gamma(\frac{1}{2}-r_5)\,\Gamma(\frac{1}{2}a_{135})\,\Gamma(1-\frac{1}{2}a_{135})\,\Gamma(\frac{1}{2}a_{345})\,\Gamma(1-\frac{1}{2}a_{345})},$$

and (27) follows at once.

SOME COMPLEXES OF CONICS IN SPACE OF THREE DIMENSIONS.

By C. G. F. JAMES.

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In a previous paper* I have given a brief sketch of the behaviour of complexes, or systems ∞^3, of curves in general, and passed on to consider a special group of complexes of space cubics. The possibility of the existence of *singular points*, or points through which ∞^2 curves pass, was pointed out. Excluding the case of a system of curves on a surface, there will be at most a curve of such points. Similarly there will be at most a congruence of *singular chords*, chord to ∞^2 curves of the system, if we exclude the case of a system of plane curves, of which ∞^2 lie in each plane of a certain developable. The case of a congruence can only occur if the curves lie by ∞^2's on the surfaces of a system ∞^1 of ruled surfaces, but it cannot occur for plane curves, since we cannot construct a scroll each generator of which meets a plane curve twice. So also in this case the existence of a scroll of singular chords implies that the planes of the curves shall touch a fixed surface. We divide singular points and chords into two types, according as the corresponding system of curves does not lie, or lies, on a surface, the former being the general case. ∞^1 curves pass through an arbitrary point, and ∞^1 have an arbitrary line as chord.

* "On Complexes of Cubics in Ordinary Space", *Proc. Camb. Phil. Soc.*, 21 (1923), 610–624.

In this paper we consider a group of complexes of conics, arising in the first place from the representation of a conic by matrix equations,*

$$\begin{Vmatrix} a_x^2 & c_x & a \\ b_x^2 & d_x & b \end{Vmatrix} = 0, \quad \text{or} \quad \begin{Vmatrix} a_x^2 & b_x & c_x \\ A_x & b & c \end{Vmatrix} = 0. \tag{1}$$

In these equations, as everywhere in this paper, a_x^n denotes a homogeneous polynomial in $x_1 \dots x_4$ of order n; a_x, etc., denote linear forms, and a, etc., constants. We shall employ the usual notation for polar forms, etc. To this is prefixed a consideration of these systems in general.

1. The characteristics which I have adopted† for the general case do not agree with the established convention‡ for the case of conic systems, and these latter I use throughout the discussion. The *order* of the complex is the number of conics in an assigned plane, and the *class* is defined as the class of the cone formed by the planes of those conics which pass through a given point, *or* the number of conics through a point of a line meeting it elsewhere. This number is therefore also the difference between the order of the surface of conics through a point P, and the order of multiplicity of P on it; *or* the order of multiplicity of a line l on the surface of conics, having it as chord, in the former case provided the conics form on that surface a linear system.

The complex will contain ∞^2 degenerate conics, the separate lines describing a congruence; and a finite number (in general) of conics formed by a line counted twice, or double lines as we shall call them for brevity, provided the conics are given as loci of points.

A peculiarity of complexes of plane curves, including conics, consists in the possible existence of *singular planes*, or planes containing ∞^1 or ∞^2 curves of the system. These may be finite or infinite in number.

2. *Historical Note.*—There is a fairly well developed theory of complexes of circles,§ which can be projectively generalised to give complexes with a bi-directrix conic, or one met twice by every curve of the system.

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* First given by Godeaux, *Bull. Acad. R. de Belge* (1908), 896.

† *Loc. cit.*, § 1.

‡ Montesano, "Una estensione del problema", etc., *Annali di Mat.* (3), 1 (1898), 321.

§ Coolidge, *A Treatise on the Circle and the Sphere*, Oxford (1916), Ch. 13, 14.

The characteristics used are, however, different, and naturally the discussion has diverged somewhat from the more general theory. A circle complex of order 4 (in our sense) has been considered by Bordiga,* but its discussion is attached to the projective rather than to the metrical theory.

The complex of the first order and class has been considered by various authors, and finally exhaustively by Montesano,† who shows that there is one general case from which all can be derived. The remaining cases yet considered are all linear, and have been mainly deduced by synthetic methods. A linear complex of lines enables us to assign uniquely and rationally a point in every plane, while a bilinear congruence enables us similarly to assign a line. Thus Montesano‡ has considered the complex (1, 3) of conics, for which four of the points on each conic are thus assigned by as many linear complexes, the fifth condition being that these four points shall have on their conic a constant cast. Godeaux§ has considered a complex (1, 4) defined in a similar manner. We may similarly define many others, but their discussion is not generally very manageable. Thus the surface of conics through a point is usually of too high an order to be of tangible interest. As an illustration we might mention the complex (1, 6), for which the conic in each plane is determined by its five poles for as many linear complexes.

3. The object of this paper is, in the main, the consideration of *certain complexes which contain an infinite number of congruences of one or more of the Stuyvaert types*,‖ namely, such congruences as are given by matrices as in (1), in which the elements contain, at most linearly, homogeneous parameters a_1, a_2, a_3, the congruences being themselves linear, in the sense that one curve passes through an arbitrary point. There are six such types. They were first considered from this point of view by Godeaux¶ (in part only); and, later, completely by the author.** They are included

* "Di un complesso di cerchi del quatro ordine", *Atti Ist. Veneto*, 63 (1903-4), 733.

† "I complessi lineari di coniche . . .", *Atti Accad. Napoli* (2), 15 (1914).

‡ "Una estensione . . .", 314.

§ "Sur un complexe . . .", *Bull. Acad. R. de Belge* (1910), 725.

‖ See "Analytical Representation of Congruences of Conics", *Proc. Camb. Phil. Soc.*, 21 (1922), 150; for the bibliography of these congruences. This paper will be referred to as A.R.

¶ "Sur quelques congruences . . .", *Archiv der Math. u. Phys.* (3), 16 (1910), 101.

** A.R.

in the general classification previously given by Montesano.* The cases we shall choose will be derived from the first two types only. The first type is that in which the parameters occur in one row only; the second that in which they enter into two columns only. The complexes in question are derived by inserting a fourth parameter a_4 in those elements in which a_1, a_2, a_3 occur, and also linearly. This process gives five complexes, each of which contains ∞^3 congruences of the corresponding type, given by assigning a linear relation $\Sigma \mu_i a_i = 0$. The first of these is the complex (1, 1); three of the others will be considered in turn, and will be denoted by G_1, G_2, G_3 respectively, being of types (4, 1), (0, 0), (4, 2), the former numbers denoting as usual the orders, and the latter the classes of the systems. We shall use each of these to serve as types for complexes of a similar nature. The fifth case is included in the generalised type for G_2, and consists of all conics meeting a fixed line twice and lying on the quadrics of a net. No further reference will be made to it. The discussion is divided into two parts. In the first part (§ 4) is considered the representation on a space $S_{(a)}$ of any system whose equations involve rationally four parameters (a). It will be seen that a statement of this representation enables a large proportion of the detailed discussion of any special case to be suppressed. We shall then (in § 5 *et seq.*) proceed to the consideration of the examples which have been chosen, completing those portions for which the parametric representation is not suitable, and examining various subsidiary questions which arise.

4. This representation is obtained by regarding the equations of our complex as equations in (a), the (x) being regarded as parameters and the (a) as coordinates in a space $S_{(a)}$. The relation of the two spaces is symmetrical. There exist, in general, singular points in $S_{(a)}$ to which correspond ∞^1 conics lying on a surface. Such surfaces we shall call *special*. They are, in general, infinite in number, the corresponding singular points lying on a curve. So also we must allow, in the general case, for the existence of points (finite in number) to which correspond ∞^2 curves. Conversely there may exist curves of the complex represented by an infinite number of sets of parameters. It is assumed that the representation is birational. This is not necessarily the case, but it will be true in all cases to which we apply it.

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* "Su i varii tipi di congruenze lineari di coniche nello spazio", *Rend. Accad. Napoli*, 34 (1895), 93–110, 155–181.

We draw up a table giving in the right-hand column the constructs whose points represent those conics satisfying the conditions on the left.

To pass through a point,	Curve of order n_1.
Its plane to pass through a point,	Surface of order n_2.
To have a line as chord,	Curve of order n_3.
To pass through a point of a singular curve $e_{(i)}$,	Surface of order n_i'.
To pass through an isolated singular point Q_i,	Surface of order n_i''.
To meet an arbitrary line,	Surface of order n_4, having a nodal n_3-tic curve.
To degenerate,	Surface of order d.
To touch a plane,	Surface of order t.

In this table $e_{(i)}$ stands for one of the singular curves $e_{(1)}$, $e_{(2)}$, etc., and Q_i for one of the singular points Q_1, Q_2, etc. Expressions such as $(n_i)^2$, $(n_i)(n_j)$, will denote the order of the intersection of the constructs for the separate factors residual to the singular system. Thus

$$n_3 = (n_2)^2. \tag{I}$$

In most cases $n_i' = n_i'' = n_2$, for every i, but this is not necessarily the case.

The congruence of conics given by a general linear relation $\mu_\alpha = 0$ is a congruence* $[n_1, (n_2)^2]$. Thus for each of G_1, G_2, G_3 we have n, equal to unity. The other properties of these congruences can be similarly expressed. The order of the complex is $(n_2)^3$, and the class $(n_1)(n_2)$. This second result is given by considering the conics meeting one line twice, and a second line, meeting the first, and simplifying by using (II) (below). The incidence formulæ of Schubert† give

$$(n_2)(n_4) = 2(n_2)^2 + n, \tag{II}$$

$$m = (n_4)^3 - 8(n_2)^3 - 3(n_1)(n_4), \tag{III}$$

m being the class of the congruence of lines of degenerate conics. Its order, n, is $(n_1)(d)$. (II) may be obtained directly by considering the

* *Kalkül der Abzählenden Geometrie*, 92, 95.

† The first number in the bracket denotes the order, the second the class of the congruence, or the number of its curves having an arbitrary line as chord.

conics whose planes pass through a point, while they meet a line through the point; and (III) by considering the conics meeting three coplanar lines.

From the same source of reference

$$2(t) = (n_4) + (d) = 4(n_4 - n_2) \qquad \text{(IV)}$$

$$2(n_1)(n_2) = (n_2)^2(t) - 2(n_2)^3, \qquad \text{(V)}$$

the two sides of the second equation giving *the number of conics touching a line.* This and the second equality in (IV) are restricted to systems containing at most ∞^1 double lines. In fact, from (II) and (IV) is derivable the equation

$$2(n_1) = (n_2)(t) - 2(n_2)^2, \qquad \text{(V}a\text{)}$$

which includes V. *The number of conics touching a general curve* of order m and rank r is *

$$2m(n_1)(n_2) + \tfrac{1}{2}r(n_1)(t).$$

The equations I, II, IV, and Va are to be completed by formulæ of the same form between the multiplicities of the singular points in $S_{(\alpha)}$ on the constructs indicated by the separate terms, the intersections of the singular curves with the curves given by the separate terms in I, II, and Va, and their multiplicities on the surfaces of IV. Since the singular system can be arbitrarily assigned, subject to such a system being possible, and also the nature of the surfaces (n_2), and that of the curves (n_1), subject to their being complete intersections of the (n_2), residual to the singular system, it can be seen that there are no independent relations between $n_1, \ldots, t$ other than those given.

Further (Schubert, *loc. cit.* 96), *the number of conics touching a given plane at a given point* is $\frac{1}{2}(n_1)(t)$. If q is the order of the system of conics through a point P on the surface filled, then $\frac{1}{2}(n_1)(t)/q$ gives the multiplicity of P on it. In most cases, including all those of this paper, q is unity.† The conics touching the lines of a pencil are represented by points of a curve of order $(n_2)(t)$, those touching a plane at points of a line by points of a curve of order $\frac{1}{2}(n_4)(t)$ (*loc. cit.*), on which the points representing the conics in the plane are nodes, in general. Considering its intersection with the surface (n_2) for a point in the plane we deduce that *the order of the surface locus of points of contact from P to the conics is equal to*

$$\tfrac{1}{2}(n_2)(n_4)(t) - 2(n_2)^3.$$

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* Severi, "Sopra le coniche che toccano e seccano una o più curve gobbe", *Atti Accad. Torino*, 36 (1901), 74. The formula in question is on 75.

† Cf. James, "On complexes . . .", §§ 1–3.

Similarly from its intersection with the surface (d) it is found that *the order of the vertex locus of degenerate conics is*

$$\tfrac{1}{4}\{(n_4)(d)(t)-2\tau m\},$$

where τ is the number of times a line of the congruence of lines of degenerate conics belongs to such a conic, and is usually equal to unity.

The class of the envelope of their planes is $w = (n_2)^2(d)$.

Finally, when q is unity, the genus p of a plane section of T_y, the surface generated by conics of the complex through (y), is given by*

$$(n_1)(t) = 2(p+1),$$

provided T_y is rational, and similar statements hold for every surface of conics of the complex not containing ∞^1 degenerate conics.

We now have sufficient data to express all the simpler constructs associated with a given complex in terms of its representative system. It would scarcely be profitable to do this in detail, but we shall give several examples in the course of the second part. We shall not in that part make any detailed reference to the systems of conics in our complexes, which satisfy certain tangential conditions (*e.g.* to touch a plane), as the only practicable way is to determine the surfaces (t) from IV, or directly, and to apply the theory of this paragraph. As a last example we can show that, *for the system of degenerate conics meeting a line,*

the order of the scroll of lines meeting g is $(n_1)(d)+m$;

the order of the scroll of lines not meeting g is $(d)(n_4)^2-(n_1)(d)-m$;

the order of the vertex locus is $\tau m+2v$; and

the class of the envelope of their planes $\tau n+2w$.

The last two statements are given by applying formulæ of Schubert (*loc. cit.*, 29) to the construct formed by a line (g) of a degenerate conic, its vertex (p), and plane (e), the symbols in brackets being the symbols of the formulæ in question.

5. The complex G_1 is given by a matrix equation

$$\begin{Vmatrix} a_x^2 & \sum_1^4 a_i a_{ix} & \sum_1^4 a_i a_i \\ b_x^2 & \sum_1^4 a_i b_{ix} & \sum_1^4 a_i b_i \end{Vmatrix} = 0,$$

* Severi-Löffler, *Vorlesungen über Algebraischer Geometrie* (1921), 168.

or, as we may take it without loss of generality,

$$(G_1) \qquad \begin{Vmatrix} a_x^2 & \sum_1^4 a_i a_{ix} & a_1 \\ b_x^2 & \sum_1^4 a_i b_{ix} & a_2 \end{Vmatrix} = 0. \qquad (2)$$

The planes of the conics fill the entire plane space, while the conics lie by ∞^2's on the quadrics of a pencil

$$a_x^2 + \lambda b_x^2 = 0,$$

where $a_1 = -\lambda a_2$, as throughout the discussion of this system. The complex contains ∞^3 congruences of type Γ_4,* obtained by assigning a linear relation $\Sigma \mu_i a_i = 0$. Thus all the conics of G_1 meet c_4, the base of the pencil, four times. The complex contains an unlimited sequence of linear congruences of the general type considered in association with Γ_4, namely, one such for every rational scroll through the curve v_4 (§6). An interesting case is that of the ∞^8 congruences of class 8, in which the scroll is generated by lines meeting v_4 and two fixed lines. The singular curve of order 17 and genus 8 is the residual intersection of the surfaces of conics having the lines respectively as chords (§12).

Throughout the rest of the paper c_n^p will denote a curve of order n and genus p. The symbol $\equiv$ will be used to denote incidence. Thus $R \equiv P^r(c_m)^s$ denotes that a surface R passes r times through a point P, and s times through the curve c_m. Similarly $c_n \equiv P^r(c_m)^s$ denotes that c_n has P as an r-fold point, and meets c_m s times outside P, etc.

For a set of values of the parameters for which $a_1 = a_2 = 0$ we have a special (cubic) surface (§4). There is a pencil of such surfaces, namely,

$$a_3(a_x^2 b_{3x} - b_x^2 a_{3x}) + a_4(a_x^2 b_{4x} - b_x^2 a_{4x}) = 0, \qquad (3)$$

of base c_4 and the c_5^2, meeting c_4 eight times

$$\begin{Vmatrix} a_x^2 & a_{3x} & a_{4x} \\ b_x^2 & b_{3x} & b_{4x} \end{Vmatrix} = 0. \qquad (4)$$

The system of singular c_7^3 of the congruences Γ_4 is given by taking the intersection of the pencil (3)† and the cubic system of envelopes of the plane

$$a_2\left(\sum_1^4 a_i a_{ix}\right) - a_1\left(\sum_1^4 a_i b_{ix}\right) = 0,$$

* A.R., § 26, 164.
† A.R., § 29.

subject to $\Sigma \mu_i a_i = 0$, and residual to their double directrices

$$\begin{Vmatrix} a_{1x} & b_{1x} & \mu_1 & 1 & 0 \\ a_{2x} & b_{2x} & \mu_2 & 0 & 1 \\ a_{3x} & b_{3x} & \mu_3 & 0 & 0 \\ a_{4x} & b_{4x} & \mu_4 & 0 & 0 \end{Vmatrix} = 0,$$

or
$$\mu_3 a_{4x} - \mu_4 a_{3x} = \mu_3 b_{4x} - \mu_4 b_{3x} = 0. \tag{5}$$

Thus *the singular septimics describe a complex, whose curves lie by* ∞^2*'s on the cubics in* (3), *etc.*

6. Let us consider next *the planes of the conics lying on a given quadric* (λ) *of the pencil.* They form a star, of centre

$$a_{3x} + \lambda b_{3x} = a_{4x} + \lambda b_{4x} = a_{2x} - \lambda(a_{1x} - b_{2x}) - \lambda^2 b_{1x} = 0, \tag{6}$$

describing a rational quartic v_4 of the second species, having the lines $a_{3x} = b_{3x} = 0$, and $a_{4x} = b_{4x} = 0$ as trisecants. Thus its trisecant regulus is conjugate to that of the line (5). We shall denote this vertex by P_λ, its coordinates x_λ being given by quartic polynomials in λ. We shall denote the result of substituting (x_λ) in a form a_x^n by $a_{(\lambda)}^n$.

Each plane of space contains four conics of G_1, one corresponding to each intersection with v_4. Hence G_1 *is of the fourth order.* In ∞^2 planes, those through a tangent to v_4, two of these conics coincide; in ∞^1, its osculating planes, three coincide; and all four coincide in the four stationary planes of v_4. There are no singular planes.

7. *The conics which pass through a point* (y) fill the quadric of the pencil through (y) and pass through a second fixed point on it, residual section by the line $P_\lambda(y)$. Hence *the complex is of the first class.* The tangent plane at (y) to this quadric is

$$(a_x a_y)\, b_y^2 - (b_x b_y)\, a_y^2 = 0, \tag{8}$$

and this in (y) represents the locus of points of contact of tangents from (x) to conics of G_1, a cubic surface $R^3 \equiv c_4, (x), g$, g being the polar line of (x) for the pencil.

Each point of c_4 is singular for the complex, and of the first type. There are outside c_4 a finite number of singular points, which we shall call the points Q. These are the nine points of v_4, which lie on the corresponding quadrics. They are of the second type, since all the conics through such a point lie on the quadric through it.

8. *The degenerate conics in the complex* are given by planes which touch a quadric of the pencil, and pass through the corresponding point of v_4. On each such quadric are ∞^1 degenerate conics, formed by an arbitrary line of either regulus, and one of the other, uniquely determined by the first. *The vertex locus* is the section of the quadric by the polar plane of P_λ.

$$a_x^2+\lambda b_x^2 = (a_x a_{(\lambda)})+\lambda(b_x b_{(\lambda)}) = 0, \tag{9}$$

and thus for the whole system a surface $R^{11} \equiv (c_4)^5$, $9Qi$, containing the lines on the tangent planes at these nine points Q_i to the quadrics through them, and therefore touching these planes there. The congruence formed by the lines of degenerate conics is the congruence (2, 6) of chords of c_4.

The tangent planes to the four cones of the pencil of quadrics from the corresponding points on v_4 each contain a double line of the complex, so that there are eight such. These lie on R^{11}, the tangent plane along each being fixed.

The planes of the degenerate conics envelope a surface Σ^{11} of the eleventh class. This is seen by a correspondence argument on v_4. It has nine pairs of pencils, axes the generators through the singular points Q of the corresponding quadrics; and four single pencils, whose axes join the vertices of the cones of the pencil to the points on V_4 corresponding to these cones. It has a double developable, envelope of those planes which touch two quadrics of the pencil and pass through the corresponding points on v_4. To find its class, let us consider those of its planes which pass through a point P on v_4. We may establish a correspondence between points L and M on v_4, such that a plane through LP touches the quadrics corresponding to M and P. To a point L corresponds four points M, and to a point M twelve points L. The 16 coincidences R give the planes in question. Now v_4 does not lie on the developable in question, and hence *the double developable is of the sixteenth class.* The correspondence (P, R) is involutory, and the 32 coincidences give the stationary planes of the developable. This number may be verified by determining those planes, tangent to a quadric (λ) at points of c_4, which contain the tangent at the corresponding point of v_4 (directly, or using the representation of c_4 in elliptic functions). The developable has a number of tritangent planes, also tritangent planes for Σ^{11}, given by those planes which touch three quadrics of the pencil, and pass through the corresponding points of v_4. If we express that the plane $u_x = 0$ passes through (x_λ) and touches the corresponding quadric we obtain the equations

$$(a_0 a_1 \dots a_4 \lambda 1)^4 = 0; \quad (b_0 b_1 \dots b_3 \lambda 1)^3 = 0,$$

the a being linear and the b quadratic in (u). The planes (u) for which these equations have three common roots are given by

$$\begin{vmatrix} a_0 & 4a_1 & 6a_2 & 4a_3 & a_4 \\ b_0 & 3b_1 & 3b_2 & b_3 & \\ & b_0 & 3b_1 & 3b_2 & b_3 \end{vmatrix} = 0,$$

There are no redundancies in this representation, but I have not determined the number of planes represented.

9. The points of R^{11} are such that at each of them ∞^1 conics touch the same line. The lines in question generate a congruence (10, 8), for which each point of v_4 is vertex of a quadric cone of lines. The points Q are such that their cones break up into pencils of plane-double rays, or lines counting twice amongst the eight lines in a plane, but not in general more than once among the 10 lines through a point.*

10. We have seen that all conics of G_1 through a point P pass of themselves through a second point P'. We have thereby obtained an *involutary birational transformation of space* T, given analytically by

$$(T) \qquad x_i' = x_i\{a_{(\lambda)}^2 b_x^2 - a_x^2 b_{(\lambda)}^2\} - 2\{(a_x a_{(\lambda)})\, b_x^2 - a_x^2 (b_x b_{(\lambda)})\}\, x_{\lambda i}, \qquad (10)$$

where $\lambda = -a_x^2/b_x^2$, and $x_{\lambda i}$ denotes the ith coordinate of (x_λ). It is of order 19, having R^{11} as surface locus of united points. To a plane corresponds a 19-tic surface $T^{19} \equiv 9Q_i^2, (c_4)^9$. The fixed base of the system of these surfaces consists of the points Q_i (doubly), the curve c^4 (ninefold), and the residual section of the surfaces

$$a_{(\lambda)}^2 b_x^2 - a_x^2 b_{(\lambda)}^2 = (a_x a_{(\lambda)})\, b_x^2 - a_x^2 (b_x b_{(\lambda)}) = 0,$$

consisting, in fact, of the 18 lines through the points Q_i lying on the corresponding quadrics. The intersection of two surfaces T^{19} gives the curve corresponding to the line of intersection, l, of the corresponding planes. This is a curve $c_{19,\,l}^0 \equiv Q_i^2, (c_4)^{36}, l^{11}$. It is given analytically by writing $x = \alpha + \mu\beta$ in (10), (α) and (β) being points of l.

The correspondents of the various elements of the fundamental system are as follows. To a point Q corresponds all points of the quadric of the pencil through it, to a point of one of the 18 lines all points of the same line, and to a point P, or (z), of c_4 the locus of its projection on to the

* Zeuthen, *Lehrbüch der abzählenden Methoden der Geometrie*, Leipzig (1914), 70.

quadrics of the pencil, from the corresponding points of v_4, a rational curve $c^0_{9,z}$, $\equiv 9Q_i$, P^5, $(c_4)^{12}$. It is given by

$$x_i = z_i(a^2_{(\lambda)}+\lambda b^2_{(\lambda)}) - 2x_{\lambda i}\{a_z(a_{(\lambda)})+\lambda(b_z b_{(\lambda)})\}. \qquad (11)$$

Its 12 intersections with c_4 outside P are the points where c_4 meets the cone $\{P(v_4)\}$, and the multiplicity of P is given by a correspondence argument. Its three chords through P are the chords of v_4 through P. Finally as P describes c_4 these curves generate a surface of order 36, $R^{36} \equiv (c_4)^{17}$, $9Q_i^4$.

The quadrics of the pencil and the conics of the complex are self-correspondent in T.

11. We consider next *the congruence of conics through a point P of* c_4. It is linear, and of the fourth class. Its conics lie in the planes of a series of pencils whose axes describe the cone $\{P(v_4)\}$, its generators being in projective correspondence with the pencil of quadrics. It is thus of known general type,* but is somewhat specialized by the existence of the fundamental point P. Its conics satisfy the incidence conditions expressed by $P(c_4)^3\ c^0_{9,z}$.

11*a*. Now four conics of G_1 pass through three points of c_4, while on each quadric of the pencil a single curve of G_1 passes through two prescribed points. Hence c_4 is four-fold on the surface of conics through two points (z) (w) of c_4. Considering its section by any quadric of the pencil, we see that it is of the ninth order,

$$R^9_{z,w} \equiv (c_4)^4,\ c_{9,z},\ c_{9,w},\ 9Q_i,\ (z)^5,\ (w)^5,\ g;$$

g being the line joining (z), (w). The surface contains the degenerate conic containing g and 10 other degenerate conics. Its equation is

$$|\, a_{2\xi}+\lambda(b_{2\xi}-a_{1\xi})-\lambda^2 b_{1\xi} \quad a_{3\xi}+\lambda b_{3\xi} \quad a_{4\xi}+\lambda b_{4\xi}\,| = 0, \qquad (12)$$

where ξ stands for x, z, w in the three rows, and $\lambda = -a_x^2 | b_x^2$. Such is therefore the surface of conics of the congruence through a second point of c_4. The conics through a point of $c^0_{9,z}$ fill the corresponding quadric. We shall see that the surface of those of its conics which meet a line is of order 20.

11*b*. *Degenerate conics of the congruence.* As for the general type of congruence mentioned these are given by the tangent planes to the quadric (λ) through the corresponding generator g of the cone of axes.

* Montesano, "Su i varii tipi . . .", 98; or, A.R., § 37.

The vertex locus is given by

$$a_x^2+\lambda b_x^2 = (a_x a_{(\lambda)})+\lambda(b_x b_{(\lambda)}) = (a_x a_z)+\lambda b_x b_{(z)} = 0, \qquad (13)$$

and is therefore* a curve $c_{13}^6 \equiv P^5(c_4)^{19}$, the tangents at P being conjugate to those of $c_{9,z}^0$ in the planes containing them with respect to the quadrics of the pencil touching at P the planes in question. The curve does not meet $c_{9,z}^0$ except at P, and does not pass through the points Q. It is hyperelliptic, and the lines joining the pairs of the fundamental involution generate a sextic scroll, having as five-fold line the tangent to c_4 at P, and given by elimination from the last two equations in (13). The surface formed by the degenerate conics breaks up into the cubic cone $P(c_4)$, and a scroll of order $(2r+1)$, r being the multiplicity of c_4 on it. To find this number we take a second point R on c_4. The cone $R(c_4)$ has 15 generators meeting c_{13}^6 outside c_4, and these only give generators of the scroll through R. Hence $r = 15$, and we have a scroll $R^{31} \equiv (c_4)^{15}, c_{13}^6, (c_{9,z}^0)^2$. It is of genus six, thus supplying a verification of that of c_{13}, the two being necessarily equal.

12. We pass on to the consideration of *the* ∞^1 *conics having a line* l *as chord.* These conics cut on the line an ordinary involution, and hence two conics touch the line. The surface filled is of the ninth order: $c_l^9 \equiv l,\ c_{19,l}^0,\ (c_4)^4,\ 9Q_i$. It contains 11 degenerate conics, given by planes through l and P_λ, and touching the quadric λ. There are six conic tropes (§ 6). The equation of c_l^9 is given by (12), with (z) and (w) any two points on the line. For each point of intersection of l and v_4 the quadric of the pencil, which corresponds to this point, breaks away from the surface.

13. We may now consider *the congruence of conics of* G_1 *whose planes pass through a point* (y); and, associated therewith, the congruence of polar lines of (y) for these conics. The former includes as a special case the congruence of §11, and, like it, is of the fourth class and the same general type. Analytically it is given by the relations

$$\left.\begin{aligned} a_x^2+\lambda b_x^2 &= 0, \\ a_2(a_{2x}+\ldots-\lambda^2 b_{1x})+a_3(a_{3x}+\lambda b_{3x})+a_4(a_{4x}+\lambda b_{4x}) &= 0, \\ a_2(a_{2y}+\ldots-\lambda^2 b_{1y})+a_3(a_{3y}+\lambda b_{3y})+a_4(a_{4y}+\lambda b_{4y}) &= 0. \end{aligned}\right\} \qquad (14)$$

The residual singular curve given by the elimination of a_2, a_3, a_4, and λ is

* Using the formula given in my paper "On the intersection of constructs ....", *Proc. Camb. Phil. Soc.*, 21 (1923), 435. This paper will be referred to as I.C.

a curve* $c^4_{9,y} \equiv (y), (c_4)^{16}, 9Q_i$. The conics through a point of this curve all pass through a second point on it; and those through a point (w) of c_4 fill a surface again given by (12), replacing (z) by (y).

Each one of the ∞^3 *curves* $c^4_{9,y}$ *is self-correspondent in the involution* T.

The congruence of polar lines of (y) is given by (14), replacing the first equation by

$$(a_x a_y) + \lambda(b_x b_y) = 0,$$

so that it has a singular line, f (common to the polar planes of (y) for the pencil of quadrics); and the lines of the congruence in a plane through f form a pencil whose centre describes the second singular curve, a $c^0_{5,y} \equiv (y), f^4, (c^4_{9,y})^{10}$. The last-named points of intersection are the united points of the involution on $c^4_{9,y}$ cut by the conics of the congruence, or those points where the curve meets R^{11} outside c_4 and the points Q. Both curves lie on the cone $\{y(v_4)\}$.

12. In the parametric representation on $S_{(\alpha)}$ (§4) the only singular points are the points of $\alpha_1 = \alpha_2 = 0$ (§ 5). In the following table the second number in each bracket gives the multiplicity of this line on the construct, when it is a surface, and the number of intersections with the construct, when it is a curve. n_3 and t are deduced from the others, using (I) and (IV) respectively, these others being deduced directly from the preceding discussion.

n_1	n_2	n_3	n' (for c_4)	n'' (for Q)	n_4	d	t
(1, 1),	(2, 1),	(3, 2),	(2, 1),	(1, 1),	(5, 3),	(7, 5),	(6, 4).

To these we may add the construct for the condition to lie on a given quadric (λ) a plane through $\alpha_1 = \alpha_2 = 0$. This table enables us to determine the main features of a number of sub-systems of G_1 for which direct analysis would be laborious, for example the congruence (2, 10) of conics touching a plane, the order being given by the intersections of a curve (n_1) and a surface (t), and the class by those of a curve (n_3) and the same surface. The focal surfaces of this congruence are known, namely, the plane and its correspondent in T (§10). Thus we need not consider this and similar problems in detail.

13. As a last example let us consider the interesting *congruence* (2, 9) *of conics of* G_1 *meeting a line* g. Its conics satisfy the incidence conditions expressed by $(c_4)^4$, g, $c^0_{19,g}$ (§10), so that no focal surface exists. We know

* I.C., Formula XI, 452.

the surface of conics through a point of g or $c^0_{19,g}$. Using the parametric representation or directly, it can be shown that the surface of conics through a point $P(z)$ of c_4 is an $R^{20} \equiv g,\ c^0_{19,g},\ c^0_{9,z},\ (c_4)^9,\ 9Q^2_i$. Each of the ∞^1 conics of G_1 having g as chord will be, in general, a conic of contact of a singular tangent plane on any surface of conics of congruence which contains it. Thus R^{20} has 4 such conics, each on the plane $\{P.g\}$. The section by this plane is completed by g and three lines of degenerate conics on the surface. Using § 12 it is found* that there are 21 other degenerate conics on the surface, for which one line meets g and one passes through P.

13*a*. *Degenerate conics in the congruence.*—The lines of these conics which meet g form a scroll of order 8, $S^8 \equiv (c_4)^3,\ g^2,\ c_{(1)},\ (c_{(2)})^2,\ c_{(4)}$, the residual lines a scroll $S^{44} \equiv (c_4)^{21},\ (c^0_{19,g})^2,\ c_{(1)}\ (c_{(3)})^2,\ c_{(4)}$, the multiplicity of c_4 on the latter being given by the last result, and its order then follows by considering its section with an arbitrary quadric of the pencil. The remaining curves will be sufficiently explained by the following diagram, showing the distribution of lines on such a quadric, the lines of the two

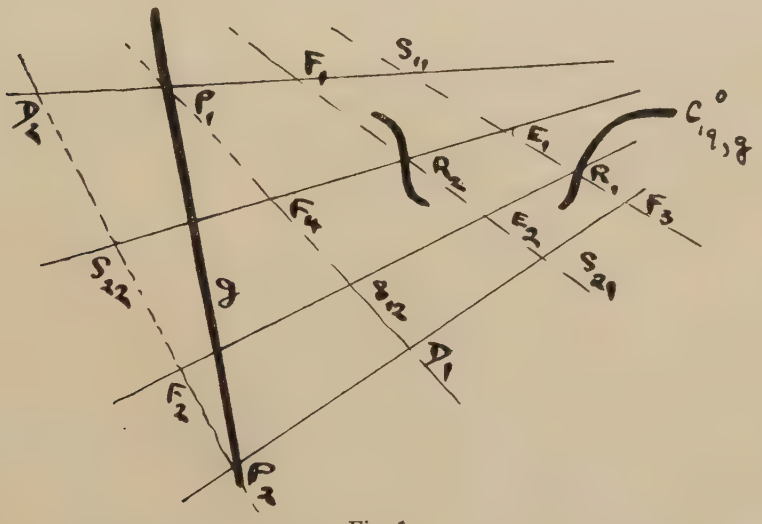

Fig. 1.

reguli being represented by continuous and discontinuous lines respectively. Thus $c_{(1)} \ldots c_{(4)}$ are the loci of the points S, D, E, F respectively.

* I.C., Formula I, 445.

We shall later give the diagrams for the various special cases which arise, from which all the intersections of these curves with each other, and with g and $c^0_{19,g}$, can be read off. Further they will give the united points of the involutions on these curves, and thence their genera.* At each such point we must distinguish between a true coincidence, arising from a tangency with the quadric, and one arising from an actual node. This is done systematically by the use of a continuity argument, which we shall not give in the individual cases. In the former case we pass from a real to an imaginary pair in the passage through such a point, in the latter from a real to a real or from an imaginary to an imaginary pair.

The figures in question are conveniently given together at the end of the section. We shall denote by M_i the 11 intersections of g with R^{11} (§ 8), and with $c^0_{19,g}$. S^{44} passes through these points, and has at them the same generators as S^8. It also shares with S^8 the 11 pairs of degenerate conics having g as chord. A verification of the order of S^{44} is given by its intersections with g, thus determined.

$c_{(1)}$ *is a curve of order* 28, having actual nodes at the 11 points M_i, and the four cone vertices H_i, and at certain 11 points N_i (Fig. 4). It is of genus 3, meets c_4 52 times, and lies on R^{11}, being the vertex locus for the congruence. Its equations are given by eliminating λ and μ between

$$a_x^2+\lambda b_x^2 = (a_x a_{(\lambda)}+\lambda(b_x b_{(\lambda)}) = (a_x a_\xi)+\lambda(b_x b_\xi) = a_\xi^2+\lambda b_\xi^2 = 0,$$

where $(\xi) = (\alpha)+\mu(\beta)$ is a point describing g.

$c_{(2)}$ is the quintic of genus two given by

$$\begin{Vmatrix} (a_\alpha a_\xi) & (a_\beta a_\xi) & a_\xi^2 \\ (b_\alpha b_\xi) & (b_\beta b_\xi) & b_\xi^2 \end{Vmatrix} = 0,$$

(α) and (β) being points on g, $c_{(2)} \equiv g^2(c_4)^6$ being generated by the intersection of the polars of g for the various quadrics of the pencil, and these quadrics.

To determine $c_{(3)}$ we note first that the line R_1R_2 describes a rational scroll of order 9, having g as five-fold line (since a plane through g contains four generators, one for each intersection with v_4). The points of $c_{(3)}$ are then cut on the quadrics by the polars of the lines of this scroll for

* Severi-Löffler, *loc. cit.* $c_{(1)}$, $c_{(4)}$, S^8, and S^{44} must have the same genus.

the corresponding quadrics, and it is therefore of order 23. It is of genus two, having nodes at the points Q_i, and meeting c_4 in 44 points.

Finally $c_{(4)}$, as the remaining intersection of S^8 and S^{44}, *is of the* 28*th order* and third genus, meeting c_4 54 times and having nodes at the points H_i.

We now give the promised diagrams.

I. *For a quadric containing a point* Q_i. $(i = 1 \ldots 9)$.

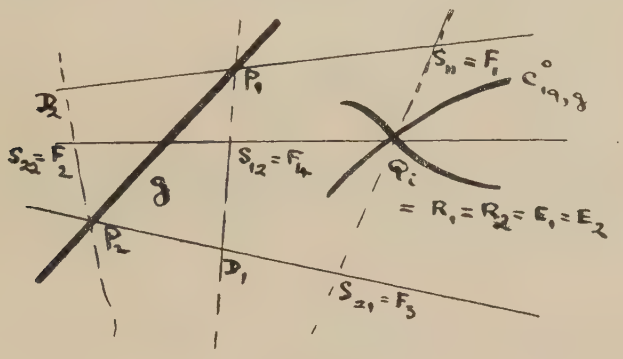

Fig. 2.

Thus $c_{(1)}$ meets $c_{(4)}$ four times for each point Q_i, and $c_{(3)}$ has nodes at the points Q_i. The 18 generators through the nine points Q_i are double generators of S^{44}.

II. *For a quadric meeting* g *in one of the* 11 *points* M_i.

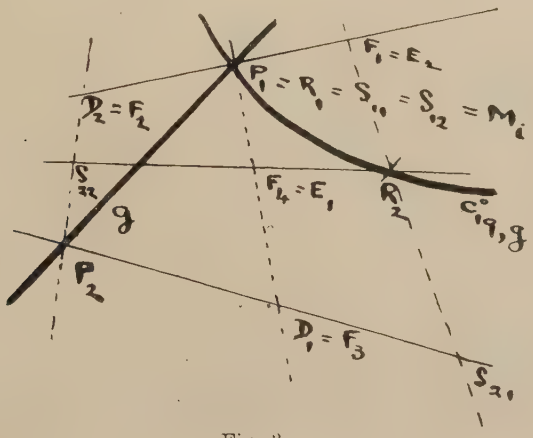

Fig. 3.

Thus $c_{(2)}$ and $c_{(3)}$ each meet $c_{(4)}$ in two points for each M_i, and $c_{(1)}$ has nodes at the points M_i, where it also meets g and $c^0_{19,g}$.

III. *A quadric containing a degenerate conic of G_1 having g as chord.*

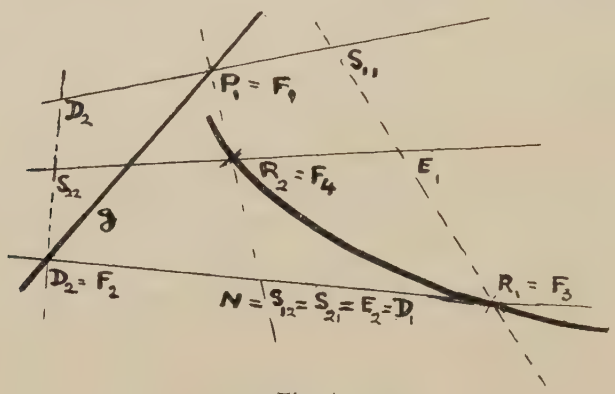

Fig. 4.

Thus $c_{(4)}$ meets $c^0_{19,g}$ twice for each such case, while $c_{(1)}$, $c_{(2)}$, and $c_{(3)}$ concur in a point N, a node on the first curve.

IV. *For a cone of the pencil, vertex H_i ($i = 1 \ldots 4$).*

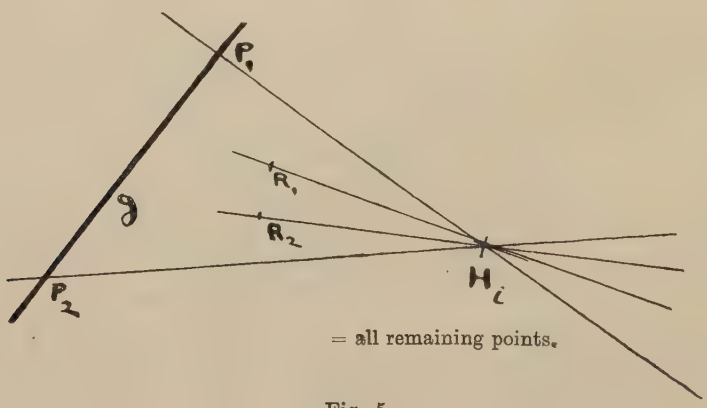

Fig. 5.

$c_{(1)}$ and $c_{(4)}$ both have nodes at H_i, each branch containing a coincidence of the corresponding involutions. $c_{(2)}$ and $c_{(3)}$ have at H_i a coincidence of the corresponding involutions.

V. *For one of the two quadrics which touch g, and therefore $c^0_{19,g}$.*

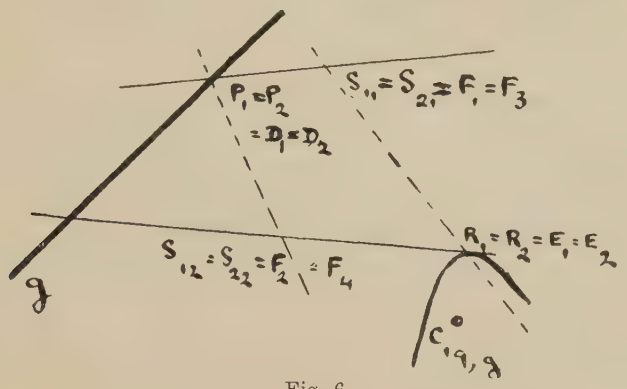

Fig. 6.

$c_{(2)}$ meets g in each point of contact, and these points are coincidences of the involution on $c_{(2)}$. Similarly $c_{(3)}$ meets $c_{19,g}$, etc. Finally $c_{(1)}$ meets $c_{(4)}$ in two points for each case, which are coincidences of the involutions on each. All these coincidences arise from contacts with the quadrics in question.

The curves $c_{(1)}$, $c_{(4)}$ are self-correspondent, and $c_{(2)}$, $c_{(3)}$ correspond to each other in T.

13*b*. To complete the discussion of the congruence of conics of G_1 meeting g, we must mention that, using the parametric representation of G_1, or directly, it can be shown that *the surface of conics of G_1 meeting two skew lines, g, h, is a surface* $F^{44} \equiv g^2, h^2, (c^0_{19,g})^2, (c^0_{19,h})^2, (c_4)^{20}$. It has 18 double conics, having either g or h as chord, and 8 conic tropes, arising from quadrics of the pencil touching g or h.

14. In precisely the same way *the congruence* $(2m, 9m)$ *of conics of G_1 meeting an m-tic curve, c^p_m,* can be considered. In general, the remaining directrix curve is a $c^p_{19m} \equiv (c_4)^{36m}, 9Q^{2m}_i, (c_m)^{11m}$. A congruence $(1, 4)$ is removed for each intersection with c_4, and the order of the singular curve is reduced by 9 for each such intersection, etc.

Similarly *the conics of G_1 meeting twice a curve c_m of rank R* fill a surface of order $22m^2-13m-R$, passing $(2n-1)$ times through c_m and $(10m^2-6m-\frac{1}{2}R)$ times through c_4. This is seen by allowing c_m to break up into m lines with $\frac{1}{2}R$ intersections. Each intersection with c_4 reduces the order by $20m-11$, the multiplicity of c_4 by $9m-5$, and that of c_m by unity. These numbers give, in fact, the surface of conics through a point of c_4 meeting a c_m through it elsewhere.

As an example the circles of G_1 (in a metrical space) generate a surface $F^{60} \equiv (c_2)^3$, $(c')^3$, $(c_4)^{27}$, c_2 being the isotropic circle and c', of order 38, its correspondent in T.

15. *Generalisations of the complex* G_1.—The first suggested is given by replacing v_4 by an arbitrary rational curve v_n, not meeting c_4. The discussion is largely parallel to that just given, so that we may confine ourselves to stating the analogous enumerative results.

The system is of class n, with $2n+1$ singular points Q. The vertex locus for *degenerate conics* is an $R^{2n+3} \equiv (c_4)^{n+1}$, the congruence of their lines is the same congruence, but the lines are differently paired. The class of the envelope of their planes equals $2n+3$. It has a double developable of class $4n$, with $8n$ stationary planes (§8).

The order of *the analogous involution* T is $4n+3$. To a plane corresponds a surface $T^{4n+3} \equiv (c_4)^{2n+1}$, Q_i^2, and to a line l a curve $c^0_{4n+3,\,l} \equiv l^{2n+3}$, $(c_4)^{8n+4}$, Q_i^2. To a point $P(z)$ on c_4 correspond the points of a curve $c^0_{2n+1,\,z} \equiv (c_4)^{3n}$, Q_i^2, P^{n+1} (§10).

The last-named curve is the residual directrix curve for *the congruence* $(1, n)$ *of conics through a point* P of c_4. Through two points (z) (w) of c_4 pass ∞^1 conics on a surface

$$R^{2n+1} \equiv (c_4)^n,\ c^0_{2n+1,\,z},\ c^0_{2n+1,\,w},\ w,\ (z)^{n+1},\ (w)^{n+1},\ (z)(w).$$

The curve locus of vertices of degenerate conics in the congruence is $c^{n+2}_{2n+5} \equiv P^{n+1}$, $(c_4)^{3n+7}$. The conics in question generate the cone $\{P(c_4)\}$ and a scroll $R^{6n+7} \equiv (c_4)^{3n+3}$, c^{n+2}_{2n+5}, $(c^0_{2n+1,\,z})^2$ (§ 11).

The conics of the complex having a line l *as chord* fill a surface of order $2n+1$, containing $2n+3$ pairs of lines of degenerate conics. It has $2(n-1)$ conic tropes (§ 12).

The residual singular curve of *the congruence of conics* $(1, n)$ *whose planes pass through a point* (z) is a $c^n_{2n+1,\,z} \equiv (z)$, $(c_4)^{4n}$, Q_i. That of the congruence of polar lines of (z) a $c^0_{n+1} \equiv (z)$, f^4, $(c^n_{2n+1,\,z})^{2n+2}$ (§ 11).

The congruence of conics meeting a line g is $(2, 2n+1)$, the residual singular curve being $c^0_{4n+3,\,g}$. Its conics through a point $P(z)$ of c_4 fill a $R^{4(n+1)} \equiv g$, $c^0_{4n+3,\,g}$, $c^0_{2n+1,\,z}$, $(c_4)^{2n+3}$, Q_i^2. The lines of degenerate conics meeting g generate a scroll $S^8 \equiv g^2$, $(c_4)^3$, $c_{(1)}$, $(c_{(2)})^2$, $c_{(4)}$; the others $S^{4(2n+3)} \equiv (c_4)^{4n+5}$, $(c^0_{4n+3,\,g})^2$, $c_{(1)}$, $(c_{(3)})^2$, $c_{(4)}$; where $c_{(1)}$ is a $c^3_{4n+12} \equiv (c_4)^{8n+20}$; $c_{(2)}$ a $c_5^2 \equiv (c_4)^8$; $c_{(3)}$ a $c^2_{4n+7} \equiv (c_4)^{8n+12}$; and $c_{(4)}$ a $c^3_{4(n+3)} \equiv (c_4)^{8n+20}$. The rest of the discussion is precisely as for G_1. The conics meeting two skew lines fill a surface of order $4(2n+3)$, etc., etc. (§ 13).

16. These results may be extended to the case when v_n meets c_4; and particularly interesting is the case when each point of v_n lies on the corre-

sponding quadric, in which case v_n becomes a curve of singular points, v_n must then meet c_4 $(2n-1)$ times.

We may then further extend the discussion to the complex $(\beta n, \alpha)$, arising when v_n and the pencil of quadrics are in (α, β) correspondence, and finally we may replace the series of stars, whose centres describe v_n, by a series of surfaces regarded as envelopes. In particular we might mention the complex generated by the sections of quadrics of a pencil by planes tangent to the quadrics of a projective pencil or range, of respective characteristics (3, 2) (1, 2), which can be discussed in a similar manner.

17. We will now pass on to a complex (0, 0), which will serve as a type for a group of such systems. Congruences of order zero fall into two classes, those in which the planes of the conics envelop a surface, and those in which the envelope is a developable. In the former case the class is equal to the class of the surface, in the latter to zero. The group we are concerned with arises from the simplest case, in which the developable reduces to a pencil. We start with the complex

$$(G_2) \qquad \begin{Vmatrix} \sum_1^4 a_i a_{ix}^2 & a_x & a_1 \\ \sum_1^4 a_i b_{ix}^2 & b_x & a_2 \end{Vmatrix} = 0, \tag{16}$$

containing ∞^3 congruences (1, 0) of type Γ_3, associated with the fundamental chord* $a_x = b_x = 0$, which we shall call l. It also contains an unlimited number of linear congruences of the same general type†, given by assigning a uni-directrix curve, since the conics of G_2 lie by ∞^2's in the planes of a pencil. There is a pencil of special surfaces (§4):

$$a_3(a_x b_{3x}^2 - a_{3x}^2 b_x) + a_4(a_x b_{4x}^2 - b_x a_{4x}^2) = 0, \tag{17}$$

of base l and a c_8^7. The singular c_{10}^{10} of the congruences Γ_3 form a complex lying by ∞^2's on the surfaces of this pencil.

The conics of G_2 in a plane through l belong to a net, or linear system ∞^2, in general without base points. A base point of such a net would be singular for G_2, and conversely a singular point, other than those of l, would be a base point of the net in the plane through it and l.

The conics through a general point (y) form a pencil and pass through three other points, so that G_2 defines an involutory correspondence (3, 3)

* A.R., §14, 157.

† Montesano, "Su i varii tipi . . .", 93.

of space. No conics have an arbitrary line as chord. The conics in a definite plane λ, where $a_1 = -\lambda a_2$, are cut by a net of cubic surfaces (base l and 20 points) or by a net of quadrics (with 8 base points):

$$a_2(a_{2x}^2 - \lambda a_{1x}^2 + \lambda b_{2x}^2 - \lambda^2 b_{1x}^2) + a_3(a_{3x}^2 + \lambda b_{3x}^2) + a_4(a_{4x}^2 + \lambda b_{4x}^2) = 0,$$

which it will be convenient to write

$$a_2 A_{2x}^2 + a_3 A_{3x}^2 + a_4 A_{4x}^2 = 0. \tag{18}$$

The singular points of G_1 are given by those base points which lie in the corresponding planes, namely the 24 points Q_i given by the results of substituting $\lambda = -a_x | b_x$ in $A_{2x}^2 = 0$, $A_{3x}^2 = 0$, $A_{4x}^2 = 0$. They are singular points of the second type. Each point of l is singular of the first type.

18. *The congruence of conics passing through a point $P(z)$* of l is of characteristics (1, 0), being of the generalised type cited, with the speciality that in each plane (λ) one base point of the pencil is fixed at (z). The remaining three describe the curve

$$\begin{Vmatrix} A_{2x}^2 & A_{3x}^2 & A_{4x}^2 \\ A_{2z}^2 & A_{3z}^2 & A_{4z}^2 \end{Vmatrix} = 0, \tag{19}$$

in which λ is replaced by $-a_x | b_x$. The congruence is, in fact, given by

$$\sum_2^4 a_i A_{ix}^2 = \sum_2^4 a_i A_{iz}^2 = a_x + \lambda b_x = 0.$$

The singular curve is of the twelfth order, meeting l in 9 points in all, some of which lie at P. To find this number, let us observe that the tangent line at P to the c_4^1 base of the pencil of quadrics, which cut the plane (λ) in the conics it contains, is given by

$$\| (A_{ix} A_{iz}) \quad A_{iz}^2 \| = 0, \quad i = 2, 3, 4.$$

Now if this line contains a further point of the plane λ it lies entirely therein, and one of the three remaining base points falls into P. We can take in the plane a line selected rationally, say a line meeting two assigned lines g_1, g_2. Its points will be given by

$$x = \alpha_\lambda + \mu \beta_\lambda,$$

the α_λ and β_λ being linear functions of λ. Substituting, we require solutions of a matrix equation, for which that term in each element which is

highest in both λ and μ is expressed, these elements being otherwise arbitrary, namely the equation

$$\begin{Vmatrix} \lambda^3\mu & \lambda^2\mu & \lambda^2\mu \\ \lambda^2 & \lambda & \lambda \end{Vmatrix} = 0.$$

These solutions are six in number, but one arises redundantly from the line through P meeting g_1 and g_2. Thus our singular curve is a $c_{12,z} \equiv P^5, l^4, Q_i$. There can be no lines through P, and elsewhere chords of the curve, so that $c_{12,z}$ is of genus nine.*

18a. The degenerate conics in the congruence generate a surface composed of the 7-tic cone $\{P(c_{12,z})^1\}$ and the scroll of lines which do not pass through P. This will be a $R^{3+r} \equiv l^r(c_{12})^2 c'_m$, the last-named curve being the locus of diagonal points of the quadrangles whose vertices are the base points of the various pencils. A continuity argument shows that this curve must have two branches at P for each branch of $c_{12,z}$, and since it has a simple point on l for each point of c_{12} on it outside P, it is of the 17th order. It must be of the same genus as $c_{12,z}$, namely nine. Considering the intersection of the two scrolls gives $r = 10$, and thus the latter is an $R^{13} \equiv l^{10}, (c_{12,z})^2, c_{17,z}$. R^{13} is also of genus 9.

18b. The conics of the congruence through a second point (w) of l, or the conics through two points of l, generate the surface given by writing $\lambda = -a_x|b_x$ in

$$|A^2_{ix} \quad A^2_{iz} \quad A^2_{iw}| = 0, \tag{20}$$

a surface $R^6 \equiv l^4, c_{12,z}, c_{12,w}$. Using this result it is seen that those meeting a line g fill an $R^8 \equiv l^6, c_{12,z}, g$.

19. *Degenerate conics of the complex* G_2.—Those in a plane of (l) envelop a curve of the third class, breaking up into a pencil and tangential conic for a plane through a point Q. Thus the order of the congruence of lines of degenerate conics is three. Each point of l is the vertex of a septimic cone of lines of the congruence (18a). By expressing that $z+\mu x$ lies on the plane λ and on one of the corresponding quadrics we obtain the equation of this cone in the form

$$|A^2_{ix} \quad (A_{ix}A_{iz}) \quad A^2_{iz}| = 0 \quad (\lambda = -a_x|b_x).$$

* All these statements may be verified directly by the method of I.C.

Thus the congruence is of the seventh class. Writing $z = \xi + \tau\eta$, (ξ) and (η) being points of l, and taking the τ discriminant of the last equation, we obtain the focal surface, locus of the curves of the third class. This is therefore a surface of order 28, having l as 22-fold line.

Expressing that the plane $a_x + \lambda b_x = 0$ touches the quadric (18) at (ξ) gives

$$a_i + \lambda b_i \equiv a_2(A_{2i}A_{2\xi}) + a_3(A_{3i}A_{3\xi}) + a_4(A_{4i}A_{4\xi}) \quad i = 1, \ldots, 4.$$

Writing $\lambda = -a_\xi \mid b_\xi$, and eliminating a_2, a_3, a_4, we obtain in (ξ) the equation of *the locus of vertices of degenerate conics*, namely

$$\left| a_i b_\xi - a_\xi b_i \quad (A_{2i}A_{2\xi}) \quad (A_{3i}A_{3\xi}) \quad (A_{4i}A_{4\xi}) \right| = 0, \quad i = 1, \ldots, 4, \quad (21)$$

in which $\lambda = -a_\xi \mid b_\xi$. This is a surface $U^8 \equiv l^5, Q_i^2$, the multiplicity of l being given by the fact that the residual section by a plane λ is a cubic curve, the Jacobian of the net of conics in the plane.

19*a*. To add the condition that the conics in question shall meet a line g, points (α) (β) on g are taken so that $a_\alpha = b_\beta = 0$, $b_\alpha = -a_\beta = 1$, and thus for points on g

$$x = \alpha + \lambda\beta.$$

Expressing that this point lies on (18) gives the further condition

$$a_2 A^2_{2,\,\alpha+\lambda\beta} + a_3 A^2_{3,\,\alpha+\lambda\beta} + a_4 A^2_{4,\,\alpha+\lambda\beta} = 0.$$

Hence *the curve locus of vertices of degenerate conics meeting a line* is given by adding to (21) a row

$$0, \quad A^2_{2,\,\alpha+\lambda\beta}, \quad A^2_{3,\,\alpha+\lambda\beta}, \quad A^2_{4,\,\alpha+\lambda\beta}.$$

This is a* $c^{12}_{23} \equiv l^{20}$. It meets g in eight nodes (the points gU^8).

19*b*. Similarly adding the condition that the degenerate conics shall pass through a point (z) of l, and hence adding to (21) a row

$$0, \quad A^2_{2z}, \quad A^2_{3z}, \quad A^2_{4z},$$

we obtain the equation of $c^9_{17,\,z}$ (§ 18*a*).

20. *The congruence of conics of G_2 meeting a line g* is (1, 0). There exists a singular curve locus of residual base points of pencils in the planes

* I.C., Formulae XI, and XIV (Case I).

of l, and corresponding to g in the (3, 3) involution. Taking (α), (β) as in §19a, its equation is obtained in the form

$$\begin{Vmatrix} A^2_{2x} & A^2_{3x} & A^2_{4x} \\ A^2_{2,\,\alpha+\lambda\beta} & A^2_{3,\,\alpha+\lambda\beta} & A^2_{4,\,\alpha+\lambda\beta} \end{Vmatrix} = 0, \quad \lambda = -a_x \,|\, b_x,$$

a c^{19}_{16} breaking up into g and a $c^{12}_{15,g} \equiv l^{12}, g^8, Q_i$.*

The surface of conics of the congruence through a point (z) of l is an $R^8 \equiv l^6, c_{15,g}, c_{12,z}, g$ (§18b), and using this it is found that the surface of those meeting a second line h is an $R^{10} \equiv l^8, c_{15,g}, c_{15,h}, g, h$; and from this, in turn (cf. §14), that the conics of G_2 meeting a curve c_m of rank R twice generate a surface of order $10\binom{m}{2} - \frac{1}{2}R$, passing $8\binom{m}{2} - \frac{1}{2}R$ times through l, and $(m-1)$ times through c_m.

20a. Since the congruence of lines of degenerate conics is (3, 7), the lines meeting g of degenerate conics meeting g form a scroll $S^{10} \equiv l^7, g^3, c_{15,g}, c_{23,g}$. The residual lines generate an $S^{r+3} \equiv l^r, (c_{15,g})^2, c_{23,g}$, and the consideration of its intersection with S^{10} gives $r = 13$.

21. The study of G_2, and in particular of its sub-systems defined by tangency conditions, may be completed by the use of its parametric representation (§4). There is a singular line $\alpha_1 = \alpha_2 = 0$, and the representation is similar to that of G_1 (§12), and, in fact, using the notation of §12, we have the table

n_1	n_2	n_3	n' (for l)	n'' (for Q)	n_4	d	t
(1, 1)	(1, 1)	—	(2, 1)	(1, 1)	(4, 3)	(8, 5)	(6, 4).

Thus the congruence of conics touching a plane is (2, 0), which result, in this case, can also be seen directly.

22. *A generalization of the complex* G_2 is given by assuming that A^2_{2x}, A^2_{3x}, A^2_{4x} (18), involve λ to the powers m_2, m_3, m_4. We shall denote Σm_i by μ, and shall confine ourselves to stating the main enumerative results.

The number of singular points Q is $4\mu+8$ (§ 17). *The congruence of conics through a point* $P(z)$ of l has as residual directrix curve a

* I.C., Formula XI, etc., $c_{15,g}$ and c_{23} (§ 19a) must have the same genus, equal to that of the scroll S^{10} (§ 20a.)

$c^{2\mu+1}_{2\mu+4,\,z} \equiv P^{\mu+1}$, l^{μ} of its lines of degenerate conics, those through P form the cone $\{P(c_{2\mu+4})\}$, the remainder generate an

$$R^{2\mu+5} \equiv l^{2\mu+2},\ (c_{2\mu+4})^2,\ c^{2\mu+1}_{3\mu+5},$$

the last named curve being the locus of the vertices. The conics of the system through two points (z), (w) of l generate an

$$R^{\mu+2} \equiv l^{\mu},\ c_{2\mu+4,\,z},\ c_{2\mu+4,\,w}\ (\S\ 18).$$

The degenerate conics of the complex form a congruence $(3, \mu+3)$, with a focal $(4\mu+12)$-tic surface, having l as $(4\mu+6)$-fold line. The vertex locus is $U^{\mu+4} \equiv l^{\mu+1}$, Q_i^2; for those meeting a line g, a $c^{2\mu+4}_{3\mu+1,\,g} \equiv l^{\mu+8}$, $g^{\mu+4}$ (the points on g being nodes) (§ 19).

The conics of the complex meeting a line g form a congruence $(1, 0)$, for which the residual singular curve is a $c^{2\mu+4}_{2\mu+7,\,g} \equiv l^{2\mu+4}$, $g^{\mu+4}$, Q_i. Those conics through (z) on l generate an $R^{\mu+4} \equiv l^{\mu+2}$, $c_{2\mu+7,\,g}$, g; those meeting a second line h an $R^{\mu+6} \equiv l^{\mu+4}$, $c_{2\mu+7,\,g}$, $c_{2\mu+7,\,h}$, g, h. The degenerate conics generate two scrolls, the first, formed by their lines meeting g, being an $S^{\mu+6} \equiv l^{\mu+3}$, g^3, $c_{2\mu+7,\,g}$, $c_{3\mu+11,\,g}$; and the second, formed by the remainder, is an $S^{2\mu+8} \equiv l^{2\mu+5}$, $(c_{2\mu+7,\,g})^2$, $c_{3\mu+11,\,g}$ (§ 20).

23. The last complex with which we shall be concerned is given by

(G_3)
$$\begin{vmatrix} a_x^2 & \sum_1^4 a_i a_{ix} & \sum_1^4 a_i b_{ix} \\ A_x & a_1 & a_2 \end{vmatrix} = 0, \qquad (22)$$

and is of the fourth order and second class. It contains ∞^3 congruences of Pieri's third species,* given by relations $\Sigma \mu_i a_i = 0$, and ∞^1 bilinear congruences,† given by $a_1 + \lambda a_2 = 0$, λ being assigned. We do not propose to give a detailed discussion of this complex, but will confine ourselves to stating its main features. The singular curves of the bilinear congruences are composed of the conic c_2, $(a_x^2 = A_x = 0)$, and curves c_5^2, which form a pencil on the quadric R^2 given by

$$a_{3x} b_{4x} - a_{4x} b_{3x} = 0.$$

The pencil has 12 base points, of which four lie on c_2, and will be called $G_1 \ldots G_4$, and the remainder, $Q_1 \ldots Q_8$, on the curve v_4^0, locus of centres of the stars of planes of the bilinear congruences (λ). *The points*

* "Sopra alcune congruenze di coniche", *Atti Accad. Torino*, 28 (1893), 293; or, A.R., § 40.
† Montesano, "Un sistema lineare . . .", *ibid.*, 27 (1892), 660; or, A.R., § 5.

of c_2 and the points Q_i are the only singular points of the complex. The planes are arranged, with respect to λ, exactly as in the case of G_1, so that the four conics in each plane are respectively associated with its intersections with v_4.

24. *The degenerate conics in the complex* fall into two systems. Those of the first are composed of a generator of that regulus of R^2 whose lines are trisecant to v_4^0 (and to the curves c_5^2), together with any incident line in $A_x = 0$. Those of the second are formed by lines of a congruence (4, 8) of lines meeting c_2, and at least one c_5^2 twice. The class is given by the fact that the cone of lines for each point of c_2 is of the fourth order. (Cf. §19.) The envelope of these cones gives the focal surface $F^{16} \equiv (c_2)^8$.

25. *The conics of G_3 through an arbitrary point $P(y)$* generate a quartic surface $T_y^4 \equiv (c_2)^2 P^2$. The four lines on T_y^4 through P give the four lines of the above congruence of conics through P (§24). The planes of the conics through P must envelop the elsewhere-tangent cone V from P, a quadric cone,* whose equation is therefore found without difficulty. Through a point of $A_x = 0$ only degenerate conics pass.

The locus of points of contact of tangents from (x) to conics of the complex G_3 is given by the equation of the tangent cone at (y) to T_y^4, regarded as an equation in (y). It is found to be an $R_x^6 \equiv (c_2)^2, (x)^2$. Similarly the equation of the cone V for (y), regarded as an equation in (y), (x) being fixed, represents the *focal surface of the congruence* (2, 4) *of conics whose planes pass through* (x), for both interpretations are equivalent to seeking pairs of points (x), (y) such that the two conics through (y) whose planes pass through (x) shall coincide. This surface is of the same character as the last. The congruence in question includes the congruence of conics through a point of c_2, but for a point Q or G the corresponding bilinear congruence breaks away, leaving a congruence (1, 3) of conics through such a point. Now the conics of a congruence (λ) through a point Q_i form a cubic surface† passing through c_2 and having Q_i as a node. It can be verified that, as λ varies, these surfaces form a pencil whose residual base curve, a $c_7^1 \equiv Q_i^4, (c_2)^6$, gives the residual singular curve of the congruence (1, 3) in question, being met by each conic in two points outside Q_i. It passes through the remaining points Q. The case of a point G may be considered similarly.

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* For these statements see Jessop, *Quartic Surfaces* (Cambridge), (1916), 62 *et seq.*

† A.R., § 9.

The conics of G_3 which have a line as chord describe a surface $R^{10} \equiv l^2, (c_2)^4$.

26. The complex G_3 is included in the general type

$$\sum_1^3 a_i A_{ix} = \sum_1^3 a_i c_{ix}^2 = 0$$

where the terms are also polynomials in λ of orders ξ_i, $\xi_i + k$ respectively ($\lambda = a_4 | a_1$, say). It contains ∞^1 bilinear congruences, the loci of the centres of the stars, formed by the planes of the respective systems, being a rational curve of order $\Sigma\xi_i$, which is therefore the order of the complex. Its class is $\Sigma\xi_i + k$, *in general*. In the case of G_3 the equation giving the values of λ, for which a conic passes through (y), while its plane passes through (z), contains a redundant factor λ, so that the class is reduced by unity.

THE UNIFORMITY OF THE JACOBIAN ELLIPTIC FUNCTIONS

By THEODORE W. CHAUNDY.

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OF the possible modes of developing the theory of the Jacobian elliptic functions, that which proceeds from the inversion of the elliptic integral, *i.e.* from the definition of $z = \mathrm{sn}\, u$ by the relation

$$u = \mathrm{sn}^{-1} z = \int_0^z (1-z^2)^{-\frac{1}{2}}(1-k^2 z^2)^{-\frac{1}{2}}\, dz,* \tag{1}$$

is in many ways the most natural and most elegant. But it has always been a very troublesome business to pass from this definition to a proof of the uniformity of the function, without which, of course, the whole theory is held up. I here explain a possible proof which I shall introduce by first establishing the one-valuedness of the circular function $z = \sin u$ directly from the definition

$$u = \sin^{-1} z = \int_0^z (1-z^2)^{-\frac{1}{2}}\, dz. \tag{2}$$

To assert the uniformity of $z(u)$ is to assert that, if $u_1 - u_2 = 0$, then $z_1 - z_2 = 0$, *i.e.* in this case that, if

$$\int_{z_1}^{z_2} (1-z^2)^{-\frac{1}{2}}\, dz \tag{3}$$

vanishes, then z_1 is z_2.

It makes for clearness to suppose the plane of z a Riemann-plane, two-sheeted for the surd $(1-z^2)^{\frac{1}{2}}$. Then any deformation of the path of integration between two fixed end-points, each in its proper sheet, adds (at most) a multiple of 2π to the integral, and it is enough to prove that

* I suppose $0 < k^2 < 1$. I do not see my way to include the case of complex moduli.

the integral (3), taken along *selected* paths, can never be an exact multiple of 2π, unless z_1 is z_2.

These selected paths will be arcs of the confocal ellipses and hyperbolas whose foci are the branch-points, $z = \pm 1$, of the integrand. For, if θ, θ' are the vectorial angles of z referred to poles $z = \pm 1$ and initial line Ox, then the argument of the complex expression

$$(1-z^2)^{-\frac{1}{2}}dz$$

is $$\psi - \tfrac{1}{2}(\theta+\theta'-\pi).$$

This is an even (odd) multiple of $\frac{1}{2}\pi$ at any point of a confocal ellipse (hyperbola), since the tangent there is the internal (external) bisector of the angle between the focal distances. Hence the integral (3) is purely real (imaginary) taken along a confocal ellipse (hyperbola). Moreover the element of the integral maintains a constant sign on an ellipse (a constant sign, multiplied by i, on a hyperbola). It can therefore never vanish along any arc of a confocal and can equal an exact multiple of 2π only for exactly-completed circuits of an ellipse.

Now the ellipse (say) through z_1 meets the hyperbola through z_2 in two points; call them z_{12}, z'_{12}. The minor arcs joining z_{12}, z'_{12} enclose a single branch-point. Thus one of the paths $z_1 z_{12} z_2$, $z_1 z'_{12} z_2$ crosses the connecting-cut between the Riemann sheets, the other avoids it. Whether, therefore, z_1, z_2 lie on the same or opposite sheets, we can join them by one of the paths $z_1 z_{12} z_2$, $z_1 z'_{12} z_2$, say the former. Then $\int_{z_1}^{z_{12}}$ and $\int_{z_{12}}^{z_2}$ are respectively the real and imaginary parts of $u_2 - u_1$. Hence by our hypothesis the first integral equals $2n\pi$ and the second integral vanishes. This, as we have seen, necessitates

$$z_1 = z_{12} = z_2$$

and establishes the uniformity of $\sin u$.

To define curves analogous to these confocals for $\operatorname{sn} u$ I take "isotropic axes", *i.e.* I write $z, z' \equiv x \pm iy$ and consider the bicircular quartics

$$az^2z'^2 - b(z^2+z'^2) + 2czz' + d = 0, \tag{4}$$

where a, b, c, d are real constants.

Solving this as a quadratic in z' we have

$$(az^2-b)z' + cz = \pm\{c^2z^2 + (az^2-b)(bz^2-d)\}^{\frac{1}{2}}.* \tag{5}$$

* We see at once from this form that the branch-points of the surd are "ordinary" foci of the quartic.

The sign on the right is invariable for continuous progression along a chosen quartic, so long as we do not pass through a branch-point of the surd. This condition we shall see later is always satisfied.

Again from (4) by differentiation

$$\{(az^2-b)z'+cz\}\,dz'+\{(az'^2-b)z+cz'\}\,dz=0.$$

By (5) and its symmetrical analogue we have

$$(6)\quad \{c^2z^2+(az^2-b)(bz^2-d)\}^{-\frac{1}{2}}\,dz=\pm\{c^2z'^2+(az'^2-b)(bz'^2-d)\}^{-\frac{1}{2}}\,dz',$$

where, by the foregoing argument, the double sign is unambiguous along any chosen quartic.

Along a quartic for which the positive (negative) sign is to be taken every element of the integral

$$(7)\qquad \int\{c^2z^2+(az^2-b)(bz^2-d)\}^{-\frac{1}{2}}\,dz$$

is purely real (imaginary), and therefore the integral itself is purely real (imaginary). I shall distinguish such quartics as r-ovals (i-ovals).

It follows also (as in the discussion of $\sin u$) that the integrals taken round the r-ovals and i-ovals are respectively monotonic and i times monotonic functions.

In order that the integral (7) may represent $\mathrm{sn}^{-1}z$, we must choose the constants a, b, c, d, defining the quartics, to satisfy the quadratic identity in z^2

$$(8)\qquad c^2z^2+(az^2-b)(bz^2-d)\equiv(1-z^2)(h^2-z^2)$$

(where it is convenient to write $h\equiv k^{-1}$). Putting $z^2=0,\ \infty,\ 1$, we have

$$\left.\begin{aligned} bd&=h^2, \quad ab=1,\\ c^2&=(a-b)(d-b).\end{aligned}\right\}$$

To avoid irrationals in the general solution of this triad of equations I define

$$\surd(a\theta)\equiv\surd(a-b)+\surd(d-b).$$

This readily gives a, b, c, d proportional to

$$4\theta,\quad -\theta^2+2(h^2+1)\,\theta-(h^2-1)^2,\quad \theta^2-(h^2-1)^2,\quad 4h^2\theta.$$

By substitution in (4) and reduction we have the Cartesian equation to the quartics as

$$(9)\qquad x^2+\frac{(h+1)^2y^2}{\theta}=\frac{(x^2+y^2-h)^2}{(h-1)^2-\theta}.$$

I find it more convenient to change the parameter θ by the substitution

$$\theta\lambda^2-(h-1)^2(\theta+\lambda^2)+(h^2-1)^2=0,$$

giving the quartics as

$$\frac{x^2(h-1)^2}{\lambda^2-(h-1)^2}+\frac{y^2(h+1)^2}{\lambda^2-(h+1)^2}=\frac{(x^2+y^2-h)^2}{4h}. \tag{10}$$

The intersection of the quartics (λ), (μ) gives parametrically the coordinates of the point (μ) of the quartic (λ) as

$$\left.\begin{aligned} x^2 &= \frac{(h+1)^2[\lambda^2-(h-1)^2][\mu^2-(h-1)^2]}{4(\lambda\mu-h^2+1)^2}, \\ -y^2 &= \frac{(h-1)^2[\lambda^2-(h+1)^2][\mu^2-(h+1)^2]}{4(\lambda\mu-h^2+1)^2}. \end{aligned}\right\} \tag{11}$$

From these parametric forms it is not difficult to determine the character and shapes of the curves. But the quartics, if the present theory is sound, are otherwise well known, since evidently the curves for which $\mathrm{sn}^{-1}z$ is purely real or purely imaginary are the conjugate equipotentials given by

$$x+iy=\mathrm{sn}\,(u+iv),$$

figures of which are readily available.*

It is enough for my purpose to mention that the quartics form two families: one of ovals enclosing the branch-points $z=\pm1$, but not the branch-points $z=\pm h$; the other of ovals enclosing either both the branch-points $z=1$, h or both the branch-points $z=-1$, $-h$.

In the fundamental theory of elliptic contour-integrals† we know that complete contours enclosing the branch-points ±1 give the real period $4K$, and complete contours enclosing either pairs of branch-points, $1, h$ or $-1, -h$, give the imaginary period $2iK'$. The two families of quartics are thus just the r-ovals and the i-ovals respectively.

As with confocal conics, through any point of the plane passes one oval of each family: an r-oval and an i-oval intersect in two points the arcs to which isolate each branch-point from its fellows. We can thus pass from any z_1 to any z_2 (on either Riemann-sheet) by a path $z_1 z_{12} z_2$

* See, for instance, Dixon, *Elliptic Functions* (1894), Appendix A, 133, Fig. 4A. The figures were also discussed and drawn (as Prof. Love has pointed out to me) by Siebeck in *Journal für Math.*, 57 (1860), 359. He there also established in geometrical form the property of these curves which I here use.

† As given, let us say, in Forsyth, *Theory of Functions* (1900), §104, 213 and on.

composed of an r-arc followed by an i-arc, and, exactly as for $\sin u$, it follows that

$$\int_{z_1}^{z_2} = 4mK+2niK'$$

necessitates $z_1 = z_{12} = z_2$, and hence that $z = \operatorname{sn} u$ is one-valued.

This discussion does not immediately include (i) values of u corresponding to branch-points, (ii) infinities of $\operatorname{sn} u$, (iii) infinite values of u. As to (i) we easily show that the integral is convergent at branch-points, so that these values of u are definite. Moreover the pair of segments $(-1, 1)$ and $(h, \infty, -h)$ of Ox forms the limiting r-oval, $\lambda = h+1$, and the alternate pair of segments $(-h, -1)$ and $(1, h)$ the limiting i-oval, $\lambda = h+1$. The branch-points are now admissible to the general theory.

For (ii) we remark that the infinities of $z(u)$ are the values of $\int_0^\infty$ taken to *any* point at infinity. The integral along any arc of the infinite circle vanishes, and it is therefore enough to go to infinity in a selected direction, say along Oy, which is the i-oval, $\lambda = h-i$. The general theory is now applicable to the infinities of $\operatorname{sn} u$. The function is one-valued at them, and they are therefore *non-essential* singularities.

For (iii) write (as we may)

$$\operatorname{sn}(4mK+2niK'+u_0) = \operatorname{sn} u_0$$

and take the integers m, n to infinity in any ratio. It is evident that $\operatorname{sn} u$ may have any value at $u = \infty$, which is therefore an essential singularity.

It is not difficult to complete the proof of the *regularity* of $\operatorname{sn} u$. Since along any of the ovals the absolute value of the integral is a monotonic function of the arc, we can secure that the arc is arbitrarily small by sufficiently restricting the corresponding absolute value of the integral. Hence *a fortiori* we can secure $|z-z_0| < \epsilon$ by making $|u-u_0| < \eta(\epsilon, z_0)$. This establishes the *continuity*, and extends to the branch-points, but not, of course, to the infinities of $z(u)$.

By the ordinary theory of the complex integral* we can prove that

$$\frac{\delta u}{\delta z} \to (1-z^2)^{-\frac{1}{2}}(1-k^2z^2)^{-\frac{1}{2}} \quad (z^2 \neq 1, k^{-2})$$

as $\delta z \to 0$ *in any direction.*

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* See, for example, Watson, "Complex integration and Cauchy's theorem" (1914), 43.

Since $z(u)$ is continuous, $|\delta z| \to 0$ is secured by $|\delta u| \to 0$, and therefore

$$\frac{\delta z}{\delta u} \to (1-z^2)^{\frac{1}{2}}(1-k^2z^2)^{\frac{1}{2}}$$

as $\delta u \to 0$ in any direction. This proves $z(u)$ analytic, save at branch-points. At branch-points we can prove that $|\delta u|$ is of the order of $|\delta z|^{\frac{1}{2}}$ and therefore that

$$\frac{\delta z}{\delta u} \to 0, \quad \text{as} \quad \delta u \to 0.$$

The function is therefore analytic also at branch-points.

We have now shown that $\operatorname{sn} u$ is regular save at isolated poles in the finite part of the u-plane and at its essential singularity $u = \infty$.

We may prove a like result for $\operatorname{cn} u$, $\operatorname{dn} u$, defining them as $(1-z^2)^{\frac{1}{2}}$, $(1-k^2z^2)^{\frac{1}{2}}$.

We are already on a Riemann surface and they are therefore uniform and analytic functions of z, itself a regular function of u. Hence they too are regular functions of u. Or we may proceed directly from the integral definitions

$$\operatorname{cn}^{-1} z = \int_z^1 (1-z^2)^{-\frac{1}{2}}(k'^2+k^2z^2)^{-\frac{1}{2}}\,dz,$$

$$\operatorname{dn}^{-1} z = \int_z^1 (1-z^2)^{-\frac{1}{2}}(z^2-k'^2)^{-\frac{1}{2}}\,dz.$$

We shall need to modify the identity (8), replacing h^2 in turn by $1-h^2$ and $1-h^{-2}$. Corresponding transformations in (9) will give the Cartesian equations to the bicircular quartics appropriate to cn and dn. They are, of course, the conjugate potential curves $x+iy = \operatorname{cn}(u+iv)$, $\operatorname{dn}(u+iv)$, and are available in the authorities quoted for $x+iy = \operatorname{sn}(u+iv)$.

I add my thanks to the referees from whose suggestions this paper largely takes its present shape.

ON A CLASS OF FACTORIAL SERIES

By T. Arnold Brown.

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1. Our object in this paper is twofold. In the first place we investigate the conditions under which an analytic function can be represented by the direct factorial series

$$(1.1)\qquad e_0+e_1z+e_2\frac{z(z-1)}{2!}+e_3\frac{(z+1)z(z-1)}{3!}+\ldots$$
$$+e_{2\nu}\frac{(z+\nu-1)(z+\nu-2)\ldots(z-\nu)}{(2\nu)!}$$
$$+e_{2\nu+1}\frac{(z+\nu)(z+\nu-1)\ldots(z-\nu)}{(2\nu+1)!}+\ldots,$$

which is a generalisation of Gauss's interpolation formula.* In the second place we exhibit a remarkable relationship between the factorial series (1.1) and a generalised form of a certain interpolation formula discussed, in a different connection, by de la Vallée Poussin.†

As regards the first of these problems a certain amount of information is already available, for Faber‡ has treated the more general problem of determining the necessary and sufficient conditions that an integral

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* Cf. I. Bendixson, "Sur une extension à l'infini de la formule d'interpolation de Gauss", *Acta Math.*, 9 (1886), 1–34.

† Ch.-J. de la Vallée Poussin, "Sur la convergence des formules d'interpolation entre ordonnées équidistantes", *Bull. de l'Acad. Roy. de Belgique, Classe des Sciences* (1908), 319–410. See also H. Hahn, "Über das Interpolationsproblem", *Math. Zeitschrift*, 1 (1918), 115–142, and Maria Theis, "Über eine Interpolationsformel von de la Vallée Poussin", *ibid.*, 3 (1919), 93–113.

‡ G. Faber, "Beitrag zur Theorie der ganzen Funktionen", *Math. Annalen*, 70 (1910), 48–68.

function $G(z) = \Sigma c_\nu z^\nu$ may be expressible in the form

$$(1.11) \quad e_0 + e_1(z-a_0) + e_2 \frac{(z-a_0)(z-a_1)}{a_1-a_0} + e_3 \frac{(z-a_0)(z-a_1)(z-a_2)}{(a_1-a_0)(a_2-a_0)} + \ldots,$$

where $|a_0| \leqslant |a_1| \leqslant |a_2| \ldots$ and $\lim_{\nu\to\infty} |a_\nu| = \infty$.

He obtained in particular the following result.*

In order that the integral function $G(z)$ may admit the expansion (1.11), *with*

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(|e_\nu|)} < 1,$$

it is necessary that

$$(1.12) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(|a_1 a_2 \ldots a_{\nu-1} c_\nu|)} < \infty,$$

and sufficient that

$$(1.13) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(|a_1 a_2 \ldots a_{\nu-1} c_\nu|)} < \tfrac{1}{2}.$$

Expressed in our notation these conditions become respectively

$$(1.14) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{\{[(\nu-1)/2]!\}^2 |c_\nu|} < \infty,$$

$$(1.15) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{\{[(\nu-1)/2]!\}^2 |c_\nu|} < 1,$$

where $[(\nu-1)/2]$ denotes the integral part of $(\nu-1)/2$.

Now, conditions (1.14) and (1.15) may be written more simply

$$(1.141) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(\nu!\,|c_\nu|)} < \infty,$$

$$(1.151) \quad \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(\nu!\,|c_\nu|)} < 1,$$

and the first implies that $G(z)$ is of order 1 and finite type, the second that $G(z)$ is of order 1 and type less than 1.†

We show in §7 that 1 may be replaced by $\log(3+2\sqrt{2}) = 1{\cdot}762\ldots$ in condition (1.151), while the most general result of this kind is expressed by Theorem **F**.

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* *Loc. cit.*, 53 and 55.

† A. Pringsheim, "Elementare Theorie der ganzen transcendenten Funktionen von endlicher Ordnung", *Math. Annalen*, 58 (1904), 257–342 (273).

The arguments which we employ depend upon the ideas of Pincherle,* Lindelöf,† and Carlson ‡ ; indeed, the last named writer has developed the corresponding theory for the factorial series

$$\Sigma e_\nu \frac{z(z-1)(z-2)\dots(z-\nu+1)}{\nu!}, \tag{1.2}$$

which is a generalisation of Newton's interpolation formula.§ In spite of the fact that the two series (1.1) and (1.2) may both be regarded as limiting cases of Lagrange's interpolation formula, there is a marked difference in their behaviour. We may observe that the series (1.1), if it converges at all, must converge over the whole z-plane, while the series (1.2) may admit a half-plane of convergence.‖ Moreover, the same function may be represented by two different expansions of the form (1.2), though this property is not shared by (1.1).¶

When we come to establish the connection between the interpolation formulae of de la Vallée Poussin and of Gauss, we are led naturally to consider the following type of convergence problem. Let $L(z)$ denote the Lagrangian polynomial which is equal to the analytic function $f(z)$ for a finite number of real equidistant values of z contained in an interval (a, b) ; then the problem is to determine the region of the z-plane within which $L(z)$ "converges" to $f(z)$, as the number of given values is indefinitely increased. Runge** has shown that this region is determined by the singular points of $f(z)$ and that it may contain only a part of the given interval (a, b). We solve in §10 the analogous problem for Gauss's formula in the unlimited interval $(-\infty, \infty)$. Since we are dealing with an infinite series instead of a polynomial, our problem is essentially more complicated. We are in effect replacing a single passage to the limit by a repeated one. De la Vallée Poussin also investigates a repeated limit problem. He finds the conditions which the real function $f(x)$ must satisfy in order that his interpolation formula (§8) should converge to $f(x)$ in a finite interval.

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* S. Pincherle, "Sur les fonctions déterminantes", *Annales de l'école normale supérieure* (3) 22, (1905), 9–68.

† E. Lindelöf, *Le calcul des résidus*, Paris (1905), Chap. 5.

‡ F. Carlson, "Sur une classe de séries de Taylor", Dissertation, Uppsala (1914).

§ "Sur les séries de coefficients binomiaux", *Nova Acta regiae societatis scientiarum Upsaliensis*, (4) 4 (1914–17), No. 3 (1915), 1–61.

‖ Cf. E. Landau, "Über die Grundlagen der Theorie der Fakultätenreihen", *Sitzungsberichte d. k. bayer. Akad. d. Wissens. zu München, Math.-phys. Kl.*, 36 (1906), 151–218.

¶ See, *e.g.*, Carlson's memoir, 6.

** C. Runge, "Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten", *Zeitschrift f. Math. u. Phys.*, 46 (1901), 224–243 (243).

2. Let $f_0, f_1, f_2, \ldots, f_\nu, \ldots$

be a given sequence of numbers, such that

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{(|f_\nu|)} < \infty ;$$

then the inverse power series

$$\sum_{\nu=0}^{\infty} f_\nu/x^\nu,$$

together with its continuations, defines a one-valued branch of an analytic function in a certain *étoile*,* including infinity, in the x-plane. This branch will be denoted by $\phi(x)$ and will be called the generating function corresponding to the set of values $\{f_\nu\}$.

Now it may happen that a closed contour can be defined which contains in its interior all the singularities of $\phi(x)$, but neither intersects the negative real axis nor encloses the origin. Such a contour will be said to be of the Γ-type, or simply a Γ contour. In case no contour of the Γ-type exists, we define the contour $\Gamma_{(a)}$ to be any simple curve which joins the points $ae^{-\pi i}$, $ae^{\pi i}$, encloses all the singularities of $\phi(x)$, and meets the negative real axis in the sole point $x = -a$.

We write also $x = re^{i\theta}$ and we define x^z as $e^{z \log x}$, where

$$-\pi \leqslant \theta \leqslant \pi.\dagger$$

This being so, we have the following

LEMMA a.—*If a contour of the Γ-type exists, the integral*

$$\frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx$$

defines an integral function of order 1 *and finite type which takes the values $f_1, f_2, f_3, \ldots$, for $z = 1, 2, 3, \ldots$.*

For let C be any closed contour which contains in its interior the circle of convergence of the inverse power series $\phi(x) = \Sigma f_\nu/x^\nu$. Then

$$f_\nu = \frac{1}{2\pi i}\int_C x^{\nu-1}\,\phi(x)\,dx \quad (\nu = 0, 1, 2, \ldots). \tag{2.11}$$

* G. Mittag-Leffler, "Sur la représentation d'une branche uniforme de fonction monogène (première note)", *Acta Math.*, 23 (1900), 43-62.

† This will give rise to no ambiguity in what follows.

Now deform C into a contour of the Γ-type. The value of the integral (2.11) is not thereby altered except possibly for $\nu = 0$. Consider the function

$$f(z) = \frac{1}{2\pi i} \int_\Gamma x^{z-1} \phi(x)\, dx, \tag{2.12}$$

which is plainly an integral function, such that

$$f(\nu) = f_\nu, \quad \text{for} \quad \nu = 1, 2, 3, \ldots.$$

We have also

$$|f(z)| \leqslant \frac{1}{2\pi} M_\Gamma(x^z) \int_\Gamma \left| \frac{\phi(x)}{x}\, dx \right|,$$

where $M_\Gamma(x^z)$ denotes the maximum value of $|x^z|$ as x describes Γ. Therefore

$$|f(z)| < k M_\Gamma(x^z),$$

where k is a positive number, and it follows that a number A exists such that

$$|f(z)| < k e^{A\rho} < e^{(A+\epsilon)\rho}$$

for all sufficiently large values of $|z| = \rho$. $f(z)$ is therefore of order 1 and finite type,* and the lemma is established.

It will be convenient to introduce at this stage two theorems which are particular cases of Carlson's more general results and which have been proved independently by Hardy.†

THEOREM **A**.—*If* (1) $f(z)$ *is regular at all points inside the angle* $-\alpha \leqslant \operatorname{am} z \leqslant \alpha$, *where* $\alpha \geqslant \pi/2$; (2) $|f(z)| < Ae^{k|z|}$, *where* $k < \pi$ *throughout this angle*; (3) $f(\nu) = 0$ *for* $\nu = 1, 2, 3, \ldots$; *then* $f(z)$ *is identically zero.*‡

THEOREM **B**.—*The necessary and sufficient condition that* $\psi(x) = \Sigma f_\nu x^\nu$ *should be a one-valued branch of an analytic function, regular in the region exterior to the curve*

$$(\log r)^2 = \vartheta^2 - \theta^2 \quad (0 < \vartheta < \pi), \tag{2.2}$$

and including infinity, but not in any more extensive region of the same

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* Pringsheim, *loc. cit.*, 264.

† G. H. Hardy, "On two theorems of F. Carlson and S. Wigert", *Acta Math.*, 42 (1920), 327–339. Full references to both theorems will be found in this note.

‡ *Vide* Carlson's *Dissertation*, 58; Hardy, *loc. cit.*, 328; and M. Lerch, "Sur un point de la théorie des fonctions génératrices d'Abel", *Acta Math.*, 27 (1903), 339–352.

*character, is that there should be an integral function $f(z)$, of order 1 and type $\mathfrak{H}$, which assumes the values $f_1, f_2, f_3, \ldots$, for $z = 1, 2, 3, \ldots$.**

Any closed contour in the x-plane, which is such that any point x and its inverse point $1/x$ together lie outside, on, or inside it, will be said to possess the I-property. Now this property is evidently possessed by the curve (2 . 2), and we deduce the following

Corollary.—*Theorem* **B** *is true when the function $\psi(x)$ is replaced by $\phi(x) = \Sigma f_\nu / x^\nu$.*

3. We proceed to prove a number of easy lemmas relating to the convergence of the series (1 . 1). Let $R = p + \frac{1}{2}$ be half an odd integer, and suppose the circle $|z| = R$ in the z-plane perforated by a series of small circles of radius $r(< \frac{1}{2})$ with their centres at the points $z = 0, \pm 1, \pm 2, \ldots, \pm p$. Let S denote the closed set of points consisting of the region bounded by these circles together with their circumferences; then our first lemma is as follows.

Lemma β.—

$$\lim_{\nu \to \infty} \sqrt[2\nu]{\left\{ \frac{|(z+\nu-1)(z+\nu-2)\ldots(z-\nu)|}{(2\nu)!} \right\}} = \tfrac{1}{2}$$

uniformly for all points of S.

We have

$$\frac{(z+\nu-1)(z+\nu-2)\ldots(z-\nu)}{(2\nu)!} = \frac{\Gamma(z+\nu)}{\Gamma(z-\nu)\,\Gamma(2\nu+1)}$$

$$= (-1)^\nu \frac{\sin \pi z \, \Gamma(\nu+z)\,\Gamma(\nu+1-z)}{\pi \Gamma(2\nu+1)}.$$

Now there exist positive numbers A_1 and A_2 such that for all points of S

$$A_1 \leqslant |\sin \pi z| \leqslant A_2, \tag{3 . 11}$$

and using the asymptotic property of the gamma-function we find that

$$\lim_{\nu \to \infty} \sqrt[2\nu]{\left\{ \left| \frac{\Gamma(\nu+z)\,\Gamma(\nu+1-z)}{\Gamma(2\nu+1)} \right| \right\}} = \tfrac{1}{2} \tag{3 . 12}$$

uniformly for all points of S.

The required result follows from (3 . 11) and (3 . 12).

* Carlson, *loc. cit.*, 54; Hardy, *loc. cit.*, 338.

Applying Cauchy's test for convergence to the factorial series (1.1) and having regard to the previous lemma, we obtain our second lemma, which is as follows:—

LEMMA γ.—*In order that the factorial series* (1.1) *should converge uniformly for all points of* S, *it is necessary that*

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|e_\nu|} \leqslant 2,$$

and sufficient that

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|e_\nu|} < 2.$$

We observe finally that each term of the series in question is a regular function of z at all points of S; and appealing to a classical theorem we conclude that the series, when convergent, represents a function regular for all points of S. Since R may be as large and r as small as we please, we may state our third lemma as follows:—

LEMMA δ.—*If the series* (1.1) *converges, it represents an integral function of* z.

4. Suppose that the series (1.1) converges, and let f_ν be the value of its sum for $z = \nu$, where ν is any positive or negative integer or zero. The sequence $\{f_\nu\}$ is thus defined by the following set of equations:—

$$(4.11) \qquad f_\nu = e_0 + e_1\nu + \sum_{n=1}^{\nu-1} ({}_{\nu+n-1}C_{2n}\, e_{2n} + {}_{\nu+n}C_{2n+1}\, e_{2n+1}) \quad (\nu \geqslant 0),$$

$$(4.12) \qquad f_{-\nu} = e_0 - \sum_{n=1}^{\nu} ({}_{\nu+n-1}C_{2n-1}\, e_{2n-1} - {}_{\nu+n}C_{2n}\, e_{2n}) \quad (\nu \geqslant 0),$$

which is equivalent to the set

$$(4.13) \qquad e_{2\nu} = \Delta^{2\nu} f_{-\nu}, \qquad e_{2\nu+1} = \Delta^{2\nu+1} f_{-\nu},$$

where $$\Delta^n f_\nu = f_{n+\nu} - {}_nC_1 f_{n+\nu-1} + {}_nC_2 f_{n+\nu-2} + \ldots + (-1)^n f_\nu$$

$$(n = 0, 1, 2, \ldots).$$

Consider now the generating function $\phi(x)$ defined in a certain domain of the x-plane by the following expansion:—

$$(4.2) \qquad \frac{\phi(x)}{x} = \frac{f_0}{x-1} + \frac{\Delta f_0}{(x-1)^2} + \sum_{n=1}^{\infty} \left\{ \frac{x^{n-1}\Delta^{2n} f_{-n}}{(x-1)^{2n+1}} + \frac{x^n \Delta^{2n+1} f_{-n}}{(x-1)^{2n+2}} \right\},$$

and let

$$\lambda = \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{\{|\Delta^\nu f_{-[\nu/2]}|\}}.$$

We require the following lemma:—

Lemma ϵ.—*The domain of convergence of the expansion* (4.2) *is bounded by the curve*

$$C(r, \theta; \lambda) = r^2 - (2\cos\theta + \lambda^2) r + 1 = 0. \tag{4.21}$$

For if we apply Cauchy's test to the series of moduli we see that the latter converges or diverges according as

$$\lambda^2 \left| \frac{x}{(x-1)^2} \right| < \text{ or } > 1.$$

We conclude, therefore, that the boundary of the domain of convergence is given by the equation

$$\lambda^2 \left| \frac{x}{(x-1)^2} \right| = 1,$$

which assumes the form (4.21) when expressed in polar coordinates.

This curve evidently possesses the I-property, and consists of a single loop enclosing the point $r = 1$, $\theta = 0$, provided $0 < \lambda < 2$.

We may obtain also the following

Corollary.—*If the series* (4.2) *converges absolutely for a value* x' *of* x *such that*

$$\left| \frac{x'}{(x'-1)^2} \right| = \frac{1}{\lambda'^2}, \quad \textit{where} \quad 0 < \lambda' < 2,$$

then it converges uniformly in any finite closed region exterior to the curve

$$C(r, \theta; \lambda) = 0,$$

and on its boundary, where $\lambda' < \lambda < 2$.

This region may be as extensive as we please, and since each term of the series is a regular function of x in the region, the fundamental property of the expansion (4.2) may be expressed by

Lemma ξ.—*If* $\lambda < 2$, *the series* (4.2) *defines an analytic function regular in the domain exterior to the curve* (4.21).

When $\lambda \geqslant 2$ the above domain splits up into two domains, $D^{(1)}$ and $D^{(2)}$, one inside and the other outside the unit circle. The preceding argument then leads to the

Corollary.—*If* $\lambda \geqslant 2$, *the series* (4.2) *defines at most two analytic functions, one regular in* $D^{(1)}$ *and the other regular in* $D^{(2)}$.

We show in Fig. 1 (below) the curves $C(r, \theta\,;\ \lambda) = 0$, for $\lambda = 1$ and $\lambda = 2$.

We proceed now to prove a theorem which plays the same part in the theory of Gauss's formula as Montmorts' theorem on the transformation of a power series plays in the theory of Newton's formula.* If D_λ denote the domain exterior to the curve $C(r, \theta\,;\ \lambda) = 0$, where λ is unrestricted

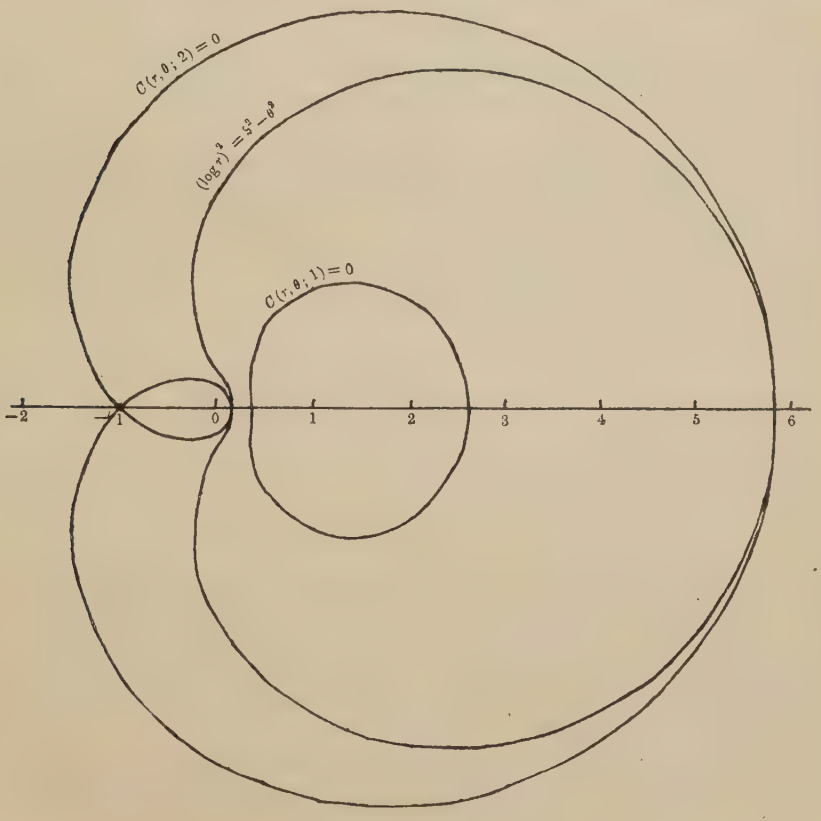

FIG. 1.

(so that D_λ includes both the origin and the point at infinity in the x-plane), and if we write

$$\Lambda = \overline{\lim_{\nu \to \infty}} \sqrt[\nu]{\{|f_\nu|\}}, \qquad 1/\Lambda' = \overline{\lim_{\nu \to \infty}} \sqrt[\nu]{\{|f_{-\nu}|\}},$$

the theorem in question is as follows.

* Chrystal's *Algebra*, 2 ed., 2, 407.

THEOREM C.—*The series* (4 . 2) *represents the one-valued branch of an analytic function*

$$\sum_{\nu=0}^{\infty} f_\nu / x^{\nu+1}$$

in the part of D_λ *exterior to the circle* $|x| = \Lambda$, *and the one-valued branch*

$$-\sum_{\nu=1}^{\infty} f_{-\nu} x^{\nu-1}$$

in the part of D_λ *interior to the circle* $|x| = \Lambda'$.

The proof of this theorem is immediate, for if we develop each term of the series (4 . 2) in negative powers of x, we find for the coefficient of $x^{-\nu-1}$ the expression

$$f_0 + \nu \Delta f_0 + \sum_{n-1}^{\nu-1} \left\{ \frac{(2n+1)(2n+2)\dots(\nu+n-1)}{(\nu-n-1)!} \Delta^{2n} f_{-n} + \frac{(2n+2)(2n+3)\dots(\nu+n)}{(\nu-n-1)!} \Delta^{2n+1} f_{-n} \right\}.$$

The term in brackets may be written

$${}_{\nu+n-1}C_{2n} \Delta^{2n} f_{-n} + {}_{\nu+n}C_{2n+1} \Delta^{2n+1} f_{-n},$$

so that, by (4 . 11) and (4 . 13), the above expression reduces to f_ν.

If we develop each term of the series (4 . 2) in positive powers of x, we find in a similar way that the coefficient of $-x^{\nu-1}$ is $f_{-\nu}$.

The theorem follows, therefore, from the theory of double series and Lemma ζ with its corollary.

Now we have seen that the series (4 . 2) defines, in its domain of convergence, a single analytic function when $\lambda < 2$, so that we have the following

COROLLARY.—*A sufficient condition that the two series* $\Sigma f_\nu / x^{\nu+1}$ *and* $-\Sigma f_{-\nu} x^{\nu-1}$ *should represent the same branch of an analytic function is that*

$$\overline{\lim_{\nu \to \infty}} \sqrt[\nu]{\{|\Delta^\nu f_{-[\nu/2]}|\}} < 2.$$

The result expressed by this corollary will be required in the sequel, and is not without intrinsic interest, for it provides a contribution to the classical theory of functions defined by Taylor's series.

5. The following lemma will enable us to prove the analogue for Gauss's formula of Pincherle's theorem regarding Newton's formula.*

LEMMA η.—*Any function of x regular in D_λ can be expressed in the form*

$$\frac{e_0}{x-1}+\frac{e_1}{(x-1)^2}+\sum_{n=1}^{\infty}\left\{\frac{x^{n-1}e_{2n}}{(x-1)^{2n+1}}+\frac{x^n e_{2n+1}}{(x-1)^{2n+2}}\right\} \tag{5.1}$$

in this domain.

Let $f(x)$ be regular in D_λ and suppose for simplicity that $\lambda < 2$. Let x be any point in this domain and let

$$\left|\frac{(x-1)^2}{x}\right| = \lambda_2,$$

so that $\lambda_2 > \lambda$. Choose as contour Γ the curve $C(r, \theta;\ \lambda_1) = 0$, where $\lambda_1 < 2$ and $\lambda < \lambda_1 < \lambda_2$, and consider the identity

$$\begin{aligned}\frac{1}{z-x} = &-\frac{1}{x-1}-\frac{z-1}{(x-1)^2}\\ &-\sum_{r=1}^{n}\left\{\frac{x^{r-1}(z-1)^{2r}}{z^r(x-1)^{2r+1}}+\frac{x^r(z-1)^{2r+1}}{z^r(x-1)^{2r+2}}\right\}+\frac{x^n(z-1)^{2n+2}}{(z-x)\,z^n(x-1)^{2n+2}}.\end{aligned} \tag{5.2}$$

Multiplying each term of (5.2) by $f(z)/2\pi i$ and integrating with respect to z round Γ in the negative sense, we have by Cauchy's theorem

$$f(x) = \frac{e_0}{x-1}+\frac{e_1}{(x-1)^2}+\sum_{r=1}^{n}\left\{\frac{x^{r-1}e_{2r}}{(x-1)^{2r+1}}+\frac{x^r e_{2r+1}}{(x-1)^{2r+2}}\right\}+R_n,$$

where

$$e_{2r} = -\frac{1}{2\pi i}\int_\Gamma \frac{(z-1)^{2r}}{z^r}f(z)\,dz,$$

$$e_{2r+1} = -\frac{1}{2\pi i}\int_\Gamma \frac{(z-1)^{2r+1}}{z^r}f(z)\,dz,$$

$$R_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\,x^n(z-1)^{2n+2}}{(z-x)\,z^n(x-1)^{2n+2}}\,dz,$$

and all these integrals are taken in the negative sense.

The lemma will be established if we can show that $R_n \to 0$ as $n \to \infty$.

Let δ be the lower bound of $|z-x|$ and M the upper bound of $|f(z)|$ for all points z on Γ. Then if L be the length of Γ, we have

$$|R_n| \leqslant \frac{ML}{2\pi\delta}\left|\frac{x}{z}\right|^n\left|\frac{z-1}{x-1}\right|^{2n+2}$$

* Pincherle, *loc. cit.*, 65.

which tends to zero since

$$\left|\frac{(z-1)^2}{z}\right| = \lambda_1 < \left|\frac{(x-1)^2}{x}\right| = \lambda_2.$$

Since $\lambda_2 - \lambda$ may be as small as we please, the lemma is established.

We are now in a position to prove

THEOREM D.—*The necessary and sufficient condition that an integral function $f(z)$ be expressible in the form* (1 . 1), *with the condition*

$$\lambda = \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|e_\nu|} < 2,$$

is that it should be of the form

$$\frac{1}{2\pi i}\int_\Gamma x^{z-1}\phi(x)\,dx,$$

where $\phi(x)/x$ is a regular function of x in D_λ and Γ is as above defined.

The condition is sufficient. For suppose that $\phi(x)/x$ has the prescribed property; then it can be expanded in the form (5 . 1) by Lemma η. We have therefore

$$f(z) = \frac{1}{2\pi i}\int_\Gamma x^z \frac{\phi(x)}{x}\,dx$$

$$= \frac{1}{2\pi i}\int_\Gamma x^z \left\{\frac{e_0}{x-1} + \frac{e_1}{(x-1)^2} + \sum_{n=1}^{\infty}\left(\frac{x^{n-1}e_{2n}}{(x-1)^{2n+1}} + \frac{x^n e_{2n+1}}{(x-1)^{2n+2}}\right)\right\} dx.$$

Now x^z is a regular function of x on Γ, and the series which appears under the integral sign is uniformly convergent by the corollary to Lemma ϵ, so that, integrating term by term, we find

$$f(z) = e_0 + e_1 z + \sum_{n=1}^{\infty}\left\{\frac{(z+n-1)(z+n-2)\ldots(z-n)}{(2n)!}e_{2n} + \frac{(z+n)(z+n-1)\ldots(z-n)}{(2n+1)!}e_{2n+1}\right\},$$

which is Gauss's formula.

The condition is also necessary. For suppose that $f(z)$ admits the expansion (1 . 1) and that

$$\lambda = \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|e_\nu|} < 2.$$

Then the corresponding series (5 . 1) defines a function regular in D_λ by

Lemma ξ. If we denote this function by $\phi(x)/x$ it follows from the first part of the proof that the integral

$$\frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx$$

admits precisely the same expansion as $f(z)$, and the theorem is completely proved.

6. The two lemmas which follow will lead us to the proof of our second main theorem, which expresses the conditions which the integral function $f(z)$ must satisfy without explicit reference to the generating function $\phi(x)$.

The first lemma is as follows :—

LEMMA θ.—*If $\phi(x)/x$ is regular in D_λ and $\lambda < 2$, then the integral*

$$\frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx \tag{6.1}$$

defines an integral function $f(z)$ of order 1 and finite type, which assumes the values f_0, f_1, f_{-1}, for $z = 0, 1, -1, \ldots$, where $f_0, f_1, f_2, \ldots$, and $f_{-1}, f_{-2}, \ldots$, are the coefficients in the expansions of $\phi(x)/x$ in the neighbourhoods of the point at infinity and the origin respectively.

We may take the contour Γ to be the curve $C(r, \theta\,;\, \lambda') = 0$, where $\lambda < \lambda' < 2$. Then Lemma α implies that the integral (6.1) defines an integral function $f(z)$ of order 1 and finite type and such that

$$f(\nu) = f_\nu \quad \text{for} \quad \nu = 0, 1, 2, \ldots.$$

We have also

$$f(z) = \frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx = -\frac{1}{2\pi i}\int_\Gamma x^{-z-1}\,\phi\left(\frac{1}{x}\right)dx,$$

on making the substitution $(x, 1/x)$ and utilising the fact that Γ possesses the I-property. Observing that the expansion of $\phi(1/x)$ in the neighbourhood of the point at infinity is of the form

$$-\sum_{\nu=1}^{\infty} f_{-\nu}/x^\nu,$$

and appealing for a second time to Lemma α, we conclude that

$$f(\nu) = f_\nu \quad \text{for} \quad \nu = -1, -2, \ldots.$$

The proof of the lemma is therefore complete.

We now come to

Lemma 1.—*The curve*

$$(6.21) \qquad (\log r)^2 = \mathfrak{S}^2 - \theta^2 \quad \text{where} \quad \mathfrak{S} = \log(3+2\sqrt{2}), \\ = \operatorname{arc\,cosh} 3,$$

is contained within the curve

$$(6.22) \qquad r^2 - 2(\cos\theta + 2)r + 1 = 0.$$

Both curves are shown in Fig. 1 (p. 157). Now it is evident, in the first place, that the curves touch each other where $\theta = 0$, $r = e^{\pm\mathfrak{S}}$, and that the greater radius vector of (6.22), which corresponds to the value $\theta = \mathfrak{S}$, is greater than unity. In the second place, since the first curve lies entirely within the angle $|\theta| \leqslant \mathfrak{S}$ and since both curves are symmetrical with respect to the axis $\theta = 0$, it is sufficient to prove the result for $0 < \theta < \mathfrak{S}$.

Since both curves possess the I-property, they intersect (or touch) if, and only if,

$$F(\theta) = 2 + \cos\theta - \cosh\sqrt{(\mathfrak{S}^2 - \theta^2)} = 0.$$

But we have $$F'(\theta) = -\sin\theta + \frac{\theta \sinh\sqrt{(\mathfrak{S}^2-\theta^2)}}{\sqrt{(\mathfrak{S}^2-\theta^2)}};$$

and $$F(0) = F'(0) = 0.$$

Also, since

$$\frac{\sin\theta}{\theta} < 1 < \frac{\sinh\sqrt{(\mathfrak{S}^2-\theta^2)}}{\sqrt{(\mathfrak{S}^2-\theta^2)}} \quad \text{for} \quad 0 < \theta < \mathfrak{S},$$

it follows that $F'(\theta) < 0$ for these values of θ.

Hence $F(\theta) \neq 0$ throughout the interval $0 < \theta < \mathfrak{S}$, which proves the lemma.

7. We now proceed to prove

Theorem **E**.—*In order that the integral function $f(z)$ can be expressed by Gauss's formula, with $\lambda < 2$, it is necessary that $f(z)$ should be of order* 1 *and finite type, and sufficient that it be of order* 1 *and type less than* $\log(3+2\sqrt{2})$.

To prove the first part of the proposition we suppose $f(z)$ expressible in the form (1.1); then, by Theorem **D**, $f(z)$ is of the form

$$\frac{1}{2\pi i}\int_\Gamma x^{z-1}\phi(x)\,dx,$$

where $\phi(x)/x$ is a regular function of x in D_λ, and $\lambda < 2$. Lemma θ implies, therefore, that $f(z)$ is of order 1 and finite type.

Now let $f(z)$ be an integral function of order 1 and type $\vartheta < \log(3+2\sqrt{2})$. Then the generating function

$$\frac{\phi(x)}{x} = \sum_{\nu=0}^{\infty} \frac{f(\nu)}{x^{\nu+1}}$$

is regular in the domain exterior to the curve

$$(\log r)^2 = \vartheta^2 - \theta^2 \tag{7.1}$$

by the corollary to Theorem **B**.

But since $\vartheta < \log(3+2\sqrt{2})$ there exists, by Lemma ι, a $\lambda < 2$, and such that the curve

$$C(r, \theta; \lambda) = 0 \tag{7.2}$$

encloses (7.1). Let Γ denote the curve (7.2). Then the integral

$$\frac{1}{2\pi i} \int_\Gamma x^{z-1} \phi(x)\, dx$$

satisfies the condition that it should be expressible by Gauss's formula, with $\lambda < 2$. It represents also by Theorem **B** an integral function of order 1 and type ϑ which assumes the values $f(0)$, $f(1)$, $f(2)$, ... for $z = 0, 1, 2, \ldots$.

Appealing finally to Theorem **A**, we conclude, since $\vartheta < \pi$, that this integral is no other than the given function $f(z)$. Theorem **E** is therefore established.

While the theorem which we have just proved supplies all the information required for applications of the interpolation theory, it is natural to inquire whether it is not possible to obtain conditions which are at once necessary and sufficient.* This question may be answered in the affirmative. To obtain the result we have merely to appeal to Theorem **D** and to investigate the maximum value of $|x^z|$ as x describes the contour $C(r, \theta; \lambda) = 0$, where $\lambda < 2$. Let us suppose that $|z| = \rho$ has a fixed value; then we have to find the maximum value $M(\psi)$ of the expression

$$\theta \sin\psi - \cos\psi \log r,$$

* By a suitable choice of the tabular interval h, it is possible to express any integral function of order 1 and finite type in the form

$$e_0 + e_1 \frac{z}{h} + e_2 \frac{z(z-h)}{2!\, h^2} + e_3 \frac{(z+h)\, z(z-h)}{3!\, h^3} + \ldots .$$

where r and θ are connected by the relation

$$r^2-(2\cos\theta+\lambda^2)r+1=0.$$

We find easily that

$$M(\psi)=\sin\psi\arccos\tfrac{1}{2}\{\sqrt{(4-\lambda^4\sin^2\psi\cos^2\psi)}-\lambda^2\sin^2\psi\}$$
$$-\cos\psi\log\tfrac{1}{2}[\sqrt{(4-\lambda^4\sin^2\psi\cos^2\psi)}+\lambda^2\cos^2\psi$$
$$-\lambda\cos\psi\sqrt{\{\lambda^2\cos 2\psi+2\sqrt{(4-\lambda^4\sin^2\psi\cos^2\psi)}\}}].$$

Appealing finally to a theorem on integral functions proved by Carlson,* we obtain the required result in the following form:—

THEOREM **F.**—*The necessary and sufficient condition that $f(z)$ can be expressed by Gauss's formula, with $\lambda<2$, is that $f(z)$ should be an integral function such that*

$$|f(\rho e^{i\psi})|<e^{\{M(\psi)+\epsilon\}\rho}$$

for every positive ϵ and all sufficiently large values of ρ.†

Since $M(\psi)=0$ for all ψ, when $\lambda=0$, we may add the

COROLLARY.—*The necessary and sufficient condition that $f(z)$ can be expressed by Gauss's formula, with $\lambda=0$, is that $f(z)$ should be an integral function of order* 1 *and type* 0.

8. The next theorem which we wish to prove reveals a remarkable property of the factorial series (1.1).

THEOREM **G.**—*If the series*

$$(8.11)\qquad f_0+z\Delta f_0+\frac{z(z-1)}{2!}\Delta^2 f_{-1}+\frac{(z+1)z(z-1)}{3!}\Delta^3 f_{-1}+\dots$$

converges and $\overline{\lim_{\nu\to\infty}}\sqrt[\nu]{|\Delta^\nu f_{-[\nu/2]}|}<2,$

then its sum can be expressed in the form

(8.12)

$$\frac{\sin\pi z}{\pi}\left\{a^z\sum_{\nu=0}^{\infty}(-1)^\nu\frac{f_\nu}{a^\nu(z-\nu)}+a^{-z}\sum_{\nu=1}^{\infty}(-1)^\nu\frac{f_{-\nu}}{a^\nu(z+\nu)}-\int_{1/a}^{a}x^{z-1}\phi(-x)\,dx\right\},$$

---

* *Vide Dissertation*, 54.

† The corresponding theorem for Newton's formula is proved by Carlson in his memoir on binomial coefficient series, 60.

where $\phi(x)$ *is defined by either of the series*

$$\sum_{\nu=0}^{\infty} f_\nu/x^\nu, \qquad -\sum_{\nu=1}^{\infty} f_{-\nu}x^\nu,$$

and the sum of the series (8 . 12) *is independent of the real number* a.

Let us denote the sum of the series (8 . 11) by $f(z)$; then we know by Theorem **D** that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx,$$

where $\phi(x)$ is defined by (4 . 2) and Γ is the contour $C(r, \theta\,;\ \lambda) = 0$, λ being such that

$$\overline{\lim_{\nu\to\infty}}\ \sqrt[\nu]{|\,\Delta^\nu f_{-[\nu/2]}\,|} < \lambda < 2.$$

Suppose a cut made along the negative real axis in the x-plane, and choose a real number $a > 0$ and such that

$$1/a < \text{Min}\ (\Lambda,\ \Lambda') \leqslant \text{Max}\ (\Lambda,\ \Lambda') < a,$$

where Λ and Λ' are as defined in §4.

Consider the contour Γ' formed by (1) the circle $r = a$, described in the positive sense and meeting the upper and lower edges of the cut at B and B' respectively; (2) the circle $r = 1/a$, described in the negative sense and meeting the upper and lower edges of the cut in A and A' respectively; and (3) the two straight lines BA and $A'B'$.

Denoting the circles (1) and (2) by C_1 and C_2 respectively, we have

$$f(z) = \frac{1}{2\pi i}\int_\Gamma x^{z-1}\,\phi(x)\,dx = \frac{1}{2\pi i}\int_{\Gamma'} x^{z-1}\,\phi(x)\,dx$$

$$= \frac{1}{2\pi i}\left\{\int_{C_1} - \int_{C_2} - \int_A^B + \int_{A'}^{B'}\right\} x^{z-1}\,\phi(x)\,dx,$$

where the integrals round C_1 and C_2 are both taken in the positive sense.

But, by Theorem **C**,

$$\phi(x) = \sum_{\nu=0}^{\infty} f_\nu/x^\nu \quad \text{on} \quad C_1,$$

$$= -\sum_{\nu=1}^{\infty} f_{-\nu}x^\nu \quad \text{on} \quad C_2,$$

and each series is uniformly convergent on the corresponding contour.

Now $\phi(x)$ is one-valued in D_λ so that

$$\phi(re^{\pi i}) = \phi(re^{-\pi i}) = \phi(-r).$$

Therefore

$$f(z) = \frac{1}{2\pi i} \sum_{\nu=0}^{\infty} f_\nu \int_{C_1} x^{z-\nu-1}\, dx + \frac{1}{2\pi i} \sum_{\nu=1}^{\infty} f_{-\nu} \int_{C_2} x^{z+\nu-1}\, dx$$

$$- \frac{\sin \pi z}{\pi} \int_{1/a}^{a} r^{z-1}\, \phi(-r)\, dr.$$

Observing that

$$\frac{1}{2\pi i} \int_{C_1} x^{z-\nu-1}\, dx = (-1)^\nu \frac{a^{z-\nu} \sin \pi z}{\pi(z-\nu)},$$

and

$$\frac{1}{2\pi i} \int_{C_2} x^{z+\nu-1}\, dx = (-1)^\nu \frac{a^{-z-\nu} \sin \pi z}{\pi(z+\nu)},$$

we are led immediately to the result expressed by the theorem.

It is possible that the interpolation formula (8.12) might be of considerable use in practical mathematics. The constant a plays the rôle of a convergence factor, and by a suitable choice of this number it may be possible to ensure that the expansions which occur are highly convergent. A particular case of this formula has been known for some time.*

Thus, suppose that, for all points z belonging to the set S defined in § 3, the two series

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu f_\nu}{z-\nu}, \qquad \sum_{\nu=1}^{\infty} \frac{(-1)^\nu f_{-\nu}}{z+\nu}$$

are convergent. Then, observing that a^z and a^{-z} are continuous functions of a and applying Abel's theorem on power series, we may let $a \to 1$, and we obtain the formula

$$\frac{\sin \pi z}{\pi} \sum_{\nu=-\infty}^{\infty} \frac{(-1)^\nu f_\nu}{z-\nu}, \tag{8.2}$$

which was discussed by de la Vallée Poussin.†

Now Whittaker has shown that a certain class of integral functions which admit the expansion (8.11) can be expressed in the form (8.2).‡

* *Vide* E. Borel, "Sur la recherche des singularités d'une fonction définie par un développement de Taylor", *Comptes Rendus*, 127 (1898), 1001–1003 (1002).

† Many similar formulae have been defined; see, *e.g.*, F. G. Teixeira, "Sur la convergence des formules d'interpolation de Lagrange, de Gauss, etc.", *Journal für Math.*, 126 (1903), 116–162 (141 *et seq.*).

‡ E. T. Whittaker, "On the Functions which are represented by the Expansions of the Interpolation Theory", *Proc. Royal Soc. Edinburgh*, 35 (1915), 181–194 (192).

But it may happen that the series (8 . 11) converges (with $\lambda < 2$) although (8 . 2) does not. The function cosh ($z \log 3$), for example, can be expressed in the first form but not in the second. The convergence of the series (8 . 11), however, implies that of (8 . 12) for an infinity of values of a, and the fundamental importance of the latter formula is due to the fact that it represents the unique integral function, of least croissance, which takes the values f_ν for integral values of its argument.

9. The problem of determining general conditions under which a function $f(z)$ can be expressed by Gauss's formula, with $\lambda = 2$, would appear to be difficult. In this case the series (4 . 2) may represent two different functions, each of which is regular in one of the two connex domains formed by the "exterior" of the curve $C(r, \theta\,;\ 2) = 0$. The convergence of Gauss's formula depends upon the "order" of these two functions on this curve. We shall confine ourselves in the present paper to the discussion of a particular case of special interest.

Suppose that $g(z)$ is an analytic function which satisfies the two following conditions :—

(1) $g(z)$ is one-valued and possesses only a finite number of singular points :

(2) to every ϵ there corresponds a positive number K, such that

$$|g(\rho e^{i\psi})| < e^{\epsilon\rho} \quad \text{for} \quad \rho > K.$$

Suppose also that

$$g(\pm\nu) = O(\nu^{-\alpha}), \quad \alpha > 0, \tag{9 . 1}$$

and that the two series

$$\sum_{\nu>0} |g(\nu+1)-g(\nu)|, \qquad \sum_{\nu>0} |g(-\nu-1)-g(-\nu)| \tag{9 . 2}$$

are convergent.

We write

$$f_\nu = (-1)^\nu g(-\nu),$$

and we make the stipulation that, if $g(z)$ has a pole for any integral value of z, the corresponding value of f_ν is to be taken equal to zero.

If we make a cut along the segment $(0, -1)$ of the real axis in the

x-plane, the generating function

$$\phi_1(x) = \sum_{\nu=0}^{\infty} f_\nu/x^\nu = \sum_{\nu=0}^{\infty} (-1)^\nu g(-\nu)/x^\nu$$

is regular in the cut plane.*

We have also, for $|x| < 1$,

$$\phi_1(x) = -\sum_{\nu=1}^{\infty} f_{-\nu} x^\nu + \chi(x) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} g(\nu) x^\nu + \chi(x),$$

where $\chi(x)$ denotes the sum of the residues of

$$-\frac{\pi g(z) x^z}{\sin \pi z}$$

relative to all the singularities of $g(z)$, which are at a finite distance. $\chi(x)$ is evidently an analytic function of x, with the sole singularity $x = 0$ in the finite part of the plane.

This being so, we proceed to prove

THEOREM **H.**—*Gauss's formula* (8.11) *converges, with* $\lambda = 2$, *for the sequence* $\{f_\nu\}$ *in question and represents the integral function*

$$f(z) = \frac{1}{2\pi i} \int_{-1}^{(0+)} x^{z-1} \chi(x)\, dx = \frac{\sin \pi z}{\pi} \sum_{\nu=-\infty}^{\infty} \frac{(-1)^\nu f_\nu}{z-\nu}.$$

Consider $\int x^{z-1} \phi_1(x) dx$ taken round the contour $C(r, \theta\,;\ 2) = 0$. If we indent this contour by means of two arcs γ_1 and γ_2 of the circle $|x+1| = \delta$, one above and one below the real axis, the integrand is a regular function of x in and on the indented contour.

But the condition (9.1) and the convergence of the first of the two series (9.2) imply that

$$\lim (1+x) \Sigma f_{-\nu} x^\nu = \lim_{\nu \to \infty} g(\nu) = 0,$$

when $x \to -1$ through any sequence of points which lie within or on the unit circle. We have, similarly,

$$\lim \left(1 + \frac{1}{x}\right) \Sigma f_\nu / x^\nu = 0,$$

* Lindelöf, *loc. cit.*, 124.

when $x \to -1$ through any sequence of points which lie without or on the unit circle. It follows that, on the arcs γ_1 and γ_2, we have

$$\lim_{\delta \to 0} |(x+1)\,\phi_1(x)| = 0.$$

If therefore we let $\delta \to 0$, the integrals relative to the arcs γ_1 and γ_2 vanish and we have, by Cauchy's theorem,

$$\int_C x^{z-1}\,\phi_1(x)\,dx = 0, \tag{9.3}$$

where C denotes the contour $C(r, \theta;\ 2) = 0$.

Now let C_1 denote the larger and C_2 the smaller of the two $\Gamma_{(1)}$ contours formed by C; then (9.3) leads to

(9.4)

$$\frac{1}{2\pi i}\int_{C_1} x^{z-1} \sum_{\nu=0}^{\infty} \frac{f_\nu}{x^\nu}\,dx + \frac{1}{2\pi i}\int_{C_2} x^{z-1} \sum_{\nu=1}^{\infty} f_{-\nu}\,x^\nu\,dx = \frac{1}{2\pi i}\int_{C_2} x^{z-1}\,\chi(x)\,dx,$$

where the contours C_1 and C_2 are both described in the positive sense.

If we integrate term by term the series which occur in the left-hand member of (9.4), we obtain the series

$$\frac{\sin \pi z}{\pi} \sum_{\nu=-\infty}^{\infty} \frac{(-1)^\nu f_\nu}{z-\nu},$$

which evidently converges in virtue of the condition (9.1). We conclude, therefore, that this term by term integration is justified. Now Whittaker has shown that the function defined by this series can be expressed by Gauss's formula in the form (8.11) and that this last series is convergent.*

To prove that $\lambda = 2$ we observe that, if $\lambda > 2$, the series (8.11) cannot converge by Lemma γ; while, if $\lambda < 2$, we must have $\chi(x) \equiv 0$, by the corollary to Theorem **C**.

Theorem **H** is therefore established.

From this theorem and the argument of Lemma α we easily derive the

COROLLARY.—*The integral*

$$\frac{1}{2\pi i}\int_{-1}^{(0+)} x^{z-1}\,\chi(x)\,dx$$

defines an integral function of order 1 *and finite type, which assumes the values* $f_0, f_1, f_{-1}, \ldots$, *for* $z = 0, 1, -1, \ldots$.

* *Loc. cit.*, 189 and 191.

As an example consider the sequence

$$f_\nu = \frac{(-1)^\nu}{1+\nu^2} \quad (\nu = 0, \pm 1, \pm 2, \ldots),$$

where the prescribed conditions are evidently satisfied.

In this case $\chi(x)$ is the sum of the residues of

$$-\frac{\pi x^z}{(1+z^2)\sin \pi z}$$

with respect to the poles of the function

$$g(z) = \frac{1}{1+z^2}.$$

Hence
$$\chi(x) = \frac{\pi \cos(\log x)}{\sinh \pi}.$$

The function represented by (8.11) is then

$$\frac{1}{2i \sinh \pi} \int_{-1}^{(0+)} x^{z-1} \cos(\log x)\, dx = \frac{\cos \pi z}{1+z^2} + \frac{\cosh \pi}{\sinh \pi} \cdot \frac{z \sin \pi z}{1+z^2}$$

$$= \frac{\sin \pi z}{\pi} \sum_{\nu=-\infty}^{\infty} \frac{1}{(1+\nu^2)(z-\nu)}.$$

10. The case in which $f_\nu = g(-\nu)$ can be treated in a similar manner, and the details of the analysis may be left to the reader. In this case the generating function has no singular point on the contour $C(r, \theta;\ 2) = 0$ and the convergence of Gauss's formula is less precarious. We shall conclude by applying the preceding results to Runge's problem. If we seek to express, for example, the function

$$g(z) = \frac{1}{1+z^2}$$

by the factorial series

$$e_0 + e_1 \frac{z}{h} + e_2 \frac{z(z-h)}{2!\ h^2} + e_3 \frac{(z+h)z(z-h)}{3!\ h^3} + \ldots \tag{10.1}$$

where h is real and the coefficients $e_0, e_1, e_2, \ldots$, depend only upon the values of $g(z)$ for $z = 0, h, -h, \ldots$, we find that this series represents the integral function

$$f(z) = \frac{1}{1+z^2} + \frac{z}{1+z^2} \frac{\sin \pi z/h}{\sinh \pi/h}.$$

We write $z = \tau + it$ and we shall show that $f(z)$ "converges" to $g(z)$ in the infinite strip of the z-plane defined by

$$|t| < 1.* \tag{10.2}$$

To establish this, we have merely to show that the expression

$$\frac{\sin \pi z/h}{\sinh \pi/h}$$

tends to zero with h for all points z of the strip. Now

$$\lim_{h\to 0}\left|\frac{\sin \pi z/h}{\sinh \pi/h}\right| = \lim_{h\to 0}\left|\frac{\sinh \pi t/h}{\sinh \pi/h}\right| \sqrt{\left\{1+\frac{\sin^2 \pi\tau/h}{\sinh^2 \pi t/h}\right\}} = 0,$$

provided (10.2) is satisfied.

The result is therefore proved; and its generalisation to the case in which $g(z)$ is a rational function, such that the corresponding series (10.1) converges, is immediate. The region of convergence is an infinite strip whose edges are parallel to and equidistant from the real axis and whose breadth is determined by the position of the singular points of the function.

* It follows from de la Vallée Poussin's investigation that convergence takes place for every real value of z.

ÜBER TAUBERSCHE BEDINGUNGEN.

Von LUDWIG NEDER, in Leipzig.

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VORLIEGENDE Arbeit beschäftigt sich mit der Bedeutung, welche in der Theorie der *Dirichletschen Reihen*

$$(1) \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (s = \sigma + i\tau, \quad \lambda_n \geqslant 0),^1$$

den folgenden *Tauberschen Bedingungen*

$$(o) \qquad a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

$$(o_1) \qquad \Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^{p-1} |a_n|^p \quad \text{konvergiert} \quad (p > 1),$$

$$(o_2) \qquad \sum_{x \leqslant \lambda_n \leqslant 2x} |a_n| = o(1),^2$$

$$(O) \qquad a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \quad \text{und} \quad = o(1),$$

$$(O_1) \qquad \overline{\lim_{x=\infty}} \sum_{x \leqslant \lambda_n \leqslant (1+\delta)x} |a_n| < \eta(\delta), \quad \text{wobei} \quad \lim_{\delta=0} \eta(\delta) = 0,$$

$$(O_2) \qquad \sum_{x \leqslant \lambda_n \leqslant 2x} |a_n| = O(1)^2$$

zukommt.

Wir betrachten zu diesem Zweck zunächst im § 1 die *Abhängigkeitsbeziehungen*, welche zwischen unseren Tauberschen Bedingungen bestehen. Schreibt man kurz $A \leqslant B$ oder $B \geqslant A$, um anzudeuten, dass B aus A folgt (diese Relation ist transitiv), so wird sich ergeben

$$\left.\begin{matrix} o \\ o_1 \end{matrix}\right\} \leqslant o_2, \qquad \left.\begin{matrix} o_2 \\ O \end{matrix}\right\} \leqslant O_1 \leqslant O_2.$$

[1] Letzteres ohne Beschränkung der Allgemeinheit, da es hier nirgends auf die Anfangsglieder ankommt.

[2] Statt 2 könnte jede feste Zahl > 1 stehen.

Es umfasst also $\mathbf{O}_1$ und erst recht $\mathbf{O}_2$ die (mit einer einzigen Ausnahme[3] bisher allein benutzten) Bedingungen $\mathbf{o}$, $\mathbf{o}_1$, und $\mathbf{O}$, und es wird die Einführung von $\mathbf{O}_1$ bzw. $\mathbf{O}_2$ statt $\mathbf{o}$, $\mathbf{o}_1$ und $\mathbf{O}$ nicht nur eine Verallgemeinerung, sondern auch eine Zusammenfassung der bisher bekannten Sätze zur Folge haben.

Unsere dahin zielenden Ergebnisse werden in zwei Richtungen liegen:

I.

Einerseits beschäftigen wir uns mit der *Dirichletschen Multiplikation konvergent vorausgesetzter Reihen*, d.h. mit der Konvergenz des formalen Produkts Dirichletscher Reihen (1) für $s = 0$, wobei man sich auf Reihen desselben λ-Typus beschränken darf.

Es handelt sich also zunächst *im Falle zweier Reihen* um die Beziehung

$$\lim_{x=\infty} \sum_{\lambda_m+\lambda_n \leqq x} a_m b_n = \Sigma a_m . \Sigma b_n. \tag{2}$$

Dieselbe ist als richtig bekannt für folgende Paare von durch Σa_m, Σb_n zu erfüllenden Tauberschen Bedingungen: Für $\mathbf{O}+\mathbf{O}$ durch die Herren Hardy[4] und Rosenblatt,[5] für $\mathbf{o}_1+\mathbf{o}_1$ durch die Herren Hardy und Littlewood,[6] während die Zulässigkeit der dritten Kombination $\mathbf{O}+\mathbf{o}_1$ noch in Frage steht[7] und nur die von $\mathbf{o}+\mathbf{o}_1$ sichergestellt ist.[7]

Nun zeigt ein Blick auf den Landauschen Beweis des $(\mathbf{O}+\mathbf{O})$-Satzes,[5] dass (2) auch noch unter dem allgemeinstmöglichen Bedingungspaar $\mathbf{O}_2+\mathbf{O}_2$ gilt. Man braucht in der Tat nur zu beachten, dass die am Schluss gebrauchte Relation

$$\sum_{\frac{1}{2}x < \lambda_m < x-y} |a_m| = O(1)$$

unmittelbar aus $\mathbf{O}_2$ folgt.

Dieser neue Satz umfasst alle vorgenannten und erledigt auch die obige Frage ohne weiteres in bejahendem Sinne.

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[3] Herr Landau hat gelegentlich auf eine mit $\mathbf{O}_1$ gleichwertige Bedingung aufmerksam gemacht. Vgl. l.c. [23], S. 271, Vorbemerkung.

[4] "On the multiplication of Dirichlet's series", *Proc. London Math. Soc.* (2), 10 (1912), 396.

[5] Vgl. E. Landau, "Über einen Satz des Herrn Rosenblatt", *Jahresber. Deutsche Math. Ver.*, 29 (1920), 238.

[6] "Some theorems concerning Dirichlet's series", *Messenger of Math.* (New Series), 43 (1914), 134, vgl. Theorem B.

[7] Ebenda, vgl. S. 139 unten, 140 oben.

Was nunmehr die *Multiplikation von mehr als zwei Reihen* angeht, so ist darüber nur für $\lambda_n = n$ einiges bekannt:

Die Herren Hardy und Littlewood[8] haben bewiesen, dass aus

(K) $\qquad \Sigma a_n$ konvergiert und ist $= A$

und **O** die Summabilität $(C, -1+\delta)$ folgt für jedes $\delta > 0$, und dass daher für beliebig viele konvergente Reihen mit $a_n = O(1/n)$ die Cauchysche Produktreihe stets konvergiert [sogar summabel $(C, -1+\delta)$ ist]. Ferner haben sie gezeigt,[9] dass aus **K** und $\mathbf{o}_1$ die Summabilität $(C, -1+1/p+\delta)$ folgt, sodass für beliebig viele konvergente, je einer Bedingung $\mathbf{o}_1$ unterworfene Reihen die Cauchysche Produktreihe konvergiert, falls die zugehörige $\Sigma 1/p < 1$ ist.

Wir werden nachstehend im § 2 ohne Heranziehung von Summabilitätsbegriffen ganz elementar erhalten:

SATZ 1.—*Die Dirichletsche Produktreihe beliebig vieler konvergenter, der Bedingung* $\mathbf{O}_2$ *unterworfener Reihen ist stets konvergent und hat den "richtigen" Wert.*[10]

II.

Andererseits beschäftigen wir uns mit der Einführung unserer Bedingungen $\mathbf{O}_1$ und $\mathbf{O}_2$ in die sog. Abelschen und Tauberschen Sätze, von denen unter den Bedingungen **o**, $\mathbf{o}_1$ und **O** eine grosse Anzahl bekannt geworden ist. Unsere Ergebnisse lauten, wenn wir uns noch der Abkürzungen

$$\left.\begin{aligned} &\text{(L)} && f(s) = \Sigma a_n e^{-\lambda_n s} \\ &\text{(L*)} && F(s) = \Sigma a_n \frac{1-e^{-\lambda_n s}}{\lambda_n s} \end{aligned}\right\} \to A, \quad \text{wenn} \quad s \to 0,^{11}$$

bedienen, summarisch folgendermassen:

Im § 3: *Sätze, welche von* **K** *auf* **L**, **L*** *etc. schliessen (Abelsche Sätze), gelten auch noch unter der allgemeinsten Bedingung* $\mathbf{O}_2$.

Im § 4: *Sätze, welche von* **L**, **L*** *etc. auf* **K** *schliessen (Taubersche Sätze), gelten noch unter der Bedingung* $\mathbf{O}_1$, *jedoch nicht stets unter der Bedingung* $\mathbf{O}_2$.

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[8] "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1913), 411, vgl. Theorem 37 und 38.

[9] Vgl. l.c. [6], Theorem C.

[10] Natürlich bleiben daneben die zuvor genannten, unter schärferen Voraussetzungen mehr ergebenden Sätze nach wie vor von Bedeutung.

[11] Längs einer anzugebenden Punktmenge.

Da die neue Form der genannten Sätze nicht nur die bisherigen o- und O-Sätze, sondern auch die o_1-Sätze umfasst, die bisher neben den vorigen eine Sonderstellung einnahmen, und da die neuen Beweise nirgends komplizierter ausfallen als bisher im Falle O, so dürfte die Einführung der Bedingungen O_1 bzw. O_2 entschieden von Vorteil sein, zumal die neuen Sätze auch Anwendungen gestatten, wo die bisherigen versagten.[12]

Alle in den §§ 3, 4 bewiesenen Sätze gelten natürlich *a fortiori* unter der Bedingung o_2. Diese Sonderfälle sind durchaus von dem trivialen Charakter der in ihnen speziell enthaltenen o- und o_1-Sätze, und die Herstellung von nur zu ihnen führenden Sonderbeweisen bietet nach dem Muster des Hilfssatzes 2 im § 4 keinerlei Schwierigkeit. Wir halten uns damit nicht auf.

III.

Zum Schlusse sei noch darauf hingewiesen, dass man auf *die zu den Dirichletschen Reihen analogen Integrale*

$$\phi(s) = \int_0^\infty a(u)\, e^{-us}\, du$$

vermöge einer höchst einfachen Bemerkung mit einem Schlage alle in den §§ 3, 4 enthaltenen Sätze übertragen kann.

Dagegen beweist man das Analogen unseres Multiplikationssatzes 1 wohl besser direkt, was auf Grund der im § 2 benutzten Bezeichnungen mühelos gelingt.

1. Abhängigkeiten zwischen unseren Tauberschen Bedingungen.

(1) Die Beziehung $o \leqq o_2$ liegt auf der Hand. Denn aus o folgt

$$\sum_{x \leqq \lambda_n \leqq 2x} |a_n| \leqq o(1) \cdot \Sigma \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = o(1).$$

(2) Die Beziehung $o_1 \leqq o_2$ ergibt sich auf Grund der Hölderschen Ungleichung

$$|\Sigma uv| \leqq \{\Sigma |u|^p\}^{1/p} \{\Sigma |v|^{p/p-1}\}^{p-1/p}, \quad \text{wo} \quad p > 1,$$

da ihr zufolge

$$\sum_{x \leqq \lambda_n \leqq 2x} |a_n| \leqq \left\{\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^{p-1} |a_n|^p\right\}^{1/p} \left\{\Sigma \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right\}^{p-1/p} = o(1).$$

(3) Die Beziehung $o_2 \leqq O_1$ ist trivial, da aus o_2 folgt $\eta(\delta) = 0$ für $0 < \delta < 1$.

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[12] Vgl. Satz 5′ und die darauf folgende Anwendung.

(4) Die Beziehung $\mathbf{0} \leqslant \mathbf{0}_1$ ergibt sich aus der Ungleichung

$$\sum_{x \leqslant \lambda_n \leqslant (1+\delta)x} |a_n| \leqslant o(1) + \sum_{x \leqslant \lambda_{n-1} < \lambda_n \leqslant (1+\delta)x} C \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \leqslant o(1) + C\delta.$$

(5) Die Beziehung $\mathbf{0}_1 \leqslant \mathbf{0}_2$ endlich folgt daraus, dass auf Grund von $\mathbf{0}_1$ ein $\delta > 0$ vorhanden ist, zu dem $\eta = \eta(\delta) > 0$ existiert der Art, dass für $x \geqslant x_0$

$$\sum_{x \leqslant \lambda_n \leqslant (1+\delta)x} |a_n| \leqslant \eta.$$

Wählt man dann die natürliche Zahl $m \geqslant \log 2/\log(1+\delta)$, so ist für $x \geqslant x_0$

$$\sum_{x \leqslant \lambda_n \leqslant 2x} |a_n| \leqslant \sum_{\mu=1}^{m} \sum_{(1+\delta)^{\mu-1} x \leqslant \lambda_n \leqslant (1+\delta)^{\mu} x} \leqslant m\eta = O(1).$$

2. Ein Multiplikationssatz.

Wegen der Analogie mit den Integralen wollen wir a_n als Funktion von λ_n betrachten und dann abkürzend μ statt λ_n schreiben, also $a(\mu)$ statt a_n. Ferner klammern wir im Falle mehrerer Koordinaten $\mu_1, \ldots, \mu_r$ eine Summationsgebietsbezeichnung wie z.B. $\mu_1 + \ldots + \mu_r \leqslant x$ ein, um anzudeuten, dass die Koordinaten überdies den Nebenbedingungen

(3) $$\mu_\rho \leqslant x_\rho \quad \text{mit} \quad x_\rho \geqslant x/r \quad (\rho = 1, \ldots, r)$$

unterworfen sind. Dann lautet unser Satz, für die Zwecke des Beweises etwas verallgemeinert:

Satz 1.—Wenn für $\rho = 1, \ldots, r$,

$$\Sigma a_\rho(\mu) \text{ konvergent}, \quad = A_\rho \quad \text{und} \quad \sum_{x \leqslant \mu \leqslant 2x} |a_\rho(\mu)| = O(1).$$

ist, so gilt $$\sum_{(\mu_1 + \ldots + \mu_r \leqslant x)} a_1(\mu_1) \ldots a_r(\mu_r) \to A_1 \ldots A_r,$$

und zwar gleichmässig für alle Nebenbedingungen (3).[13]

Beweis.—Für $r = 1$ ist nichts zu beweisen. Sei also $r > 1$ und die Behauptung bis $r-1$ richtig. Wir ziehen aus unserer Summe die Glieder des Produkts

$$\sum_{\mu_1 \leqslant x/r} a_1(\mu_1) \ldots \sum_{\mu_r \leqslant x/r} a_r(\mu_r) \to A_1 \ldots A_r$$

[13] Wir brauchen dieselben nur in obiger Gestalt und legen keinen Wert darauf, sie zu verallgemeinern.

heraus. Dann bleibt zu zeigen, dass der Rest gleichmässig nach Null strebt. Wir teilen ihn zu diesem Zweck in r Teilsummen $\Sigma_1, \ldots, \Sigma_r$ ein, wobei Σ_ρ aus den Gliedern besteht, bei denen

$$\mu_1 \leqslant x/r, \ \ldots, \ \mu_{\rho-1} \leqslant x/r, \quad \text{aber} \quad \mu_\rho > x/r,$$

ist.[14] Da man die ersten $\rho-1$ dieser Ungleichungen mit den Nebenbedingungen (3) vereinigen kann, sind alle Σ_ρ gleichberechtigt, und es genügt zu zeigen, dass

$$\Sigma_1 = \sum_{(x/r < \mu_1 \leqslant x-\mu_2-\ldots-\mu_r)} a_1(\mu_1) \ldots a_r(\mu_r) \to 0.$$

Nun haben wir nach Landauschem Vorbild[5] für $0 < y < x/r$ die Zerlegung

$$\Sigma_1 = \sum_{(\mu_2+\ldots+\mu_r \leqslant y)} a_2(\mu_2) \ldots a_r(\mu_r) \cdot \sum_{(x/r < \mu_1 \leqslant x-\mu_2-\ldots-\mu_r)} a_1(\mu_1)$$

$$+ \sum_{(x/r < \mu_1 < x-y)} a_1(\mu_1) \cdot \sum_{(y < \mu_2+\ldots+\mu_r \leqslant x-\mu_1)} a_2(\mu_2) \ldots a_r(\mu_r) = \Sigma_1' + \Sigma_1''.$$

In Σ_1' ist die innere Summe, absolut genommen,

$$\leqslant \left\{ \text{obere Grenze von } \left| \sum_{x/r < \mu_1 \leqslant z} a_1(\mu_1) \right| \right\} \to 0 \quad \text{bei} \quad x \to \infty,$$

also $y = y(x)$ so wählbar, dass $y \to \infty$ und $\Sigma_1' \to 0$, gleichmässig. In Σ_1'' ist dann auf die innere Summe

$$= \sum_{(\mu_2+\ldots+\mu_r \leqslant x-\mu_1)} - \sum_{(\mu_2+\ldots+\mu_r \leqslant y)}$$

der Fall $r-1$ unseres Satzes anwendbar, da im Subtrahend die Nebenbedingungen wegen $y < x/r$ ohne Einfluss sind, während sie im Minuenden lauten

$$\mu_\rho \leqslant x_\rho, \quad \text{wobei} \quad x_\rho \geqslant \frac{x}{r} = \frac{1}{r-1}\left(x - \frac{x}{r}\right) > \frac{x-\mu_1}{r-1},$$

wie es sein soll. Daher strebt bei $x \to \infty$ (also $x-\mu_1 > y \to \infty$) die innere Summe von Σ_1'' gleichmässig nach Null, und es wird wegen $\mathbf{0_2}$

$$|\Sigma_1''| \leqslant o(1) \cdot \sum_{(x/r < \mu_1 < x-y)} |a_1(\mu_1)| = o(1),$$

ebenfalls gleichmässig.

[14] Man veranschauliche sich die Einteilung durch eine Figur.

3. ABELSCHE $\mathbf{O}_2$-SÄTZE.

Wir beweisen jetzt zwei Verallgemeinerungen Hardy-Littlewoodscher **O**- und $\mathbf{o}_1$-Sätze.[15] Zunächst haben wir den

HILFSSATZ 1.[16]—Unter der Voraussetzung **K** ist

$$\lim_{m=\infty} \lambda_m^{-k} \sum_{n=1}^{m} \lambda_n^k a_n = 0$$

gleichmässig für $k \geqslant 1$.

Der Beweis liegt auf der Hand.

SATZ 2.[17]—*Aus* **K** *in Verbindung mit* $\mathbf{O}_2$ *folgt* **L*** *in der ganzen Halbebene* $\sigma \geqslant 0$, $s \neq 0$.

Beweis.—Es ist bei gegebenem $\delta > 0$

$$F(s) - \sum_{\delta|s|\lambda_n \leqslant 1} a_n = \sum_{\delta|s|\lambda_n \leqslant 1} a_n \frac{1-\lambda_n s - e^{-\lambda_n s}}{\lambda_n s} + \sum_{\delta|s|\lambda_n > 1} a_n \frac{1-e^{-\lambda_n s}}{\lambda_n s}.$$

Darin gilt einerseits

$$\left| \sum_{\delta|s|\lambda_n > 1}^{\infty} \right| \leqslant \sum_{\delta|s|\lambda_n > 1} \frac{2|a_n|}{\lambda_n |s|} = \sum_{\nu=1}^{\infty} \sum_{2^{\nu-1} < \delta|s|\lambda_n \leqslant 2^{\nu}}$$

$$\leqslant \sum_{\nu=1}^{\infty} \frac{\delta}{2^{\nu-1}} . C = 4C\delta \quad \text{für jedes } s \tag{4}$$

und, wenn m den letzten Index in der anderen Summe bezeichnet, d.h. also $\lambda_m \leqslant (\delta|s|)^{-1} < \lambda_{m+1}$ ist,

$$\sum_{\delta|s|\lambda_n \leqslant 1} = \sum_{n=1}^{m} a_n \sum_{k=1}^{\infty} \frac{(-1)^k \lambda_n^k s^k}{(k+1)!} = \sum_{k=1}^{\infty} \frac{(-1)^k s^k}{(k+1)!} \sum_{n=1}^{m} \lambda_n^k a_n,$$

und daher

$$\left| \sum_{\delta|s|\lambda_n \leqslant 1} \right| \leqslant \sum_{k=1}^{\infty} \frac{(1/\delta)^k}{(k+1)!} \cdot \frac{1}{\lambda_m^k} \left| \sum_{n=1}^{m} \lambda_n^k a_n \right| \to 0 \quad \text{bei} \quad |s| \to 0,$$

nach dem Hilfssatz.—

Seien jetzt a_n', a_n'' die Fourierkonstanten einer summierbaren Funktion $\phi(t)$ und

$$a_n = a_n' + i a_n'', \quad A_n = a_n' \cos nt + a_n'' \sin nt.$$

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[15] Enthalten in der für solche Fragen in vielfacher Hinsicht abschliessenden Arbeit: "Abel's theorem and its converse", *Proc. London Math. Soc.* (2), 18 (1920), 205.

[16] Zum gleichen Zweck bei den Herren Hardy und Littlewood, "Sur la convergence des séries de Fourier et des séries de Taylor", *Comptes Rendus*, 165 (1917), 1047.

[17] Vgl. l.c. [16], Theorem II, sowie l.c. [15], Theorem T = ½ Theorem U für **O**, und ebenda ½ Theorem U_1 für $\mathbf{o}_1$.

Dann gilt

Satz 3.[18]—*Aus der Konvergenz von* ΣA_n *zur Summe* A *in Verbindung mit* $\mathbf{0}_2$ *folgt*

$$\Phi(h) = \frac{1}{2h}\int_{t-h}^{t+h} \phi(\vartheta)\, d\vartheta \to A. \tag{5}$$

Beweis.—Bei gegebenem $\delta > 0$ und für $h > 0$, $m = [(\delta h)^{-1}]$ haben wir, ähnlich wie vorhin,

$$\sum_{n=0}^{m} A_n - \Phi(h) = \sum_{n=1}^{m} A_n\left(1-\frac{\sin nh}{nh}\right) - \sum_{\delta hn>1} A_n \frac{\sin nh}{nh},$$

wobei $$\left|\sum_{\delta hn>1}\right| \leqslant 2C\delta \quad \text{für jedes } h,$$

während

$$\left|\sum_{n=1}^{m}\right| \leqslant \sum_{k=1}^{\infty} \frac{(1/\delta)^{2k}}{(2k+1)!}\cdot\frac{1}{m^{2k}}\left|\sum_{n=1}^{m} n^{2k} A_n\right| \to 0 \quad \text{bei} \quad h \to 0.$$

4. Taubersche $\mathbf{0}_1$-Sätze.

Als Verallgemeinerung eines Hardyschen $\mathbf{0}$-Satzes[19] haben wir zunächst:

Satz 4.—*Aus der Summierbarkeit* (R, λ_n, κ) *von* Σa_n *zur Summe* A *in Verbindung mit* $\mathbf{0}_1$ *folgt* $\mathbf{K}$.[20, 21]

Beweis.—Derselbe gelingt auf dem Hardyschen Wege, wenn wir zeigen können, dass bei

$$B^{\kappa}(\xi) = \sum_{\lambda_n \leqslant \xi} (\xi-\lambda_n)^{\kappa}\lambda_n a_n \quad (\kappa = 0, 1, \ldots)$$

und jedem $\delta > 0$ für alle grossen ω die Beziehung

$$|B^{\kappa}(\xi) - B^{\kappa}(\omega)| \leqslant \bar{\eta}(\delta)\omega^{\kappa+1} \quad \text{für} \quad \omega < \xi < (1+\delta)\omega \tag{6}$$

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[18] Vgl. l.c. [16], Theorem I, sowie l.c. [15], ½ Theorem X für $\mathbf{0}$, und ebenda ½ Theorem X_1 für $\mathbf{0}_1$.

[19] "An extension of a theorem on oscillating series", *Proc. London Math. Soc.* (2), 12 (1913), 174.

[20] Weil die Mittel (R, n, κ) und (C, κ) einander äquivalent sind, ist damit auch ein Cesàro-Tauber-Satz bewiesen.

[21] Ein *einseitiger* Riesz-Tauber-Satz wird im Satz 5′ enthalten sein.

mit $\bar{\eta}(\delta) \to 0$ besteht. Nun ist jene Differenz

$$= \sum_{\lambda_n \leqq \omega} \{(\xi-\lambda_n)^\kappa - (\omega-\lambda_n)^\kappa\} \lambda_n a_n + \sum_{\omega < \lambda_n \leqq \xi} (\xi-\lambda_n)^\kappa \lambda_n a_n,$$

absolut genommen,

$$\leqq (\xi-\omega)\kappa\, \xi^{\kappa-1} \sum_{\lambda_n \leqq \omega} \lambda_n |a_n| + \xi^{\kappa+1} \cdot \sum_{\omega \leqq \lambda_n \leqq (1+\delta)\omega} |a_n|$$

$$\leqq \delta\kappa(1+\delta)^{\kappa-1}\omega^\kappa \cdot 2C\omega + (1+\delta)^{\kappa+1}\omega^{\kappa+1}\eta(\delta), \tag{7}$$

letzteres weil $\mathbf{0}_1 \leqq \mathbf{0}_2$ und

$$\text{aus } \mathbf{0}_2 \text{ folgt } \sum_{\lambda_n \leqq \omega} \lambda_n |a_n| \leqq 2C\omega; \tag{8}$$

denn es ist diese Summe

$$= \sum_{\nu=1}^{\infty} \sum_{\omega 2^{-\nu} < \lambda_n \leqq \omega 2^{-\nu+1}} \leqq \sum_{\nu=1}^{\infty} \frac{\omega}{2^{\nu-1}} \sum_{\omega 2^{-\nu} \leqq \lambda_n \leqq \omega 2^{-\nu+1}} |a_n| \leqq 2C\omega,$$

wobei die Konstante C nur von den Reihengliedern abhängt. Daher enthält (7) die Behauptung (6).

Wir zeigen noch, *dass vorstehend die Bedingung* $\mathbf{0}_1$ *nicht durch* $\mathbf{0}_2$ *ersetzt werden kann*, selbst wenn man (im Falle $\lambda_n = n$) die Bedingung

$$a_n = o\left(\frac{\rho(n)}{n}\right), \quad \text{wo} \quad 0 < \rho(n) \to \infty, \tag{9}$$

noch hinzunimmt.

Denn ohne Beschränkung der Allgemeinheit darf man $\rho(n)$ monoton nicht-abnehmend und als Quadratzahl voraussetzen, und dann liefert die Definition ($\nu = 1, 2, \ldots$)

$$\begin{aligned}
a_n &= +2^{-\nu} \quad \text{für} \quad 2^\nu \sqrt{\{\rho(\nu)\}} &&\leqq n < 2^\nu \sqrt{\{\rho(\nu)\}} + 2^{\nu-1},\\
a_n &= -2^{-\nu} \quad \text{für} \quad 2^\nu \sqrt{\{\rho(\nu)\}} + 2^{\nu-1} &&\leqq n < 2^\nu \sqrt{\{\rho(\nu)\}} + 2^\nu,\\
a_n &= 0 \quad\;\; \text{für} \quad 2^\nu \sqrt{\{\rho(\nu)\}} + 2^\nu &&\leqq n < 2^{\nu+1} \sqrt{\rho(\nu+1)},
\end{aligned}$$

das Beispiel einer Reihe, die summierbar $(C, 1) = (R, n, 1)$ ist und $\mathbf{0}_2$ nebst (9) erfüllt, ohne zu konvergieren. Die Diskussion ist einfach und kann dem Leser überlassen bleiben.

Um uns nunmehr den Abel-Tauber-Sätzen zuzuwenden, zeigen wir zunächst, wie die ursprüngliche Taubersche Schlussweise im Falle unserer neuen Bedingungen abzuändern ist:

HILFSSATZ 2.—Aus $\mathbf{0}_2$ nebst $|f(s)| \leqq K$ längs einer Punktmenge $s_\mu = \sigma_\mu + i\tau_\mu \to 0$ mit

$$0 < \sigma_{\mu+1} \leqq \sigma_\mu, \quad 1/\sigma_{\mu+1} \leqq k/\sigma_\mu, \quad |\tau_\mu| \leqq k\sigma_\mu$$

folgt

$$S_n = \sum_{n=1}^{m} a_n = O(1). \tag{10}$$

Beweis.—Zunächst ist unter der Voraussetzung $\mathbf{0}_2$ die Dirichletsche Reihe (1) absolut konvergent für $\sigma > 0$, da

$$\sum_{\lambda_n > 1} |a_n| e^{-\lambda_n \sigma} = \sum_{\nu=1}^{\infty} \sum_{\nu < \lambda_n \leqq \nu+1} \leqq \sum_{\nu=1}^{\infty} e^{-\nu\sigma} \sum_{\nu \leqq \lambda_n \leqq 2\nu} |a_n|.$$

Ferner ist, wie üblich,

$$\sum_{\lambda_n \leqq 1/\sigma_\mu} a_n - f(s_\mu) = \sum_{\lambda_n \leqq 1/\sigma_\mu} a_n(1-e^{-\lambda_n s_\mu}) - \sum_{\lambda_n > 1/\sigma_\mu} a_n e^{-\lambda_n s_\mu}$$

und darin einerseits wegen (8)

$$\left| \sum_{\lambda_n \leqq 1/\sigma_\mu} \right| \leqq \sum_{\lambda_n \leqq 1/\sigma_\mu} |a_n| . \lambda_n |s_\mu| \leqq (1+k) . 2C,$$

andererseits

$$\left| \sum_{\lambda_n > 1/\sigma_\mu} \right| \leqq \sum_{\nu=1}^{\infty} \sum_{\nu < \lambda_n \sigma_\mu \leqq \nu+1} |a_n| e^{-\lambda_n \sigma_\mu} \leqq \sum_{\nu=1}^{\infty} e^{-\nu} \sum_{\nu \leqq \lambda_n \sigma_\mu \leqq 2\nu} |a_n|;$$

und der Rest der Behauptung folgt aus der für alle grossen m bei

$$1/\sigma_\mu \leqq \lambda_m < 1/\sigma_{\mu+1},$$

also $\lambda_m \leqq k/\sigma_\mu$, geltenden Beziehung

$$\left| \sum_{n=1}^{m} a_n - \sum_{\lambda_n \leqq 1/\sigma_\mu} a_n \right| \leqq \sum_{1/\sigma_\mu \leqq \lambda_n \leqq k/\sigma_\mu} |a_n| = O(1).$$

Bei den folgenden Sätzen 5–7 ist die Exponentenfolge stets der Beschränkung

$$\lambda_n/\lambda_{n-1} \to 1 \tag{11}$$

zu unterwerfen.

Als Verallgemeinerung eines ebenso bekannten als wichtigen Littlewoodschen Satzes haben wir zunächst:

SATZ 5.—*Aus* $\mathbf{0}_1$ *in Verbindung mit* **L** *längs der reellen Achse folgt* **K**.[22]

Beweis.—Herr Landau[23] hat einen entsprechenden Satz bewiesen, wobei an Stelle von $\mathbf{0}_1$ vorausgesetzt wird (10) nebst

$$\lim_{m=\infty} \left\{ \max_{(1+\delta)^{-1}\lambda_m \leqq \lambda_n \leqq (1+\delta)\lambda_m} |S_n - S_m| \right\} \leqq \psi(\delta), \quad \text{wo} \quad \psi(\delta) \to 0.$$

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[22] Unter der Beschränkung (11)!

[23] "Über einen Satz des Herrn Littlewood", *Rend. di Palermo*, 35 (1913), 265, vgl. § 4.

Darin ist der unsrige enthalten. Denn aus $\mathbf{O}_1$ folgt die letztere Beziehung ohne weiteres, und $\mathbf{L}$ in Verbindung mit $\mathbf{O}_1$ liefert (10) nach dem Hilfssatz 2.

Dass im Satz 5 die Bedingung $\mathbf{O}_1$ nicht durch $\mathbf{O}_2$ ersetzt werden kann, zeigt wieder das zum Satz 4 angegebene Beispiel.

Was die für reelle Reihen bekannten *einseitigen* $\mathbf{O}_1$-Sätze[19, 24] bzw. für komplexe Reihen ihre *Winkelraum*-Analoga[25] angeht, so ist es kaum lohnend, darin Bedingungen ähnlich $\mathbf{O}_1$ einzuführen. Wir begnügen uns daher mit folgendem Ergebnis:

SATZ 5'.—Wenn die Zahlen $\lambda_n a_n : (\lambda_n - \lambda_{n-1})$ in der komplexen Ebene einem Winkelraum der Öffnung $< \pi$ angehören[26], und $\mathbf{L}$ längs der reellen Achse gilt, so folgt $\mathbf{K}$.[22]

Beweis.—Durch Multiplikation aller $a_n = a_n' + ia_n''$ mit einem passenden $e^{-i\vartheta}$ und erforderlichenfalls Vergrösserung des Winkels kann erreicht werden, dass sein Scheitel $= -k \leqslant 0$ wird und seine Mittellinie die positivreelle Achse enthält. Dann ist also

$$\frac{\lambda_n a_n'}{\lambda_n - \lambda_{n-1}} \geqslant -k, \quad \left| \frac{\lambda_n a_n''}{\lambda_n - \lambda_{n-1}} \right| \leqslant K\left(k + \frac{\lambda_n a_n'}{\lambda_n - \lambda_{n-1}} \right).$$

Ersteres aber liefert nach dem Hardy-Littlewoodschen einseitigen $\mathbf{O}$-Satz[24] die Konvergenz von $\Sigma a_n'$, und letzteres gibt darauf für alle grossen x

$$\sum_{x \leqslant \lambda_n \leqslant (1+\delta)x} |a_n''| \leqslant Kk \Sigma \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} + K \Sigma a_n' \leqslant Kk.2\delta + o(1),$$

sodass Satz 5 die noch fehlende Konvergenz von $\Sigma a_n''$ liefert.

[ZUSATZ BEI DER KORREKTUR.—Ein etwas einfacherer Beweis ergibt sich daraus, dass, wenn die Schenkel unseres Winkelraums um $\pm(\frac{1}{2}\pi - \delta)$ gegen die positivseelle Achse geneigt sind, die beiden Reihen $\Sigma(a_n' \cos\delta \pm a_n'' \sin\delta)$ als Realteile von $\Sigma(a_n' + ia_n'')e^{\pm i\delta}$ die Voraussetzungen des einseitigen $\mathbf{O}$-satzes erfüllen, sodass die Reihen $\Sigma a_n' \cos\delta$ und $\Sigma a_n'' \sin\delta$ konvergieren.]

EINE ZWEITE ANWENDUNG des Satzes 5 (den Hinweis darauf verdankt Verf. Herrn Hardy) bildet die Feststellung der Konvergenz der Reihe

$$\Sigma p^{-1-s} \quad (\text{wo } \rho = \rho_n \text{ die } n\text{-te Primzahl})$$

für $s = i\tau, \tau \lessgtr 0$. Denn in jedem solchen Punkte ist die dargestellte Funktion bekanntlich regulär, also der Abelsche Limes gewiss vorhanden. Und es ist $\mathbf{O}_1$ erfüllt, weil für alle grossen x

$$\sum_{x \leqslant \log p \leqslant (1+\delta)x} |p^{-1-\tau i}| \leqslant \sum_{(1+\delta)^{-1}x \leqslant \log n \leqslant (1+\delta)x} 2 : (n \log n) \leqslant 4 \log(1+\delta),$$

sodass Satz 5 anwendbar ist.

EINIGE BEMERKUNGEN seien an das vorstehende Beispiel angeschlossen.

(1) Der Littlewoodsche $\mathbf{O}$-Satz versagt bei ihm, wenigstens zur Zeit. Denn seine Anwendung verlangt die Existenz eines $\alpha > 0$ mit

$$p_{n+1} - p_n \geqslant \alpha \log p_n,$$

und diese Beziehung kann gegenwärtig weder bewiesen noch widerlegt werden.

(2) Der Landausche oben angeführte Satz erfordert den Nachweis von

$$\sum_{\log p \leqslant x} p^{-1-\tau i} = O(1) \quad \text{für} \quad \tau \lessgtr 0,$$

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[24] Vgl. l.c. [6], Theorem F.

[25] Solche scheinen bisher nicht bekannt geworden zu sein.

[26] Ein Winkel $\leqslant \frac{1}{2}\pi$ ist auf Grund des einseitigen $\mathbf{O}$-Satzes trivial einer $\geqslant \pi$ offenbar nicht zulässig.

einer Beziehung, die in derselben Richtung liegt wie die zu beweisende Konvergenz und nahezu ebensoweit geht. Es ist also von praktischem Interesse, dass Satz 5 von der unbequemen Voraussetzung (10) frei ist.

(3) Ebenso wie oben ergibt sich auch die Konvergenz von Reihen wie

$$\sum \chi(p)\, p^{-1-\tau i} \quad \text{oder} \quad \sum_{p \equiv l} p^{-1-\tau i},$$

da man die Regularitätsverhältnisse der zugehörigen Funktionen auf der imaginären Achse vollkommen beherrscht.[27]

Die restlichen Sätze dieses § sind verallgemeinerungen Hardy-Littlewoodscher **O**- und **o**-Sätze[15] und folgen aus unserem Satz 5.

Bezeichnen wir als "*Kurve* Γ" eine einfache stetige, im Punkte $s = 0$ endigende Kurve der Halbebene $\sigma \geqq 0$, $s \neq 0$, so gilt:

HILFSSATZ 3.—Unter der Bedingung $\mathbf{O}_2$ ist für (10) notwendig und hinreichend, dass $F(s)$ beschränkt ist bei $s \to 0$ längs jeder oder auch nur einer "Kurve Γ."

Beweis.—Zunächst ist unter der Voraussetzung $\mathbf{O}_2$ der Ausdruck $F(s)$ in unserer Halbebene erklärt, da

$$(12) \qquad \sum_{\lambda_n > 1} \frac{|a_n|}{\lambda_n} = \sum_{\nu=1}^{\infty} \sum_{2^{\nu-1} < \lambda_n \leqq 2^{\nu}} \leqq \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu-1}} \sum_{2^{\nu-1} \leqq \lambda_n \leqq 2^{\nu}} |a_n|.$$

Weiter ist, wenn zwischen s und der natürlichen Zahl m die Beziehung $\lambda_m \leqq |s|^{-1} < \lambda_{m+1}$ festgesetzt wird,

$$F(s) - \sum_{n=1}^{m} a_n = \sum_{\lambda_n \leqq |s|^{-1}} a_n \frac{1 - \lambda_n s - e^{-\lambda_n s}}{\lambda_n s} + \sum_{\lambda_n > |s|^{-1}} a_n \frac{1 - e^{-\lambda_n s}}{\lambda_n s},$$

wobei nach (4) bzw. (8)

$$\left| \sum_{\lambda_n > |s|^{-1}} \right| \leqq 4C, \qquad \left| \sum_{\lambda_n \leqq |s|^{-1}} \right| \leqq \sum_{\lambda_n \leqq |s|^{-1}} |a_n| . \lambda_n |s| \leqq 2C.$$

SATZ 6.[28]—*Aus* $\mathbf{O}_1$ *in Verbindung mit* $\mathbf{L}^*$ *längs einer "Kurve* Γ" *folgt* $\mathbf{K}$.[22]

Beweis.—Wegen (12) ist $F(s)$ stetig in unserer Halbebene. Daher folgt Satz 6 aus Satz 5 wörtlich ebenso wie l.c. [15], Theorem Q aus Theorem F, wobei statt Lemma α unser Hilfssatz 3 einspringt.

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[27] Vgl. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, §§ 115, 124, bzw. § 118.

[28] Vgl. l.c. [16], Theorem II, sowie l.c. [15], Theorem Q = ½ Theorem U für **O**, und ebenda ½ Theorem U_1 für $\mathbf{O}_1$.

SATZ 7.[29]—*Aus* $\mathbf{O}_1$ *in Verbindung mit* $\mathbf{L}$ *längs* (*a*) *einer rektifizierbaren "Kurve* Γ*" oder* (*b*) *einer im Winkel* $\sigma > 0$, $|\tau| \leqslant k\sigma$ *verlaufenden, sonst beliebigen "Kurve* Γ*" folgt* $\mathbf{K}$.[22]

Beweis.—Zu (*a*): Wegen (12) ist bei Integration längs Γ

$$F(s) = \frac{1}{s} \Sigma \frac{a_n}{\lambda_n}(1-e^{-\lambda_n s}) = \frac{1}{s}\int_0^s F(t)\,dt,$$

sodass $\mathbf{L}^*$ längs Γ gilt, und Satz 6 anwendbar ist.

Zu (*b*): Nach Hilfssatz 2 gilt dann (10), woraus nach Abel die Beschränktheit von $f(s)$ in unserem Winkelraum folgt. Nach Herrn Lindelöf[30] gilt dann $\mathbf{L}$ längs der reellen Achse, sodass Satz 5 anwendbar ist.

Schliesslich haben wir für Fourierreihen in den Bezeichnungen vom Ende des § 3 den

SATZ 8.[31]—*Aus* $\mathbf{O}_1$ *in Verbindung mit* (5) *folgt die Konvergenz von* ΣA_n.

Beweis.—Nach Herrn Fatou gilt unter Voraussetzung (5) die Beziehung

$$\lim_{s=0} \Sigma A_n e^{-ns} = A \quad \text{für} \quad s > 0,$$

und die Behauptung folgt aus Satz 5.

Leipzig, den 12. Mai 1923.

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[29] Vgl. l.c [15], Theorem R und S für $\mathbf{O}$, und l.c. [6], Theorem A für $\mathbf{O}_1$.

[30] "Sur un principe général de l'analyse et son application . . .", *Acta Soc. Sci. Fenn.*, 46 (1920), No. 4 (1915), vgl. § 6.

[31] Vgl. l.c. [16], Theorem I, sowie l.c. [15], ½ Theorem X für $\mathbf{O}$, und ebenda ½ Theorem X_1 für $\mathbf{O}_1$.

ON THE ZEROS OF MATHIEU FUNCTIONS

By Einar Hille.

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Various problems in mathematical physics lead to the differential equation

$$\frac{d^2w}{dz^2}+(a\cos 2z+b)\,w=0, \qquad (M_0)$$

where a and b are real constants. There is no restriction in assuming $a<0$ and writing the equation in the following form:—

$$\frac{d^2w}{dz^2}+(B-A\cos 2z)w=0 \qquad (M)$$

with $A>0$. We shall refer to equation (M) as *Mathieu's equation* and call any function that satisfies (M) a *Mathieu function.** All these functions are, of course, *entire functions of z of infinite genus.*

The problem considered in the present paper is the distribution in the complex plane of the zeros of Mathieu functions. Such a problem can be studied from two different points of view, one of which might be called *the quantitative* and the other *the qualitative.* In the opinion of the writer, the former involves the determination of the number of zeros within a certain region of the plane and their approximate location by means of analytic representation, be it convergent or asymptotic. Questions of frequency are fundamental for this concept. From the latter point of view the essential problem is an *a priori* investigation of the structure of the point-set formed by the zeros of a given function. It is the question of deciding whether the function is oscillatory or non-oscillatory in a given region without attempting to determine the actual frequency. An example from the theory of Bessel's functions will clarify the distinction. The

* For an exposition of the theory of Mathieu's equation cf. Whittaker and Watson, *Modern Analysis*, Ch. 19, 404–428 (3rd ed.), Cambridge (1920), which gives further references to the literature. Prof. Whittaker reserves the name Mathieu functions for the solutions with period 2π.

famous formula of Stokes for the nth zero of $J_0(z)$ is a typical product of the former concept, whereas the latter is represented by the theorem according to which $J_m(z)$ has only real zeros when $m > -1$.

In the present paper we try to combine these two points of view with the emphasis put on the qualitative side of the question.* The quantitative methods serve us mainly for solving existence questions. When we know that the zeros exist we can proceed to study the structure of the point-set formed by them by means of the qualitative methods.

The distribution of the zeros and extrema† of an arbitrary solution shows the following main characteristics. With each half-line $x = k\pi$, $y > 0$ and $x = k\pi$, $y < 0$ ($k = 0, \pm 1, \pm 2, \ldots$) is associated an infinite set of points, here called a *string*, made up of the zeros and extrema of the solution in question. *The string is asymptotic to the associated line* in the sense that the distance of a point in the string to the line approaches zero monotonically when the ordinate tends to infinity. The points of the string are either all on the line or all to the left of the line or all to the right. Two points in the same string can never have the same ordinate and, if the string is off the line, never the same abscissa. *In the string the zeros alternate with the extrema.* The general solution has such a string for every half-line. But to every half-line there are two particular solutions which lack the associated string. Such a solution, which we call *truncated*, will lack two adjacent strings (in exceptional cases more) and tends to zero when the variable z tends to infinity between the two lines. The study of these truncated solutions is particularly instructive.

When $B \leqslant -A$ these strings account for all zeros and extrema. But if $B > -A$ we have infinitely many extraneous zeros and extrema associated with the real axis. When $B \geqslant +A$ these points can be said to form a string " parallel " to the real axis.

For our investigation we need very little of the classical theory of Mathieu functions. The necessary material is collected in §1 of the paper. On the other hand we have to devote considerable space to a development of the methods which are used for the study of the asymptotic distribution of the zeros as well as of the structure of the string. This is done in §2.

The investigation proper is started in §3 by the description of the distribution of the strings and their structure. The study can be carried

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* The asymptotic distribution of special sets of zeros of particular solutions of (M) has been investigated by several authors. We quote R. C. Maclaurin, *Trans. Camb. Phil. Soc.*, 17 (1899), 70; and W. Marshall, *Amer. J. of Math.*, 31 (1909), 319–332, and *Proc. Edinburgh Math. Soc.*, 40 (1921–22), 2–8.

† We use the term *extremum of a function* to designate a point in the complex plane where the first derivative of that function equals zero.

rather far in the case when $B \leqslant -A$; unfortunately this case seems to be of little interest for the physical applications. We cannot carry the study of the zeros of the general solution to the same degree of completeness when this restriction is not fulfilled.

We devote §4 to a study of the truncated solutions and §5 to the solutions which are real on the real axis or on one of the lines $x = k\pi$. We can solve the distribution problem for *the functions of the elliptic cylinder* completely from the qualitative point of view.

Finally, in §6 we study the variation in the structure of the strings imposed by a variation of the parameters upon which the solution in question depends. This study completes the results, especially of §3.

1. *Preliminary.*

1.1. *The solutions of* (M) *form a two-parametric linear family of entire functions.* For our purpose, however, we do not need to distinguish between different linearly dependent solutions: they have all the same zeros and extrema. Thus we can characterize our solutions by means of *one* parameter only. We select the ratio

$$\frac{w'(z_1)}{w(z_1)} = \lambda(z_1) \tag{1.11}$$

which we call *the parameter at* $z = z_1$. The point z_1 is arbitrary but fixed once and for all.

The equation (M) contains two real constants, A and B, and the function $\cos 2z$ which is real for real values of z. We say that *a solution is of real character on a straight line-segment in the complex plane if it is the product of an arbitrary real or non-real constant into a function which takes on real values only on the line-segment in question.* A solution which is of real character on the real axis admits of a real value only for its parameter at an arbitrary point x_1 on the real axis. On the other hand, if a solution has a real value for its parameter at a point on the real axis it is necessarily of real character on the real axis.

There are certain transformations of the independent variable which preserve the property of the equation (M) of having real coefficients. If we substitute $z = k(\pi/2)+iy$ (k integral) the equation (M) is carried over into

$$\frac{d^2w}{dy^2} - [B-(-1)^k A \cosh 2y]w = 0, \tag{1.12}$$

where the coefficients are real for real values of y. Let us designate the line $x = k(\pi/2)$ by l_k. Then there are solutions of (M) which are of real character on the line l_k. The parameter of such a solution at an arbitrary point on the line l_k in question is necessarily a purely imaginary number.

A solution may be of real character as well on the real axis as on one of the lines l_k, l_n say. But in that case $\lambda\{n(\pi/2)\}$, the parameter at $z = n(\pi/2)$, is necessarily either zero or infinite since it must be at the same time a real and a purely imaginary number. Thus such a solution has either a zero or an extremum at $z = n(\pi/2)$, the point of intersection of l_n with the real axis; in the former case it is an odd function of $z-n(\pi/2)$, in the latter an even function.

For particular values of A and B there may be solutions which are of real character on several or on all of the lines l_k as well as on the real axis. Then such a solution will have a zero or an extremum at each of the points where the lines l_k in question intersect the real axis.

1.2. It follows from the general theory of linear differential equations with periodic coefficients that equation (M) has two solutions of the form

$$M^+(z) = e^{-\mu z}F(z) \quad \text{and} \quad M^-(z) = M^+(-z) \tag{1.21}$$

where $F(z)$ is a periodic function of z with period π. μ is a constant depending upon A and B and is either real or purely imaginary when A and B are real.* These two solutions are in general linearly independent. This will certainly be the case if μ is either real or of the form αi where α is an irrational real number. If μ is real we can take it to be positive and we notice that

$$\lim_{z\to\infty} M^+(z) = 0, \quad \lim_{z\to\infty} M^-(z) = \infty, \tag{1.22}$$

if z tends to infinity in a direction which is parallel to the positive real axis. This excludes the possibility of linear dependence when μ is real. In case $\mu = \alpha i$ we can proceed as follows. There exists at least one value $z = \zeta$ such that $F(\zeta) = \beta$ and $F(-\zeta) = \gamma$ are both different from zero. Then form the quotient

$$M^+(\zeta+n\pi)/M^-(\zeta+n\pi) = e^{-2(\zeta+n\pi)\alpha i}\beta/\gamma.$$

This ratio evidently depends upon the value of the integer n, which it should not do if the two solutions were linearly dependent.

* There are two possible values for μ the sum of which is zero. If one is a complex number the other is the conjugate imaginary.

1.3. If $\mu = (r/s)i$ where r and s are relative primes then the two solutions $M^+(z)$ and $M^-(z)$ may be linearly dependent, but it is not necessarily the case. These solutions will admit of the period $2s\pi$, which we assume to be the primitive period. If z is increased by $s\pi$ only, the solutions take on the same value but with opposite sign. We express this fact by saying that $s\pi$ is a *semi-period* of the solutions in question. Thus ω is a semi-period of $f(z)$ if $f(z+\omega) = -f(z)$.

If $M^+(z)$ and $M^-(z)$ are linearly independent every solution of (M) will admit of $2s\pi$ as period and of $s\pi$ as semi-period. In particular, the solutions $M^+(z)+M^-(z)$ and $M^+(z)-M^-(z)$ have this property; the former solution is an even function of z, the latter an odd function. If $M^+(z)$ and $M^-(z)$ are linearly dependent, $M^+(z)$ must be either an even function of z or an odd function, and moreover must be of real character on the real axis and on the lines l_{ks} where k is an arbitrary integer. Whether or not there are other periodic solutions remains an open question.

Thus if the values of A and B are such that equation (M) admits of a periodic solution of period $2s\pi$ there will always exist at least one solution which is either an even or an odd function of z and which has the semi-period $s\pi$ and is of real character on the axis and on the lines l_{ks}.

The case when $s = 1$ is the classical case of periodic solutions studied chiefly by Whittaker and his pupils. We shall refer to these solutions as *the functions of the elliptic cylinder*. They have the important property of being either even or odd functions of z and are of real character on the real axis as well as on all the lines l_k. Using an observation made in § 1.1 we conclude that the points $z = k(\pi/2)$ are zeros or extrema of these solutions for all values of k. A set of these solutions admit of π as the primitive period.

The existence of periodic solutions with $s > 1$ has been shown by Poole,* who gave the first example proving that Mathieu's equation could have two linearly independent periodic solutions. Poole studied a group of periodic solutions existing for particular values of A and B which we can represent by the symbols

$$E(z+m\pi) \quad \text{and} \quad O(z+m\pi), \quad m = 0, 1, \ldots, s-1, \quad (1.31)$$

where $E(z)$ and $O(z)$† denote an even and an odd function of z respectively. If $2s\pi$ is actually the *primitive* period these $2s$ functions will exist simultaneously, and every two of them are linearly independent.

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* *Proc. London Math. Soc.* (2), 20 (1920–21), 374–388.

† This "O" is not to be confused with the common *ordo*-symbol. We use the same letter together with "o" in §§ 5 and 6 of the paper to denote *odd* functions. In the remainder of the paper we use these letters to denote order of magnitude.

2. *General Principles.*

2.1. We shall use this section for the development of the general principles upon which our investigation is based. These principles can serve as the foundation of a general theory concerning the distribution in the complex plane of the zeros of solutions of linear differential equations of the second order. Various aspects of such a theory have been studied by the present writer in several earlier papers.* In order to avoid too lengthy expositions we shall merely develop the general lines of the argument and refer the reader to our previous writings for fuller information.

We use three different principles for the study of the distribution of zeros, one of which essentially quantitative in nature, the other two more qualitative.

I. THE PRINCIPLE OF THE ASYMPTOTIC SINE-EQUATION.—*There exists a transformation which will carry the differential equation* $w''+G(z)w=0$ *over into a differential equation which is, at least formally, asymptotic to the equation* $W''+W=0$. *The solutions of the transformed equation are asymptotic to certain sine-functions.*

The application of this principle involves a study of the conformal mapping set up by the transformation in question and also an investigation of the sense in which the solutions of the transformed equations can be said to approach the sine-functions.

II. THE PRINCIPLE OF ZERO-FREE REGIONS.—*Let* $w(z)$ *be a solution of the differential equation* $w''+G(z)w=0$ *whose parameter at a non-singular point* (z_1) *equals* $\lambda(z_1)=w'(z_1)/w(z_1)$, *and put* $\arg\lambda(z_1)=\omega$. *Then there exists a region* $D(z_1;\omega)$ *in which* $w'(z)w(z)\neq 0$. *This region does not depend upon* $|\lambda(z_1)|$ *and is left invariant if* $G(z)$ *is replaced by* $C\,G(z)$ *where* $C>0$.

This principle flows from a peculiar integral equality associated with the differential equation which we call *Green's transform* from its similarity to a famous formula by Green. This equality can be used in a great many ways for the study of the distribution problem from a qualitative point of view both locally and in the gross. Locally we can use the equality to

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* We shall frequently have to refer the reader to the following two papers—namely, "Oscillation theorems in the complex domain", *Trans. Amer. Math. Soc.*, 23 (1922), 350–385, and "On the zeros of Sturm-Liouville functions", *Arkiv f. Mat. Astr. o. Fys.*, 16, 17 (1922), which we shall quote below under the abbreviations O.T. and S.L. respectively.

impose restrictions on the orientation of possible zeros in a given region, and to set up relations of order between the zeros.

III. THE PRINCIPLE OF CONTINUOUS VARIATION OF THE PARAMETER.—*The zeros and extrema of the solution* $w(z, \lambda)$ *of the differential equation* $w''+G(z)w=0$, *whose parameter at some points equals* λ, *are analytic functions of* λ. *If* λ *is varied continuously the zeros and extrema will also vary continuously except when* λ *approaches certain critical values. At every instant during the variation the zeros and extrema have to occupy such positions that the restrictions imposed by* PRINCIPLE II *are satisfied.*

This principle serves for extending to general solutions such relations of order and precedence between the zeros and extrema which may have been established for particular solutions, *e.g.* solutions of real character on the real axis. The principle can be extended, *mutatis mutandis*, to apply to continuous variations imposed by variations of parameters entering in $G(z)$.

2.21. We proceed to the application of Principle I which requires a study of the irregular singular point at infinity. We advise the reader to consult O.T., § 4, for further details.

We can assume $a<0$ in equation (M_0); if that is not the case from the beginning a transformation $z=u+(\pi/2)$ will effect the change. We put $a=-A, A>0$ and $b=B$. With respect to the value of B we have three possibilities: (1) $B\leqslant -A$, (2) $-A<B<+A$, (3) $+A\leqslant B$. These different possibilities we refer to as Cases 1, 2, and 3 respectively. They reflect fundamental differences in the distribution of the zeros of the solutions. Let us put

$$G(z)=B-A\cos 2z. \tag{2.211}$$

The function $G(z)$ is real on the real axis, which we shall denote by r for the sake of brevity, and on the lines $x=k(\pi/2)$, denoted by l_k in §1.1. The table below gives the signs of $G(z)$ on these lines in the three different cases

	r	l_{2n}	l_{2n+1}
1	−	−	±
2	±	−	+
3	+	±	+

(2.212)

Let us designate the semi-infinite strip above the real axis between the lines l_{2n-1} and l_{2n+1} by D_n^+, and the symmetric region below the axis by D_n^- $(n = 0, \pm 1, \pm 2, \ldots)$. We do not include the boundaries in the regions. Let us fix our attention on the region D_0^+ for a while and consider the function

$$Z(z, z_0) = \int_{z_0}^{z} \sqrt{\{G(z)\}}\, dz, \tag{2.213}$$

where $z_0 = 0$ in Cases 1 and 2, but stands for the zero of $G(z)$ in D_0^+ in Case 3. We choose that determination of the square root which will make the part of l_0 above z_0 in the z-plane correspond to the real positive axis in the Z-plane.

This function $Z(z, z_0)$ maps the region D_0^+ on a polygonal region Δ in the Z-plane. In the Cases 1 and 2 this region is non-overlapping and is bounded by five straight lines joining in order the following points: $\alpha - i\infty$, $\alpha - i\beta$, $-i\beta$, $+i\beta$, $\alpha + i\beta$, and $\alpha + i\infty$. α is positive in Case 1, negative in Case 2. The numerical values of these constants α and β are of no interest for the following developments. In order to make the mapping function single-valued in the region and to prevent overlapping in Case 3 we consider only the part of D_0^+ in which $\Im\mathfrak{m}(z) \geqq \Im\mathfrak{m}(z_0)$.* The boundary of the map of this region in the Z-plane is made up of two line-segments, namely $X = \alpha$ $(\alpha < 0)$, $|Y| \geqq \beta > 0$, and two curved arcs joining the points $\alpha + i\beta$ and $\alpha - i\beta$ respectively with the origin, making angles of $\pi/2$ with the line-segments and forming an angle of $(3\pi)/2$ with each other at $Z = 0$, the angles being measured within the region. In each case the region Δ contains a half-plane $\Re(Z) > \gamma$ where $\gamma = \alpha$ in Case 1 and 0 in Cases 2 and 3.

The curves $\mathfrak{S}$ in the z-plane which correspond to the lines

$$\Im\mathfrak{m}(Z) = \text{const.}$$

are of particular interest. We have

$$Z(z, z_0) = \sqrt{(A/2)}\, e^{-iz} + C + O(e^{-y}), \tag{2.214}$$

where the real constant C is defined by

$$C = \int_{y_0}^{+\infty} [\sqrt{(A \cosh 2y - B)} - \sqrt{(A/2)}\, e^{y}]\, dy \tag{2.215}$$

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* We put $\Re(z) = x$, $\Im\mathfrak{m}(z) = y$ if $z = x + iy$.

and $iy_0 = z_0$. Hence, a first approximation to the curves in question is given by

$$\sin x \,.\, e^y = \text{const.} \tag{2.216}$$

These curves admit of the line l_0 as an asymptote which they approach in a very rapid manner, inasmuch as for large positive values of y we have $x \sim c\, e^{-y}$. We notice that the family of curves $\mathfrak{S}$ as a whole is not dependent upon the lower limit of integration in (2.213). The curves exist, of course, even outside of D_0^+; the extension of the family can be obtained by repeated reflections in r and in the lines l_{2k+1}.

In exactly the same manner we can map any other of the regions D_k^+ or D_k^- on the region Δ in the Z-plane. In doing this it is preferable to choose the point z_0 on the axis of symmetry of the region in question. It may also be desired to map a region U_n^+ above r and between the lines l_{2n-2} and l_{2n} upon a region Υ in the Z-plane containing some half-plane $\mathfrak{Im}(Z) > \delta$. We can map U_1^+ on such a region by means of the function $Z(z, z_0)$ if we equate z_0 to $\pi/2$ in the Cases 2 and 3 but to the zero of $G(z)$ in U_1^+ in Case 1. The resulting maps in the three different cases resemble in shape what we should get by rotating the regions Δ through an angle of 90° around the origin. We shall want to know later the map of a region made up of U_n^+, U_{n+1}^+, and U_{n+2}^+. We can take $z_0 = (n+\frac{1}{2})\pi$ or $(n+\frac{1}{2})\pi + iy_0$ where $y_0\ (>0)$ is the imaginary part of a zero of $G(z)$. The resulting map is overlapping, but if we take only the part of the composite region in which $\mathfrak{Im}(z) \geqslant Y_0 > y_0$ where Y_0 is sufficiently large, the corresponding region $\mathfrak{D}$ can be spread out smoothly upon the Riemann surface of $\log Z$. The part of U_{n+1}^+ is mapped on a region Υ_0 bounded by two line-segments $Y = \beta$, $|X| \geqslant \alpha > 0$ and a curved symmetric arc joining $\alpha+\beta i$ with $-\alpha+\beta i$. We get $\mathfrak{D}$ merely by reflecting Υ_0 in the two line-segments and joining the images to Υ_0 along the segments in question.

2.22. Let us substitute in the differential equation (M)

$$Z = Z(z, z_0), \quad W = [G(z)]^{\frac{1}{2}} w \tag{2.221}$$

which carries (M) over into

$$\frac{d^2 W}{dZ^2} + [1 - \Phi(Z)]\, W = 0, \tag{M'}$$

where

$$\Phi(Z) = \phi(z) = \tfrac{1}{4}\frac{G''}{G^2} - \tfrac{5}{16}\frac{(G')^2}{G^3}, \tag{2.222}$$

the primes denoting differentiation with respect to z and not with respect

to Z. The function $\Phi(Z)$ is single-valued and analytic in Δ (and in Υ). A simple but somewhat lengthy calculation shows that

$$\lim_{Z\to\infty} Z^2\Phi(Z) = \tfrac{1}{4} \tag{2.223}$$

uniformly in Δ (and in Υ). Let us form the integral equation

$$W(Z) = W_0(Z) + \int_Z^\infty \sin(T-Z)\,\Phi(T)\,W(T)\,dT, \tag{M''}$$

where $$W_0'' + W_0 = 0,$$

and the path of integration is parallel to the positive real axis. This equation has a unique solution* in Δ which satisfies (M'); conversely every solution of (M') will satisfy an equation of type (M'') in Δ for an appropriate choice of $W_0(Z)$. This solution is asymptotic to $W_0(Z)$ in Δ in a sense which will soon be apparent. There are two fundamentally different cases to consider, namely, according as $W_0(Z)$ is *oscillatory* or *non-oscillatory*.

2.23. Let us first take the *non-oscillatory case*. We assume

$$W_{01}(Z) = e^{iZ}, \tag{2.231}$$

then the corresponding solution $W_1(Z)$ satisfies the equality

$$e^{-iZ}W_1(Z) = 1 + \frac{\Theta(Z)}{Z}, \tag{2.232}$$

where $|\Theta(Z)|$ is bounded in Δ. (Cf. O.T., §4.3, formula 4.311.) As a matter of fact we could show that the equality (2.233) holds in a considerably greater region, namely,

$$|Z| > R, \quad -\pi+\epsilon \leqslant \arg Z \leqslant 2\pi-\epsilon, \tag{2.233}$$

where R is some positive constant. This solution evidently approaches 0 in the sector $\epsilon \leqslant \arg Z \leqslant \pi-\epsilon$ and becomes infinite when $Z \to \infty$ in either of the sectors $\pi+\epsilon \leqslant \arg Z \leqslant 2\pi-\epsilon$ or $-\pi+\epsilon \leqslant \arg Z \leqslant -\epsilon$.

A similar formula holds for the solution $W_2(Z)$ corresponding to $W_{02}(Z) = e^{-iZ}$ in the sector $-2\pi+\epsilon \leqslant \arg Z \leqslant \pi-\epsilon$. It is evident that these two solutions have no zeros of large modulus in the sectors indicated, and they are the only linearly independent solutions having this property in Δ. For this reason we call $W_1(Z)$ and $W_2(Z)$ the *truncated* solutions in Δ.

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* The question of uniqueness is not touched upon in O.T., §4.2, but is easily settled by classical arguments.

2.24. This property of the truncated solution of having the same asymptotic representation throughout an angular opening of $3\pi-\delta$ is quite general and rather important. In O.T. we have shown this property only when the differential equation has polynomial coefficients (O.T., § 4.6). We shall outline a proof for the general case in order to be able to apply the result to the present equation.

On the Riemann surface of $\log Z$ we trace a curve $C(\theta_1, \theta_2)$ composed of the circular arc $|Z| = r, \theta_1 \leqslant \arg Z \leqslant \theta_2$ and the outward-pointing semi-tangents at the end-points of the arc. The region on the surface outside of this curve we denote by $\mathfrak{D}(\theta_1, \theta_2)$. In particular, we write simply $\mathfrak{D}$ if $\theta_1 = -(\pi/2)+\epsilon$ and $\theta_2 = (3\pi/2)-\epsilon$. It is evident that the region determined by formula (2.233) differs very little from $\mathfrak{D}$. Then we have the following theorem :—

THEOREM 2.24.—*Suppose that the function $\Phi(Z)$ is single-valued and analytic on $\mathfrak{D}$ and satisfies the inequality*

$$|\Phi(Z)| < \frac{M}{|Z|^2}. \tag{2.241}$$

Then the integral equation

$$W(Z) = e^{iZ} + \int_Z^\infty \sin(T-Z)\,\Phi(T)\,W(T)\,dT \tag{2.242}$$

has a unique solution which is single-valued and analytic on $\mathfrak{D}$ and satisfies the relation

$$e^{-iZ}\,W(Z) = 1 + \frac{\Theta(Z)}{Z} \tag{2.243}$$

where $|\Theta(Z)|$ is bounded on $\mathfrak{D}$.

The majorant M/Z^2 in (2.241) can be replaced by any other function $F(Z)$ having the following three properties on $\mathfrak{D}$, namely (1) $F(Z)$ is single valued and analytic, (2) $\lim_{Z\to\infty} Z\,F(Z) = 0$, and (3) the integral $\int_Z^\infty F(Z)\,dZ$ is absolutely convergent. If such a replacement is made, the factor $1/Z$ in (2.243) is to be replaced by $\int_Z^\infty F(Z)\,dZ$.

The path of integration in (2.242) is taken to be $\arg(T-Z) = 0$. It is obvious that by such paths we can reach only a part of $\mathfrak{D}$. Temporarily we shall restrict Z to lie in the region $\mathfrak{D}(\theta_1, \theta_2) = \mathfrak{D}^+$ where $\theta_1 = -(\pi/2)+\epsilon$ and $\theta_2 = (\pi/2)-\epsilon$. Now let us take a strip Δ_0 on $\mathfrak{D}^+$ in which $B_1 \leqslant Y \leqslant B_2$ $(Z = X+iY)$. Then it can be shown by the method of

successive approximations that (2.242) has a unique solution which is single-valued and analytic and moreover bounded in Δ_0. The function

$$U(Z) = e^{-iZ} W(Z) \tag{2.244}$$

satisfies the integral equation

$$U(Z) = 1 + \frac{1}{2i} \int_Z^\infty [e^{2i(T-Z)} - 1] \Phi(T)\, U(T)\, dT. \tag{2.245}$$

This equation evidently also has a unique analytic solution which is bounded in Δ_0. Suppose that $\operatorname{Max} |U(Z)| = K$ in Δ_0; then

$$|U(Z)| \leqslant 1 + \tfrac{1}{2} \int_0^\infty |(e^{2it} - 1)\, \Phi(Z+t)\, U(Z+t)|\, dt \tag{2.246}$$

$$< 1 + KM \int_0^\infty \frac{dt}{|Z+t|^2} = 1 + KM \frac{\theta}{Y}$$

$$= 1 + KM \frac{\theta}{\sin\theta} \cdot \frac{1}{|Z|},$$

where $\theta = \arg Z$. But in $\mathfrak{D}^+$, $(\theta/\sin\theta) < (\pi/\sin\epsilon)$. Thus putting $M\pi/\sin\epsilon = M_1$ we obtain

$$|U(Z)| < 1 + \frac{KM_1}{|Z|}. \tag{2.247}$$

Hence if $|Z| > 2M_1$ in Δ_0, $K < 2$, from which it follows that

$$|U(Z) - 1| < \frac{2M_1}{|Z|}.$$

This relation implies that (2.243) holds in Δ_0. But the evaluation of the integral is independent upon the numerical values of the constants B_1 and B_2. Consequently (2.243) holds for all values of Z on $\mathfrak{D}^+$, and $U(Z)$ will be single-valued and analytic throughout this region. In passing we observe in view of (2.246) that if $Y > 2\pi M$ we have $|U(Z)| < 2$, which will be useful in a moment.

It remains to extend the validity of the relation to the rest of $\mathfrak{D}$. This we accomplish by changing the path of integration in (2.242). Suppose that $Y > 2\pi M$ and consider the integral

$$\int [e^{2i(T-Z)} - 1]\, \Phi(T)\, U(T)\, dT$$

taken along the contour made up of the line-segment from $X - P + iY$ to $X + P + iY$ (where P is a great positive number) and the semicircle in the upper half-plane with this line-segment as diameter. The value of the integral is obviously zero; the contribution made by the semicircle is

$O(1/P)$ (since $|U(T)| < 2$) which approaches zero when $P \to +\infty$. Hence

$$\int_{l+} = \int_{l-},$$

where l_+ denotes the path $\arg(T-Z) = 0$ and l_- the path $\arg(T-Z) = \pi$. Thus if $Y > 2\pi M$ the function $U(Z)$ also satisfies

$$V(Z) = 1 + \frac{1}{2i}\int_{l-} [e^{2i(T-Z)} - 1]\Phi(T)\,V(T)\,dT. \qquad (2.248)$$

But this equation is exactly of the same form as (2.245) and has a unique single-valued and analytic solution which satisfies (2.243) on a region $\mathfrak{D}^- = \mathfrak{D}(\theta_1, \theta_2)$ where $\theta_1 = (\pi/2) + \epsilon$ and $\theta_2 = (3\pi/2) - \epsilon$. This region has a portion $\mathfrak{D}'$ in common with $\mathfrak{D}^+$, and $U(Z)$ is a solution of (2.248) in $\mathfrak{D}'$. $U(Z)$ and $V(Z)$ being analytic functions, they must be identical in $\mathfrak{D}'$ and $V(Z)$ must be the analytic continuation of $U(Z)$ in $\mathfrak{D}^-$. Since the regions $\mathfrak{D}^+$ and $\mathfrak{D}^-$ together make up the region $\mathfrak{D}$ we have completed the proof of the theorem.

The theorem can be somewhat sharpened. Let $\mathfrak{D}^*$ be the region

$$r \geqslant R, \quad \left|\theta - \frac{\pi}{2}\right| \leqslant \frac{3\pi}{2} - \epsilon(r) \qquad (2.249)$$

where $\epsilon(r) > 0$, $\lim_{r\to\infty} \epsilon(r) = 0$ but $\lim_{r\to\infty} r\epsilon(r) = \infty$ and $Z = re^{i\theta}$. Then $\lim_{Z\to\infty} e^{-iZ} W(Z) = 1$ in $\mathfrak{D}^*$, provided the conditions of Theorem 2.24 are fulfilled in $\mathfrak{D}^*$. This follows from the observation, already made, namely that $|U(Z)| < 2$ if $Y > 2\pi M$.

A region $\mathfrak{D}$ or $\mathfrak{D}^*$ as required in Theorem 2.24 is evidently contained in the region $\mathfrak{D}$ obtained in § 2.21 as the map of the region

$$(n-1)\pi \leqslant x \leqslant (n+2)\pi, \quad y \geqslant Y_0$$

in the z-plane. Hence the theorem applies to Mathieu's equation. We shall return to the truncated solutions in § 4 of the present paper.

2.25. The oscillatory case remains for consideration (cf. O.T., § 4.3) Without restricting the generality of the argument we can assume

$$W_0(Z) = \sin(Z-\alpha) \qquad (2.251)$$

in equation (M''). Take a strip Δ_0 in Δ such that $B_1 \leqslant Y \leqslant B_2$, $A_0 \leqslant X$, where $B_1 < \mathfrak{Im}(\alpha) < B_2$. Then

$$W(Z) = \sin(Z-\alpha) + \frac{\Theta_0(Z)}{Z} \qquad (2.252)$$

where $|\Theta_0(Z)|$ is bounded in Δ_0. It is easy to obtain the behaviour of the solution in a region of type $\mathfrak{D}^+$ as used in the proof of Theorem 2.24. If $Z \to \infty$ in either of the sectors

$$\epsilon \leqslant \arg Z \leqslant \pi - \epsilon \quad \text{or} \quad -\pi + \epsilon \leqslant \arg Z \leqslant -\epsilon,$$

$W(Z)$ becomes infinite. In the former sector

$$e^{iZ} W(Z) = -\frac{1}{2i} e^{i\alpha} + \frac{\Theta_s(Z)}{Z} \tag{2.253}$$

where $|\Theta_s(Z)|$ is bounded in the sector in question. Change i into $-i$ in (2.253) and we obtain the corresponding formula for the latter sector.

From formulæ (2.252) and (2.253) it follows that all the zeros which $W(Z)$ may have in $\mathfrak{D}^+$ and which are of large modulus have to lie in Δ_0. As a matter of fact, it can be shown that if we draw a small circle Γ_n of radius δ around each of the points $\alpha + n\pi$ there will be a zero of $W(Z)$ in each of the circles Γ_n outside of the circle Γ_R defined by

$$|Z| = KM_1\pi/\delta, \tag{2.254}$$

where M_1 is the constant in (2.247) and $K = \max|\sin(Z-\alpha)|$ in Δ_0; and there is no zero of $W(Z)$ in the part of $\mathfrak{D}^+$ outside of Γ_R. Similarly the zeros of $W'(Z)$ are approximated by the points $\alpha + (n+\frac{1}{2})\pi$.

2.31. We pass over to the application of the second principle to investigations of the zeros. Starting from the differential equation

$$w''(z) + G(z)\, w(z) = 0 \tag{2.311}$$

we multiply this equation by $\overline{w(z)}$, the conjugate of $w(z)$, and integrate along some path C joining two points z_1 and z_2 in the plane. An integration by parts gives

$$\left[\overline{w(z)} w'(z)\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w'(z)|^2 \overline{dz} + \int_{z_1}^{z_2} |w(z)|^2 G(z)\, dz = 0. \tag{2.312}$$

This we call the *Green's transform* of the differential equation (2.311). Since in the present case $G(z)$ is defined by (2.211) and $w(z)$ is an entire function we do not need to bother about the existence of the integrals as long as the path of integration is finite.

Let us designate as *principal solutions of* (2.311) at $z = z_1$ the solutions defined by

$$\left.\begin{aligned} w_1(z_1) &= 0, \quad & w_2(z_1) &= 1, \\ w_1'(z_1) &= 1, \quad & w_2'(z_1) &= 0. \end{aligned}\right\} \tag{2.313}$$

We introduce an operator $\mathcal{Q}_\gamma$ which, when applied to one of the principal solutions of (2.311) at $z = z_1$, produces the result

$$\mathcal{Q}_\gamma\{w(z), z_1; z\} = -\int_{z_1}^{z} |w'(z)|^2\, \overline{dz} + \int_{z_1}^{z} |w(z)|^2 G(z)\, dz, \quad (2.314)$$

or, shorter, simply $\mathcal{Q}_\gamma(z)$ when no confusion is likely. This function $\mathcal{Q}_\gamma(z)$ is, in view of (2.312), the product of an analytic function of z into an analytic function of $\bar{z}$ and is consequently independent of the path of integration. Then we obtain the

THEOREM 2.31.—*The necessary and sufficient condition that one of the principal solutions of* (2.31) *at* $z = z_1$ *shall vanish or admit of an extremum at* $z = z_2$ *is that*

$$\mathcal{Q}_\gamma\{w(z), z_1; z_2\} = 0. \quad (2.315)$$

2.32. As a direct application of Theorem 2.31 we give the following example. Suppose that B is a *convex region* in which $\vartheta_1 < \arg G(z) < \vartheta_2$ where $0 < \vartheta_2 - \vartheta_1 < \pi$. If z_1 and z_2 are zeros or extrema in B of an arbitrary solution $w(z)$ then either $\vartheta_+ = \arg(z_2 - z_1)$ or $\vartheta_- = \arg(z_1 - z_2)$ satisfies the inequality

$$-\vartheta_2/2 < \vartheta < -\vartheta_1/2. \quad (2.321)$$

This is shown by putting $z = z_1 + re^{i\vartheta_+}$ in (2.314) and multiplying by $e^{i\vartheta_+}$. Thus we obtain

$$e^{2i\vartheta_+} \int |w(z)|^2 G(z)\, dr = \text{a real positive quantity},$$

from which (2.321) easily follows.

As a particular case we may assume that $-\pi < \arg G(z) < 0$; then (2.321) implies that *the zeros and extrema of* $w(z)$ *in* B *can be numerated in such a fashion that their abscissæ and ordinates both form monotonically increasing sets.* If $0 < \arg G(z) < \pi$ instead, one set will be increasing while the other is decreasing.

In the paragraphs 2.3 we do not receive any information about the frequency of the supposed oscillation. By proper manipulation of the constants A and B in equation (M) we can as a matter of fact increase or decrease at will the frequency within any finite bounded region in the plane. Thus it may very well happen that the zeros lie quite close together in a certain region.

If the convex region B mentioned above reduces to the immediate neighbourhood of a point the variation of $\arg G(z)$ in B is negligible, and the formula

$$\theta_{z_1} = -\tfrac{1}{2} \arg G(z_1) \quad (2.322)$$

gives the direction from z_1 to a "nearby" zero or extremum if $w(z)$ is one of the principal solutions at $z = z_1$. The precise sense of this statement is the following. If we fix upon one of the principal solutions at z_1 and by variation of the parameters A and B in the differential equation cause a variable zero or extremum to approach z_1, then

$$\lim \arg (z_2 - z_1) \equiv \theta_{z_1} \pmod{\pi}. \tag{2.323}$$

At every finite point of the plane where $G(z) \neq 0$, formula (2.322) defines a direction uniquely save for an arbitrary multiple of π. This direction θ_{z_1}, together with the point z_1 constitutes a *lineal element* which we shall call the (local) *indicator*. The set of indicators envelop a one-parameter family of curves, the *indicatrices*, which satisfy the differential equation

$$\mathfrak{Im}\left[\sqrt{\{G(z)\}}\, dz\right] = 0, \tag{2.3241}$$

the solutions of which are given by

$$\mathfrak{Im}\left[\int_{z_0}^{z} \sqrt{\{G(z)\}}\, dz\right] = \text{const.} \tag{2.3242}$$

But this is exactly the equation of the curves $\mathfrak{S}$ mentioned in § 2.21 which correspond to the straight lines parallel to the real axis in the Z-plane by means of the transformation (2.213). We recall that in § 2.25 we showed how the zeros of the solutions of (M') in $\mathfrak{D}^+$ approach definite limiting positions, namely a set of points of distance π apart on a line parallel to the axis of reals in the Z-plane. We conclude that the zeros of the corresponding solution of (M) will also approach definite limiting positions, namely a set of points distributed according to some definite law (to which we shall return in § 3.1) along a certain curve of the family $\mathfrak{S}$.*

Through every finite point of the plane where $G(z) \neq 0$ there passes one and only one indicatrix. At a simple zero of $G(z)$ three indicatrices concur and stop, forming 120°-angles with each other; at a double-zero, which requires $A = \pm B$, two indicatrices intersect forming 90°-angles with each other. We notice, moreover, that the system of indicatrices is perfectly independent of the choice of z_0 in the transformation (2.213).

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* This property caused me to designate these curves as *asymptotic zero-curves* in O.T., § 4.5. The name *indicatrices* is chosen in order to suggest that the curves "indicate" the configuration of the zeros throughout the plane.

2.33. In § 1.1 we defined the parameter of a solution of (M) at a point z_1 as the ratio $w'(z_1)/w(z_1) = \lambda$. The result of operating on a solution of parameter λ at the point z_1 with $\mathcal{G}$ we define as follows:

$$\mathcal{G}\{w(z), z_1; z\} = -\overline{w(z_1)}\, w'(z_1) - \int_{z_1}^{z} |w'(z)|^2\, \overline{dz} + \int_{z_1}^{z} |w(z)|^2\, G(z)\, dz, \tag{2.331}$$

or, shortly, $\mathcal{G}_\lambda(z)$ if no confusion is likely to follow. If λ is equal to 0 or ∞, we have to deal with one of the principal solutions, the first term in the formula disappears and $\mathcal{G}_\lambda(z) = \mathcal{G}(z)$, or we fall back on formula (2.314).

THEOREM 2.331.—*The necessary and sufficient condition that a solution of* (2.311) *of parameter* λ *at* $z = z_1$ *shall vanish or admit of an extremum at a point* z_2 *is that*

$$\mathcal{G}\{w(z), z_1; z_2\} = 0.$$

We shall use this theorem in order to find a region such that $\mathcal{G}\{w(z), z_1; z\}$ is different from zero throughout. (Cf. S.L., § 4.) The method is based on the following classical principle: If a set of vectors $V_n = r_n e^{i\phi_n}$ is such that $\vartheta < \phi_n < \vartheta + \pi$ for all values of n and for a fixed value of ϑ, then $\Sigma V_n \neq 0$. Since a definite integral is the limit of a certain sum, the principle also applies to integration of complex functions.

Let C be a curve starting from $z = z_1$ and made up of a finite number of analytic arcs, *i.e.* having a unique tangent except possibly at a finite number of points where unique semi-tangents exist. The slope-angle of C at an arbitrary point z we denote by θ. Let us put

$$\arg \lambda = \arg [w'(z_1)/w(z_1)] = \arg [\overline{w(z_1)}\, w'(z_1)] = \omega,$$

$$\arg G(z) = \phi,$$

and choose an angle ϑ such that

$$\vartheta < \omega + \pi < \vartheta + \pi. \tag{2.332}$$

Further suppose that

$$\vartheta < \pi - \theta < \vartheta + \pi, \tag{2.333}$$

$$\vartheta < \phi + \theta < \vartheta + \pi. \tag{2.334}$$

The expression for $\mathcal{G}\{w(z), z_1; z\}$ in formula (2.331) is the sum of three terms. The argument of the first one is $\omega + \pi$. The other two terms are definite integrals and depend upon the path of integration C. At every point on the path we obtain a contribution to each integral in the form of an infinitesimal vector. These vectors together with the vector

of argument $\omega+\pi$ constitute *the vector field of the path.* If the vectors of the field satisfy the conditions (2.332–3–4) the field is said to be *definite.* Under certain circumstances we can replace the signs $<$ in the inequalities by $\leqq$, namely if either (i) the signs $<$ hold in (2.333) or (2.334) on a set of points on C of positive measure, or (ii) if (2.332) holds with the signs $<$. Then the field shall still be called definite.

A path having a definite vector field shall be called a line of influence (abbreviated an *li*) with reference to the point z_1 and the solution in question. If the field is definite the second and third terms in (2.331) will be finite vectors whose arguments lie in the same range $(\vartheta, \vartheta+\pi)$ and the sum of the three terms will be different from zero. Thus

If $z \neq z_1$ can be reached by an li with reference to the point z_1 and the solution $w(z)$, then $\mathcal{Q}_\vartheta\{w(z), z_1; z\} \neq 0$.

Let us determine all possible *li*'s corresponding to a fixed value of ϑ and give ϑ all possible values compatible with (2.332). The set of points each of which is the end-point of at least one *li* from z_1 forms a region, the *domain of influence* (abbreviated *DI*) with reference to z_1 and the solution in question, and is denoted by $DI(z_1; \omega)$ or, if $\lambda = 0$ or ∞, simply $DI(z_1)$. Evidently $DI(z_1; \omega)$ is usually contained in $DI(z_1)$ since in forming the latter we do not have to take into account the restrictive condition (2.332) on ϑ. We further note that $DI(z_1; \omega)$ depends only upon $\arg\lambda = \omega$ and not upon $|\lambda|$; in other words, all solutions whose parameters at $z = z_1$ have the same argument will have the same *DI* with reference to that point. We may replace the constants A and B in equation (M) by CA and CB respectively (C positive); every single $DI(z_1; \omega)$ will remain *invariant* under such a transformation. In view of Theorem (2.331) we obtain the following theorem which is the precise form of Principle II, namely:

THEOREM 2.332.—*The solution of* (2.31) *of parameter* λ *at* $z = z_1$ *does not admit of a zero or of an extremum in* $DI(z_1; \omega)$.

We sometimes know the argument of $\lambda(z_1)$ all along a certain curve C in the plane; *e.g.* there are solutions which are of real character on the real axis or on one of the lines l_k. In that case we may form the *DI*'s with reference to every point on C. The set of points belonging to at least one such domain forms the *domain of influence* with reference to the curve C and the solution in question which we designate by $DI(C, \omega)$. If z_1 is a point on C and z_2 is a point in $DI(C, \omega)$ then $\mathcal{Q}_\vartheta\{w(z), z_1; z_2\} \neq 0$.

It is usually not worth the trouble to try to determine the exact shape of a *DI* with reference to a given point and a given solution, but much information can be gathered even if we put rather severe restrictions on

the kind of curves we shall use to build up our li's. The simplest possible choice consists of straight lines parallel to the axes, and we shall use no other kind of curve in the later discussion.

2.41. We pass over to the third principle. Let us consider the differential equation

$$w''+G(z)\,w=0. \qquad (2.411)$$

Take a simply connected region in the plane B, where

$$k\frac{\pi}{2}<\arg G(z)<(k+1)\frac{\pi}{2} \quad (k \text{ integral}). \qquad (2.412)$$

We suppose that the boundary of B is made up of curves $\Re[G(z)]=0$ and $\Im\mathfrak{m}[G(z)]=0$. In equation (M) infinity is a point on the boundary of B, so we shall now assume that this condition is fulfilled in (2.411).

Let $z=a$ be a point in B or on its boundary and let $w_1(z)$ and $w_2(z)$ be the principal solutions at this point as defined by formula (2.313). The solution whose parameter at $z=a$ equals λ is given by

$$w(z,\lambda)=w_2(z)+\lambda w_1(z). \qquad (2.413)$$

The solution $w_2(z)$ has certain zeros, z_n, and certain extrema, ξ_m, in B where the subscripts n and m run through sets of positive integers, starting with 1.

Now let λ pass from 0 to a value λ_0 of small modulus; then the zeros and extrema will also move. If we put

$$w(z,\lambda)=0,$$

we have when $|z-z_n|$ is small

$$w(z,\lambda)=\sum_{k=0}^{\infty}a_{nk}(z-z_n)^k+\lambda\sum_{k=1}^{\infty}b_{nk}(z-z_n)^k=0,$$

where a_{n_0} and $b_{n_1}\neq 0$. Using the theorem on implicit functions we conclude that

$$z-z_n=\sum_{k=1}^{\infty}c_{nk}\lambda^k$$

where the power series converges at least when $|\lambda|$ is small. Thus the zeros are analytic functions $z_n(\lambda)$ of λ. If $G(z_1)\neq 0$ we can conclude in a similar manner that the extrema are analytic functions $\xi_n(\lambda)$ of λ, single-valued in the neighbourhood of $\lambda=0$.

We proceed to investigate the singular points of these functions. Let us consider the analytic continuation of $z_n(\lambda)$ along some path Λ from the

origin in the λ-plane. On account of the law of permanence of functional equations the relation

$$w\{z_n(\lambda), \lambda\} \equiv 0$$

will hold as long as the associated values z_n, λ are such that w is analytic in their neighbourhoods. When $\lambda \to \lambda_0$, a finite value on Λ, there are four different ways in which $z_n(\lambda)$ may conceivably behave, namely: (i) it may approach a finite non-singular value a, (ii) it may approach a finite singular value b, (iii) it may become infinite, and (iv) it may approach no limit whatever.

In case (i) $\lambda = \lambda_0$ is not a singular point of $z_n(\lambda)$. Case (ii) cannot be produced with Mathieu's equation, which has only one singular point, namely at infinity, and may be left out of consideration here. It can be shown that case (iv) cannot occur. Finally, in case (iii) $\lambda = \lambda_0$ is a singular point of $z_n(\lambda)$ and may be a branch-point of infinite order. In the case of an infinite branch-point at $\lambda = \lambda_0$ this will be a singularity common to a certain sub-set of the functions $z_k(\lambda)$ which will be permuted around $\lambda = \lambda_0$. If the differential equation admits of truncated solutions their parameters at $z = a$ will be such singular points.

The analytic continuation of $\xi_m(\lambda)$ can be handled in the same manner. There is one difference, however, with regard to case (i). If the value a should happen to be a zero of $G(z)$ of order ν the point λ_0 will be an algebraic branch-point of $\xi_m(\lambda)$ of order ν in the neighbourhood of which $\nu+1$ branches are cyclically permuted. There are no other possible singular points of the functions, since $\lambda = \infty$ is easily shown to be a regular point.

We notice that these points are the singular points of the functions inverse to

$$\lambda(z) = -w_2(z)/w_1(z); \qquad \lambda(\xi) = -w_2'(\xi)/w_1'(\xi). \tag{2.414}$$

Hence $z_n(\lambda)$ $(n = 1, 2, \ldots)$ are merely the different branches of the same analytic function $z(\lambda)$ and likewise with $\xi_m(\lambda)$. If the solutions of (2.411) are entire functions, the functions in (2.414) are meromorphic.

2.42. The essential thing for us to know is that $z(\lambda)$ and $\xi(\lambda)$ are continuous functions of λ if we except certain critical points.

Now let us vary λ from 0 along a certain continuous path avoiding the critical points. When we do so the zeros and extrema will also describe certain continuous curves. Let us concentrate our attention on the points of the set in B. At every instant of the motion these points represent the zeros and extrema of a certain particular solution of (2.411). Now the positions of these points are restricted in various ways by conditions im-

posed in view of Principle II. In order to fix the ideas, let us assume that the integer k in equation (2.412) is congruent to 2 (mod 4).

Then we can enumerate the points in B in such a way that the abscissae and ordinates form monotonically increasing sets. Moreover, according to formula (2.321) the angle of inclination ϑ of a line joining any two points of the set will satisfy the inequality $\pi/4 \leqslant \vartheta \leqslant \pi/2$ where the equality signs hold only on the boundary.

Thus if we are able to establish a certain order between the zeros and extrema of $w(z, \lambda)$ in B for $\lambda = 0$ these relations will persist for all values of λ as long as the points in question stay in B. If, for example, the relations $\Re[z_n(\lambda)] < \Re[z_m(\lambda)]$ and $\Im[z_n(\lambda)] < \Im[z_m(\lambda)]$ hold for $\lambda = 0$ they will continue to hold as long as $z_n(\lambda)$ and $z_m(\lambda)$ remain in B.

No zero or extremum can suddenly disappear in the interior of B. The only way in which we can lose a zero is by its passing the boundary of B or by its receding to infinity. In the same way, we gain new zeros and extrema only through arrivals across the boundary or from the singular point at infinity.

In the study of the zeros of Mathieu functions we are very much helped by the fact that the curves $\Im[G(z)] = 0$ are straight lines (r and l_k) and that the solution which admits of a zero or an extremum on one of these lines must be of real character on the line. Thus we can control with comparative ease the movements along and across the lines

$$\Im[G(z)] = 0.$$

The recession of a zero to infinity can occur only if we happen to hit upon one of the critical values of λ. We notice that if a zero recedes to infinity it will often, on account of the order relations, push a whole set of zeros and extrema in front of itself. If we pass through the critical value, these zeros and extrema, or at least some of them, will return to finite parts of the plane. These returning points are also subjected to the order relations, but it may be a different order from that holding among the receding points. This makes the truncated solutions so important.

One remark about the algebraic critical points. It can never happen that two zeros coalesce at a finite point a. But if a is a zero of $G(z)$ of order ν, there exists a solution such that $w^{(\mu)}(a) = 0$ $(\mu = 1, 2, \ldots, \nu+1)$. Thus $(\nu+1)$ extrema coalesce at that point. The corresponding value of λ will be an algebraic branch-point of $(\nu+1)$ branches of $\zeta(\lambda)$.

2.43. For Mathieu's equation the curves $\Re[G(z)] = P(z) = 0$ and $\Im[G(z)] = Q(z) = 0$ are given by the formulæ

$$P(z) = B - A\cos 2x \cosh 2y\,; \qquad Q(z) = A \sin 2x \sinh 2y. \quad (2.431)$$

The curves $Q = 0$ are the lines l_k and r. The curve $P = 0$ is a transcendental curve which repeats itself with period π. In each of the regions D_n^+ and D_n^- there are two branches of the curve which have the lines $x = (4n \pm 1)(\pi/4)$ as asymptotes. In Case 1 the branches in D_n^+ cross the lines l_{2n-1} and l_{2n+1} and join on to branches in D_{n-1}^+ and D_{n+1}^+ respectively; in Case 2 they cross r and join the branches in D_n^-, and finally in Case 3 they join each other.

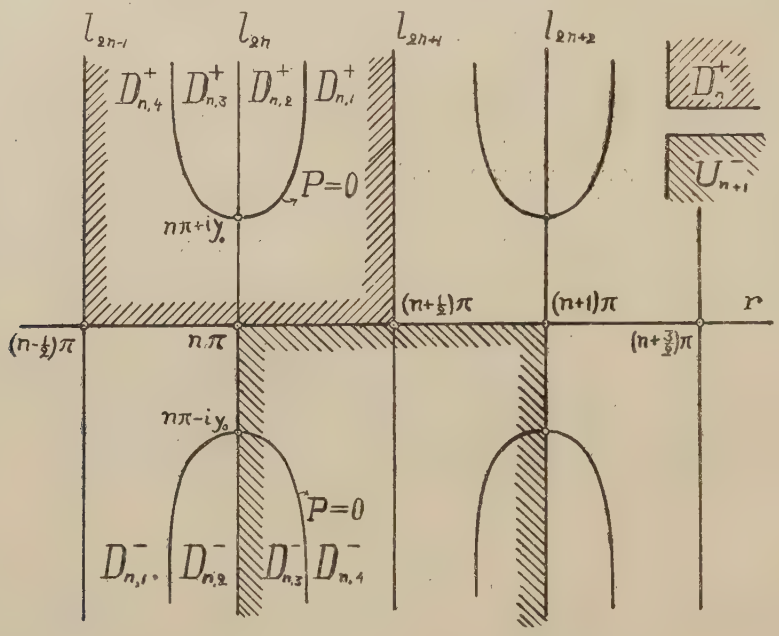

Fig. 1.

Let us designate by $D_{n,j}^+$ $(j = 1, 2, 3, 4)$ the region in D_n^+ bounded by the curves $P = 0$ and $Q = 0$ in which the argument of $G(z)$ lies in the j-th quadrant, and similarly for the region D_n^-. (See Fig. 1, which gives a schematic representation of Case 3.)

3. *The general distribution problem.*

3.1. We now proceed to discuss the distribution of the zeros of the Mathieu functions in the complex domain.

Let us first use the results of the §§ 2.2. The general solution $W(Z)$ of the differential equation (M′) is oscillatory in Δ. If we enumerate the zeros according to increasing real parts $A_1, A_2, \ldots, A_n, \ldots$, there exists a number α such that

$$A_n - a_n \to 0 \quad (a_n = \alpha + n\pi) \tag{3.11}$$

with $1/n$. When we pass back to equation (M) by means of the inverse of transformation (2.221) we notice that $w(z)$ vanishes at the points A_n in the z-plane which correspond to the points A_n in the Z-plane by the inverse of (2.213) and there are no other zeros of $w(z)$ in the region which is the map of Δ. The presence of the factor $[G(z)]^{\frac{1}{2}}$ in formula (2.221) evidently does not affect the situation, since the values of Z which correspond to the roots of $G(z)=0$ are on the boundary of Δ. We can make Δ correspond to any one of the regions D_n^+ or D_n^-. Suppose that z_0 is chosen as originally in formula (2.213) so that D_0^+ is the map of Δ. Let us put $\alpha=\sigma+i\tau$. Then the line $Y=\tau$ in the Z-plane is carried over into an indicatrix $\mathfrak{S}_\tau$ in D_0^+, namely

$$\mathfrak{Im}\left[\int_{z_0}^{z}\sqrt{\{G(z)\}}\,dz\right]=\tau, \tag{3.12}$$

and the points a_n correspond to points a_n on $\mathfrak{S}_\tau$ such that

$$\int_{z_0}^{a_n}\sqrt{\{G(z)\}}\,dz=a_n. \tag{3.13}$$

We conclude that the zeros of $w(z)$ in D_0^+ are approximated by the points a_n $(n=1, 2, \ldots)$ in the sense that

$$A_n-a_n\to 0 \tag{3.14}$$

with $1/n$. Let us at least get a rough notion of the size of these quantities a_n. By (2.215) we have

$$\sqrt{(A/2)}\,e^{-ia_n}+C+O(e^{-t_n})=a_n \quad (a_n=s_n+it_n). \tag{3.15}$$

From this formula we conclude that

$$a_n=-\frac{\tau}{|a_n|}+i\log\frac{|a_n-C|}{\sqrt{(A/2)}}+O\left(\frac{1}{n\log n}\right). \tag{3.16}$$

If we replace $|a_n|$ and $|a_n-C|$ by $n\pi$ we commit an error of the order of $1/n$ which is of the same magnitude as the distance between a_n and a_{n+1}.

The zeros of $w'(z)$ which we denote by A_n' can be handled in a similar manner. It can be shown that the presence of the factor $[G(z)]^{\frac{1}{2}}$ in formula (2.221) does not affect the asymptotic character of the distribution (cf. O.T., § 4.5). These points A_n' approach a set of points a_n' which are the images of the points $a_n'=\alpha+(n+\frac{1}{2})\pi$ by the transformation (2.213), in the sense that

$$A_n'-a_n'\to 0. \tag{3.17}$$

We notice that the point a_n' lies on $\mathfrak{S}_\tau$ between a_n and a_{n+1}. We get an asymptotic expression for a_n' by replacing a_n by a_n' in formula (3 . 16).

Let us arrange the zeros and extrema of $w(z)$ in D_0^+ into a sequence $\ldots, A_n, A_n', A_{n+1}, A_{n+1}', A_{n+2}, \ldots$. A_1 is the first A_n to occur in the sequence, whereas we remain in ignorance as to which is the first A_n' actually occurring. These points we shall say form a *string* S_0^+ in D_0^+.* At least from a certain point on, *the abscissae of the points in the string have the same signs and they ultimately approach zero as a limit; further, the ordinates increase indefinitely toward* ∞. The number of points in the string of ordinate less than T is

$$\frac{2}{\pi}\sqrt{\frac{A}{2}}\,e^T+O(1).\dagger \tag{3 . 18}$$

From a certain point on, the ordinates form a monotonically increasing sequence in which zeros and extrema alternate.

The general solution behaves in exactly the same manner in all regions D, that is to say in each of the regions D_n^+ there is a string S_n^+ and in each of the regions D_n^- a string S_n^-. For the purpose of classification we distinguish between three kinds of strings which we designate by the prefixes *l-*, *r-*, and *d-* (*laevus*, *rectus*, and *dexter*), according as the abscissae of the points in the string are ultimately less than, equal to, or greater than their limit. Thus S_n^+ is called an *l*-string if $\Re[A_{n,m}^+] < n\pi$ when $m > m_0$ and where $A_{n,m}^+$ stands for the m-th zero in the string.

There are only two linearly independent solutions of (M) which do not oscillate in D_n^+. One of these solutions approaches zero very rapidly in U_{n+1}^+; as a matter of fact in view of (2 . 214), (2 . 221), and (2 . 232) we know that

$$w_1(z)\,\exp\{i\sqrt{(A/2)}\,e^{-i(z-n\pi)}-\tfrac{1}{2}iz\} \tag{3 . 19}$$

is bounded in the region $(n-1)\pi+\epsilon \leqslant x \leqslant (n+2)\pi-\epsilon$, $y \geqslant 0$. We could replace the parallels to the lines l_{2n-2} and l_{2n} by certain curves

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* We use the word *string* to denote a set of infinitely many isolated points $c_1, c_2, \ldots, c_n, \ldots$, having the following properties: (1) the set has a unique limit point c; (2) from a certain point on, either the abscissae or the ordinates of the points form a monotonic set; (3) there exists an analytic curve C leading up to c such that the distance d_n of c_n to C approaches zero with $1/n$, and, moreover, $d_n/|\,c_{n+1}-c_n\,| \to 0$.

We can obviously exclude a finite number of points $c_1, c_2, \ldots, c_m$ without changing the character of a point set of being a string. We shall often find it hard to decide whether certain points ought to be included in a string or not (see *e.g.* below in § 3 . 24). Such questions can be settled by means of considerations concerning continuity, which is the leading notion in principle III.

† The remainder is uniformly bounded if α is restricted to a finite region.

asymptotic to these lines. [Cf. theorem (2.24) and formula (2.249).] From (3.19) it follows in particular that the strings S_n^+ and S_{n+1}^+ are missing. The other solution truncated in D_n^+ lacks the strings S_{n-1}^+ and S_n^+ and approaches zero in U_n^+. The statement that the string S_n^+ is missing is, of course, not to be interpreted to mean that there are no zeros and extrema of the solution in D_n^+; but it does mean that there are only a finite number of such points. We shall return to the truncated solution in § 4.

3.20. We have now to refine the conclusions arrived at in the preceding section. There are various problems that offer themselves for study, *e.g.*

A. *How regular is the distribution of the points in the string?*

B. *Can restrictions of the form "from a certain point on" be removed in the statements of* § 3.1?

C. *Are adjacent strings of the same character?*

These problems are all of a qualitative descriptive nature and can be studied by means of the methods developed in the §§ 2.3. In Case 1 we can give a fairly satisfactory answer to all these questions, whereas we have to content ourselves with results much more meagre in the two other cases. There is a decided difference between Case 1 on one side and Cases 2 and 3 on the other, even in other respects. There are no periodic solutions in Case 1 and the solutions with constant multipliers (periodic functions of the second kind) are of real character on the real axis. In Case 3 (and probably also in Case 2) quite the opposite holds.

3.21. Let us first attack problem A. The string S_n^+ will enter and ultimately remain in $D_{n,3}^+$ if it is an l-string, but in $D_{n,2}^+$ if it is a d-string. Topologically, the two cases are perfectly symmetric, so we can restrict ourselves to the study of l-strings. The question of r-strings will be considered in § 5, where we shall study solutions of real character.

Suppose that we have a zero or an extremum at a point $z_1 = x_1 + iy_1$, $(n-\frac{1}{2})\pi < x_1 < n\pi$, $0 < y_1$. In this region, the left half of D_n^+, we have $-\pi \leqslant \arg G(z) \leqslant 0$, where the signs of equality hold on the boundary only. Hence by a remark in § 2.32 every other zero or extremum $z_2 = x_2 + iy_2$ in the region will satisfy either $x_1 < x_2$, $y_1 < y_2$ or $x_2 < x_1$, $y_2 < y_1$.

In other words, *those of the points in an l-string* S_n^+ *which lie in the left half of* D_n^+ *form an array such that the abscissae and the ordinates*

are simultaneously increasing sequences. The most we could infer from formula (3.19) was that if the zeros and extrema were arranged in the order $\ldots, A_n, A'_n, A_{n+1}, A'_{n+1}, A_{n+2}, \ldots$ then the ordinates were ultimately steadily increasing. Here we find that if the points in the string are ordered according to increasing ordinates, the abscissae will automatically form an increasing sequence. This order may conceivably be other than the one used in § 3.1, though ultimately the order will be the same. We shall be able further to elucidate this point in later paragraphs.

3.22. Let us determine at least a part of the *DI* with respect to one of the points $z_1 = x_1 + iy_1$ in the *l*-string S_n^+ and the corresponding principal solution. Since an *l*-string S_n^+ will enter and ultimately remain in the region $D_{n,3}^+$, we shall assume that z_1 lies in this region. To begin with, the *DI* will include the two regions $(n-\frac{1}{2})\pi \leqslant x \leqslant x_1$, $y_1 \leqslant y$, and $x_1 \leqslant x \leqslant n\pi$, $0 \leqslant y \leqslant y_1$. We can further find *li*'s enabling us to cross the lines l_{2n} and r. We can always reach the regions $D_{n,2}^+$ and $D_{n,3}^-$ by means of properly chosen *li*'s. In particular, there can be no zeros or extrema on the line l_{2n} either above or below r, since a *DI* is a closed region (excepting the point z_1 itself and eventual singular points of the differential equation, here only the point at infinity). Thus the solution in question cannot be of real character on l_{2n}. In Case 1 we can also reach $D_{n,1}^+$ and $D_{n,4}^-$, and in addition a vast territory further to the right which will be determined in § 3.23.

We shall indicate how to construct an *li** leading to an arbitrary point $z_2 = x_2 + iy_2$ in $D_{n,2}^+$. We can clearly assume $y_1 > y_2$ since there are infinitely many points in the string. This assumption is not necessary, but it is convenient if we wish to avoid repetitions. Let us further agree to denote by Θ_j a perfectly arbitrary variable angle in the jth quadrant. This notation we shall preserve throughout the paper. Then the broken polygonal line $z_1 \text{——} n\pi + iy_1 \text{——} n\pi + iy_2 \text{——} z_2$ will serve our purpose. In order to show that this line is actually an *li* we have to study the variation of the angles $\pi - \theta$ and $\phi + \theta$ defined in § 2.33. On the three portions of the line we have, respectively, $\pi - \theta = \pi, \frac{3}{2}\pi, \pi$ and $\phi + \theta = \Theta_3, \pi/2, \Theta_2$. Hence if we take $\Im = \pi/2$, the formulæ (2.333) and (2.334) are fulfilled; thus the line, having a definite vector field, is a suitable *li*.

In order to reach a point in $D_{n,1}^+$ in Case 1 we proceed as follows. The line $x = x_2$ cuts the curve $P = 0$ (see § 2.43) in a point $z_3 = x_2 + iy_3$. Take the broken line $z_1 \text{——} n\pi + iy_1 \text{——} n\pi + iy_3 \text{——} z_3 \text{——} z_2$. The

* The *li*'s mentioned in §§ 3.22 and 3.23 are indicated in Fig. 2.

only difference between this path and the one treated above is the fourth portion where $\pi-\theta=\pi/2$ and $\phi+\theta=\Theta_2$.

Just a few words about the region $D^-_{n,3}$. The line l_{2n} serves as an inlet to the lower half-plane and the *li* is formed by the broken line

$$z_1 \text{——} n\pi+iy_1 \text{——} n\pi+iy_2 \text{——} z_2.$$

Let us now interpret these results. They imply that *in Case* 1 *the string* S^+_n *lies altogether in the left half of* D^+_n *if it is an l-string and in the right half if it is a d-string.* In the former case, that of an *l*-string,

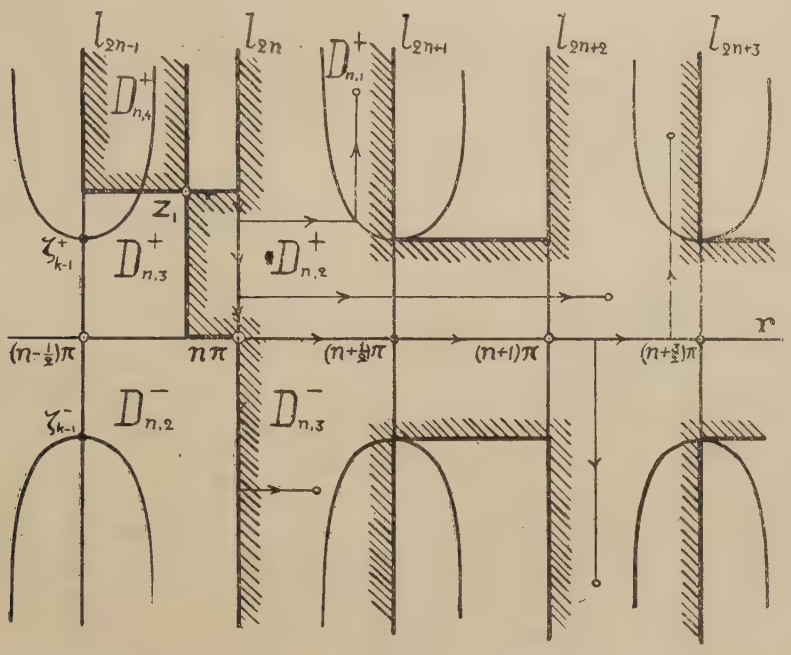

Fig. 2.

the right half of D^+_n and the dividing line l_{2n} are perfectly free from zeros and extrema. In Cases 2 and 3, on the other hand, we cannot ignore the possibility that there may be a finite number of zeros and extrema in the right half of D^+_n, namely in $D^+_{n,1}$, even in the case of an *l*-string. If we order these extraneous points according to *increasing* abscissae their ordinates will form a *decreasing* sequence. It is actually more consistent not to include the extraneous zeros in the string S^+_n. We shall return to this point in § 3 . 24.

Incidentally we can make a contribution to problem (C), namely we

have shown that if S_n^+ is an l-string then S_n^- is also an l-string. Hence *the strings S_n^+ and S_n^- have always the same prefix* (l-, d-, or r-).

3.23. Let us carry the general discussion a little further in Case 1. In the preceding paragraph we found that if z_1 is a point in $D_{n,3}^+$ then $DI(z_1)$ contains, besides regions in the left half of D_n^+, the infinite strip $n\pi \leqslant x \leqslant (n+\frac{1}{2})\pi$. But we can get very much more. Let us denote the roots of $G(z) = 0$ by ζ_k^+, $\zeta_k^- = (k+\frac{1}{2})\pi \pm iy_0$ ($k = 0, \pm 1, \pm 2, \ldots$). Then we can include in the DI the semi-infinite strip $x \geqslant n\pi$, $-y_0 \leqslant y \leqslant +y_0$, and, further, all the regions $D_{k,2}^+$, $D_{k,1}^+$, $D_{k,3}^-$, and $D_{k,4}^-$, if $k > n$, or, in other words, all the infinite strips $k\pi \leqslant x \leqslant (k+\frac{1}{2})\pi$, $k > n$. In order to reach a point z_2 in the horizontal strip we use the following broken line $z_1 \text{——} n\pi+iy_1 \text{——} n\pi+iy_2 \text{——} z_2$. The values of the angles are as follows: $\pi-\theta = \pi$, $\frac{3}{2}\pi$, π and $\phi+\theta = \Theta_3$, $\pi/2$, Θ_2 or Θ_3. In order to reach a point in one of the vertical strips we use the broken line $z_1 \text{——} n\pi+iy_1 \text{——} n\pi \text{——} x_2 \text{——} z_2$. The angles are as follows: $\pi-\theta = \pi$, $3\pi/2$, π, $\pi/2$ or $3\pi/2$ and $\phi+\theta = \Theta_3$, $\pi/2$, π, Θ_2 or Θ_3. In all cases we can take $\Im = \pi/2$. See Fig. 2 for the shape of $DI(z_1)$.

It is convenient for us to say that a string is *isolated* if the distance from an arbitrary point of the string to an arbitrary point of any other string necessarily exceeds a given positive number. Since by proper manipulation of the constants A and B we can arbitrarily increase the frequency in a given region, it is of importance to be able to separate, if not the zeros themselves, at least the strings.

From the results above we conclude that if S_n^+ is an l-string then all the strings S_k^+ with $k > n$ and all strings S_k^- with $k \geqslant n$ are also l-strings. Moreover, these strings, excepting S_n^+ and S_n^-, are all isolated, the minimum distance between two strings exceeding the smaller of the two numbers $\pi/2$ and $2y_0$. By symmetry we conclude that if S_n^+ is a d-string then all strings to the left are isolated d-strings. An r-string can be regarded as a transition case between an l-string and a d-string; this being so, we can expect that if S_n^+ is an r-string then all strings to the left are isolated d-strings and all strings to the right isolated l-strings. This statement is easily verified. We shall say that two strings are *congruent modulo $s\pi$*, if the transformation $u = z+s\pi$ will carry one string over into point-wise coincidence with the other in such a manner that zeros go into zeros and extrema into extrema.

We are now able to give a classification of the different types of solutions which occur in Case 1. For the sake of completeness we include results which will be proved later in §§4 and 5.

THEOREM 3.23.—*The differential equation*

$$w''+(B-A\cos 2z)\,w=0$$

where $A>0$ *and* $B\leqslant -A$ *has the following types of solutions:—*

(1) *Type d: solution with multiplier. All the strings are isolated d-strings congruent modulo* π. *There is only one such solution* $M^+(z)=e^{-\mu z}F(z)$ *where* μ *is a positive constant and* $F(z)$ *is a periodic function of period* π. *This solution approaches zero when* $z\to\infty$ *in a direction parallel to the positive real axis and is of real character on the real axis.*

(2) *Type d–l: the general solution. The strings* S_k^+ *and* S_k^- *are d-strings when* $k\leqslant n$, *l-strings when* $k\geqslant n+1$. *All strings except* S_n^+, S_n^-, S_{n+1}^+, *and* S_{n+1}^- *are isolated. The solution may be of real character on the real axis, in which case there is one and only one real zero or extremum, namely on the interval* $n\pi$, $(n+1)\pi$.

(3) *Type d–r–l: solutions of real character on one of the lines* l_{2k}. *The strings* S_n^+ *and* S_n^- *are r-strings. All strings to the left of these are isolated d-strings, all those to the right are isolated l-strings. The solution is of real character on* l_{2n}.

(4) *Type d–t–l: truncated solutions. The strings* S_{n-1}^+ *and* S_n^+ *are missing and there are no zeros or extrema in* D_{n-1}^+ *and* D_n^+. *The strings* S_k^+, $k<n-1$, *and* S_k^-, $k\leqslant n-1$, *are d-strings. The strings* S_k^+, $k>n$, *and* S_k^-, $k\geqslant n$ *are l-strings. All strings are isolated except* S_{n-1}^- *and* S_n^-. *The solution is of real character on the line* l_{2n-1} *and tends toward zero when* $z\to\infty$ *in* U_n^+. *There is only one such solution for each value of* n, $T_n^+(z)=T(z-n\pi)$. *The solution* $T[-z+(n-1)\pi]$ *is also truncated; its properties are the same as those of* $T(z-n\pi)$ *except that we have to interchange the plus-signs and minus-signs used as superscripts on the letters S, D, and U in the statement above.*

(5) *Type l: solution with multiplier. All strings are isolated l-strings congruent modulo* π. *There is only one such solution, namely*

$$M^-(z)=M^+(-z).$$

The statement that a string S_n^+ is isolated implies that the ordinates of the points in the string are greater than y_0.

3.24. In § 3.22 we found that in the Cases 2 and 3 there may be extraneous zeros in the region $D_{n,1}^+$ when we had an l-string in D_n^+. In this paragraph we propose to take a closer look at this phenomenon, the explanation of which we postpone to § 6 (§ 6.32) where we shall study the motion of the zeros when the parameters are varied.

Suppose that we have a zero or an extremum at a point z_1 interior to $D^+_{n,1}$ and let us find out as much as we can about the corresponding DI. To begin with, it will contain the two regions $n\pi \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$, and $x_1 \leqslant x \leqslant (n+\frac{1}{2})\pi$, $y_1 \leqslant y$. But we can also get the whole real axis and all the regions $D^-_{k,1}$ and $D^-_{k,4}$ in the lower half-plane.

This statement has to be slightly modified in Case 2 if $-A < B < 0$. Denote the roots of $G(z) = 0$ by $\xi^+_k = \xi + k\pi$ and $\xi^-_k = -\xi + k\pi$ where $\pi/4 < \xi < \pi/2$. Then we can only reach the strips $\xi^+_k \leqslant x \leqslant \xi^-_{k+1}$, $y \leqslant 0$. For our present purpose this restriction is at least partly compensated by the fact that in the case in question the DI of a point in $D^+_{n,3}$ is somewhat larger than is stated above in § 3.21. As a matter of fact, it contains the strip $n\pi \leqslant x \leqslant \xi^+_n$. Thus the point $z_1 = x_1 + iy_1$ in $D^+_{n,1}$, whose DI we are considering, may be subjected to the restriction $\xi^+_n < x_1 < (n+\frac{1}{2})\pi$, since we are only interested in the case in which there is an l-string in D^+_n.

After this digression, let us determine the li that will take us to a point $z_2 = x_2 + iy_2$ in one of the regions in the lower half-plane mentioned above as free from zeros. The broken line $z_1 \text{——} x_1 \text{——} x_2 \text{——} z_2$ will do for our purpose. The angles are $\pi - \theta = 3\pi/2$, π, $3\pi/2$ and $\phi + \theta = \Theta_4$, π or 2π, Θ_3 or Θ_4. Hence we may take $\Im = \pi$.

Let us proceed to the interpretation of this result. There is no obvious connection between the extraneous zeros in $D^+_{n,1}$ and any of the strings S^+_k. As far as we know, and later information will confirm this possibility, there may be extraneous zeros and extrema in all the regions D^+_k but we cannot at the same time have any on the real axis or in the lower half-plane. Moreover, in Case 3 the strings S^-_k are isolated; the minimum distance from a string to any other string exceeds the smaller of the two numbers $\pi/2$ and $2y_0$, if $(k+\frac{1}{2})\pi \pm iy_0$ is a root of $G(z) = 0$. In Case 2 all S^-_k are isolated from each other (minimum distance $\pi/2$) but S^+_k is separated from S^-_k only by the real axis on which no zeros or extrema can be located. Thus the phenomenon of extraneous *complex* zeros and extrema can never present itself with a solution of real character, but such solutions have *real* zeros and extrema instead, which are not accounted for by the strings.

3.3. These developments will serve to give the reader some idea of the topological nature of the configurations formed by the zeros and extrema of general Mathieu functions. An important point which has not been settled is whether or not the zeros alternate with the extrema in the string from the beginning. This question will be answered in the affirmative in § 6 of the paper.

We proceed to the study of particular solutions of interesting nature, namely the truncated solutions and solutions of real character on the real axis or on the lines l_{2n}. Among the latter we find *the functions of the elliptic cylinder*, the zeros of which are distributed in a particularly simple manner.

4. *Truncated solutions.*

4.1. We know there exist two solutions truncated in D_n^+. One of these, $T_{n+1}^+(z)$, lacks the two strings S_n^+ and S_{n+1}^+ and tends to zero when $z \to \infty$ in the region U_{n+1}^+ between the lines l_{2n} and l_{2n+2}. From formula (3.19) we conclude that if $(n-1)\pi+\epsilon \leqslant x \leqslant (n+2)\pi-\epsilon$, $y > 0$,

$$|T_{n+1}^+(z)| < K \exp[-\sqrt{(A/2)}\, e^y \sin(x-n\pi) - \tfrac{1}{2}y]. \qquad (4.11)$$

The other truncated solution $T_n^+(z)$ lacks the two strings S_{n-1}^+ and S_n^+ and tends to zero when z becomes infinite in the region U_n^+. There is a very simple relation between these two solutions. As a matter of fact, the function $T_{n+1}^+(z+\pi)$ is also a solution of (M); it lacks the strings S_{n-1}^+ and S_n^+ and tends to zero in U_n^+. But there is only one solution having these properties, if we, as usual, restrict ourselves to linearly independent solutions. Hence

$$T_{n+1}^+(z+\pi) = T_n^+(z), \qquad (4.12)$$

and in general

$$T_{n+m}^+(z+m\pi) = T_n^+(z). \qquad (4.13)$$

Thus, all the solutions which are truncated in a region D^+ in the upper half-plane can be expressed in terms of

$$T_0^+(z) = T(z) \qquad (4.14)$$

as follows

$$T_n^+(z) = T(z-n\pi). \qquad (4.15)$$

The solutions which are truncated in a region D_k^- in the lower half-plane can be expressed in a similar manner. In fact, the function $T(-z)$ is also a solution of (M); it approaches zero in U_1^- and lacks the strings S_0^- and S_1^-. Thus the family of truncated solutions is contained in the formula

$$T(\pm z+n\pi) \quad (n = 0, \pm 1, \pm 2, \ldots). \qquad (4.16)$$

Hence we can restrict ourselves to discussing the solution $T(z)$.

Are these solutions taken by pairs linearly independent? This question will be considered in § 4.31. Before we can take up that question, however, we have to solve the distribution problem.

We conclude this discussion of the analytical properties of $T(z)$ by

remarking that $T(z)$ is of real character on the line l_{-1}. This can be shown as follows. Put $z = -(\pi/2)+iy$ in equation (M). The result is

$$\frac{d^2w}{dy^2} - (A\cosh 2y + B)w = 0. \tag{4.17}$$

The coefficient of w tends to $-\infty$ when $y \to +\infty$. It is known* that such an equation possesses one and only one solution of real character for real values of y which tends to zero when $y \to +\infty$. This solution must be a constant multiple of $T(z)$.

The solution $T(z)$ will not in general be of real character on the real axis in the z-plane. According to a remark made in § 1.3, this would require that $T(z)\,T'(z)$ is zero at $z = -(\pi/2)$. But this would imply that the solutions $T(z)$ and $T(-z-\pi)$ would be linearly dependent, which is not the case in general. (Cf. § 4.21 below.)

4.20. Formula (4.11) enables us to conclude that

$$T(z_1)\,T'(z_1) \to 0, \tag{4.201}$$

and that the following integrals converge, namely

$$\int_{z_1}^{z} |\,T'(z)\,|^2\,\overline{dz} \quad \text{and} \quad \int_{z_1}^{z} |\,T(z)\,|^2\,G(z)\,dz, \tag{4.202}$$

when $z_1 \to \infty$ in U_0^+ along a path which ultimately remains at a distance from the lines l_{-2} and l_0 that exceeds a fixed positive quantity ϵ. This observation shows that, by way of extension, we can speak of the domain of influence with respect to the point $z_1 = -(\pi/2)+i\infty$ and the principal solution at that point, namely $T(z)$. We can construct the DI in the ordinary manner and the product $T(z)T'(z)$ will be different from zero at all the finite points of the region so obtained.

The regions will differ considerably in the three different cases. Let us carry through the discussion in some detail for the Case 1 and afterwards merely state the results for Cases 2 and 3.

4.21. In Case 1 $DI(z_1)$, $z_1 = -(\pi/2)+i\infty$, will contain the following regions, the boundaries included if not otherwise stated, namely: U_0^+, the horizontal strip $-y_0 \leqslant y \leqslant +y_0$, $x \neq -(\pi/2)$, the vertical strips $-(k+\frac{1}{2})\pi \leqslant x \leqslant -k\pi$ $(k = 1, 2, \ldots)$, and $k\pi \leqslant x \leqslant (k+\frac{1}{2})\pi$, $(k = 0, 1, \ldots)$. We call the reader's attention to the fact that the line-segment

* Cf. Wiman, "Ueber die reellen Lösungen der linearen Differentialgleichungen zweiter Ordnung" *Arkiv f. Mat., Astr. o. Fys.*, 12 (1917), No. 14.

$x = -(\pi/2)$, $-y_0 \leqslant y \leqslant +y_0$ is excluded from the region. The horizontal strip of course partly overlaps the other regions. The quantity y_0 has the same meaning as in § 3.23.

Let us select a couple of typical points in the indicated regions and let us show how to arrive at these points by means of *li*'s from z_1. First, take a point in the horizontal strip, namely $z_2 = x_2 + iy_2$ where $x_2 < -(\pi/2)$, say, and $-y_0 \leqslant y_2 \leqslant +y_0$. The line l_{-1} gives the easiest access to the domain under consideration. We take the broken line

$$z_1 \text{——} -(\pi/2)+iy_2 \text{——} z_2.$$

The angles are $\pi-\theta = -(\pi/2), 0$; $\phi+\theta = -(\pi/2)$ or $+(\pi/2)$, Θ_1 or Θ_4 and $\vartheta = -(\pi/2)$.

Then take a point in one of the vertical strips of the second series. We choose $z_2 = x_2+iy_2$ in $D^+_{n,1}(n > 0)$. This is the *li* that will serve our need, namely the broken line

$$z_1 \text{——} -(\pi/2) \text{——} n\pi \text{——} n\pi+iy_3 \text{——} z_3 \text{——} z_2,$$

where z_3 is the point of intersection of the line $x = x_2$ with the curve $P = 0$. The angles are as follows: $\pi-\theta = 3\pi/2$, π, $\pi/2$, π, $\pi/2$; $\phi+\theta = 3\pi/2$ or $\pi/2$, π, $3\pi/2$, Θ_2, Θ_2 and $\vartheta = \pi/2$.

These results imply the following facts about the distribution of the zeros and extrema of $T(z)$. The strings S^+_{-1} and S^+_0 are missing and there are no zeros or extrema in D^+_{-1} and D^+_0 save possibly a finite number on the dividing line l_{-1} between $-(\pi/2)-iy_0$ and $-(\pi/2)+iy_0$. All the strings to the left of l_{-1} are d-strings, all strings to the right are l-strings. All strings are isolated except S^-_{-1} and S^-_0. There are no real zeros or extrema in general (see below, however). Further, the solution is not of real character on any of the lines l_{2n} and, in general, not on the real axis.

If $T(z)$ is to be of real character on the real axis either $T(z)$ or $T'(z)$ has to vanish at $z = -(\pi/2)$. This will not be the case for arbitrary values of A and B, but it is fairly obvious, intuitively at any rate, that to a given value of A there exist infinitely many discreet values of B for which the following singular boundary problem has a so ution not identically zero, namely

$$\frac{d^2w}{dy^2} - (A\cosh 2y + B)w = 0, \qquad w(0) = 0, \qquad w(+\infty) = 0. \quad (4.211)$$

Similarly for the problem with conditions on the derivative. For these values of A and B, our equation (M) will have a truncated solution $T(z)$ which has a zero or extremum at $z = -(\pi/2)$. These solutions have the

remarkable property of approaching zero both in U_0^+ and U_0^- and are consequently deprived of four strings, namely, S_{-1}^+, S_0^+, S_{-1}^-, and S_0^-. This type of solution we call doubly truncated. The zeros are symmetrically distributed with respect to the point $z = -(\pi/2)$. In this exceptional case the solutions $T(z)$ and $T(-\pi-z)$ are linearly dependent.

4.22. Case 2 cannot be given the same complete treatment as Case 1. We shall restrict ourselves to state that the following regions are free from zeros, namely, U_0^+, $D_{-1,3}^+$, $D_{0,2}^+$, and the real axis. Furthermore, there is a region in each of the regions D_k^- in the lower half-plane the shape of which, however, depends upon the sign of B. We do not get any information about the nature of the strings. The doubly truncated solution cannot present itself, but there are various other curiosities in this case as well as in Case 3. We shall return to these exceptional solutions in §4.32.

4.23. Case 3 is somewhat easier to handle than Case 2, but the result is by no means so simple as in Case 1. In the upper half-plane we find the same zero-free region as in Case 2. In the lower half-plane we get the regions where $P(z) = \Re[G(z)] \leqslant 0$. These are the regions $D_{k,j}^-$ with $j = 2$ and 3. We cannot tell anything about the strings except that they are symmetrically distributed with respect to the line l_{-1}, and those in the lower half-plane are isolated.

4.31. Let us turn our attention for a while to the question whether or not the solutions contained in formula (4.16) are linearly independent. The answer is in general affirmative but there are many exceptions.

There are two solutions, however, which are always linearly independent, namely $T(z)$ and $T(-z)$. Suppose that this were not the case, then $T(z) = CT(-z)$. Hence $T(0) = CT(0)$, from which it follows that either $T(0) = 0$, or, if $T(0) \neq 0$, $C = 1$ and $T'(0) = 0$. But we have shown above that $T(z)$ cannot have a real zero or extremum, save $z = -(\pi/2)$ in the doubly truncated case under Case 1. Hence $T(z)$ and $T(-z)$ are linearly independent. The same kind of proof applies to show that $T(z+n\pi)$ and $T(-z+m\pi)$ are linearly independent, where, however, we have to make exception for the doubly truncated case if $m = -n-1$.

We get into real difficulties if we attempt to prove that $T(z)$ and $T(z+n\pi)$ are linearly independent. The statement is always true in Case 1 and true in general in Cases 2 and 3. To assume that $T(z)$ and $T(z+n\pi)$ are linearly dependent means that $T(z)$ is a periodic function either of the first or of the second kind. At any rate the strings would have to be congruent modulo $n\pi$.

In Case 1 we can complete the proof as follows. We know that in Case 1 all the strings to the left of l_{-1} are d-strings, all to the right l-strings, and there are no r-strings. But no translation can bring a d-string into coincidence with an l-string. Hence we are forced to the conclusion that either $T(z)$ and $T(z+n\pi)$ are linearly independent, or the function $T(z)$ has no zeros and extrema at all in the finite part of the complex plane. From the latter hypothesis it would follow that equation (M) has two linearly independent solutions, namely, $T(z)$ and $T(-z)$, which are entire functions without zeros and extrema. But this is impossible; in fact, we would have

$$T(z) = e^{g(z)}, \qquad T(-z) = e^{g(-z)},$$

where $g(z)$ is an entire function of order 1 [in view of formulæ (2.243) and (3.19)] which is not an even function of z. If we form the Wronskian of these two solutions we obtain

$$e^{g(z)+g(-z)}[g'(z)+g'(-z)] = \text{const.} \neq 0,$$

which is impossible when $g(z) \neq Cz$. This completes the proof for Case 1, but the method clearly breaks down in Cases 2 and 3.

However, in any case we can affirm that $T(z)$ and $T(z+\pi)$ are linearly independent. As a matter of fact, by formula (3.19) the function $T(z)$ approaches zero when $z \to \infty$ in U_0^+, but becomes infinite in U_1^+ and U_{-1}^+. If $T(z+\pi) = CT(z)$ a contradiction would follow, namely, $T(z)$ would have to approach zero in U_{-1}^+, which is impossible.

4.32. Let us investigate what the hypothesis $T(z+s\pi) = CT(z)$, s integral and different from ± 1, would imply for the distribution of the strings in the plane. We find that all the strings $S_{\nu-1}^+$ and S_ν^+ are missing where $\nu = ks$ and k is an arbitrary integer. The remaining strings are congruent modulo $s\pi$ and, moreover, symmetrically distributed with respect to all the lines $l_{2\nu-1}$ on which $T(z)$ is of real character. If s is an odd number this implies that the strings S_μ^+ and S_μ^- are r-strings where $\mu = (s-1/2)+ks$. Thus when s is odd $T(z)$ will also be of real character on all the lines $l_{2\mu}$. The most interesting case is that in which $s = 2$. Then all the strings in the upper half-plane are missing. Whether or not there are any zeros at all in the upper half-plane is a question which we cannot answer. In this case there are only four different truncated solutions, namely: $T(z)$, $T(z+\pi)$, $T(-z)$, and $T(-z-\pi)$.

Truncated solutions of this type actually exist. Suppose that equation (M) has two linearly independent periodic solutions of primitive period $2s\pi$

(cf. end of § 1 . 3). Then the corresponding solution $T(z)$ satisfies the equation $T(z+s\pi) = -T(z)$ and there are $2s$ truncated solutions $T(\pm z+n\pi)$ where $n = 0, 1, \ldots, s-1$, every two of which are linearly independent. There is some evidence indicating that the equation $T(z+n\pi) = CT(z)$ can hold only with $C = \pm 1$, which would show that the case just mentioned is the only case in which we can have linearly independent truncated solutions.

5. *Solutions of real character.*

5 . 1. In § 1 . 1 we called attention to the fact that equation (M) has solutions of real character on the real axis or on the lines l_k. We proceed to a closer study of some of these solutions.

The solutions which are real on the real axis behave quite differently in the three different cases, depending upon the sign of $G(z) = B-A\cos 2z$. In Case 1, $G(z)$ is negative on r. This means that the solutions are non-oscillatory on r, *i.e.* there is at most one real zero or extremum. There are two solutions which have no finite real zero or extremum. One of these solutions tends toward zero when $z \to \infty$ in a direction parallel to the real positive axis, the other approaches zero when $z \to \infty$ in the opposite direction. These are the functions $M^+(z)$ and $M^-(z)$ of § 1 . 2.

Case 2 is a transitory case. $G(z)$, which is a periodic function of a z with period π, oscillates between a positive maximum $B+A$ and a negative minimum $B-A$. The solutions of (M) which are of real character on r oscillate on the intervals where $G(z)$ is positive provided $B+A$ is large enough, and is non-oscillatory on the negative intervals. The solutions do not appear to be particularly interesting. A particular solution of this type will be studied in §§ 6 . 21 and 6 . 22.

In Case 3 $G(z)$ is positive on r and the solutions of real character on r are oscillatory. The number of zeros of a solution on the interval O, x is $(1/\pi)X+O(1)$ where

$$X = \int_0^x \sqrt{\{G(x)\}}\, dx. \tag{5 . 11}$$

We have $$\sqrt{(B-A)}x < X < \sqrt{(B+A)}x.$$

The solutions which are of real character on the lines l_{2k} are oscillatory. As a matter of fact, if we put $z = n\pi+iy$ in equation (M) it becomes

$$\frac{d^2w}{dy^2} + (A\cosh 2y - B)\, w = 0. \tag{5 . 12}$$

The coefficient of w, $H(y)$, is positive for real values of y, in Cases 1 and 2 for all real values, in Case 3 only if $|y| > y_0$. Hence every solution which is of real character on l_{2n} will have infinitely many zeros and extrema on this line. The zeros and extrema will alternate on the intervals where $H(y) > 0$. There is at most one zero or extremum on the interval $-y_0 \leqslant y \leqslant +y_0$ in Case 3. The most important solutions of this type are the periodic solutions of period π or 2π, namely, the functions of the elliptic cylinder or Mathieu functions in the restricted sense.

The solutions of (M) which are of real character on l_{2n+1} are non-oscillatory, either for all values of $y\,[z = (n+\frac{1}{2})\pi + iy]$ as in Cases 2 and 3, or if $|y| > y_0$ as in Case 1. The most important solutions of this type are the truncated solutions treated in § 4 of the present paper. We shall not give this kind of solution any further consideration in the present section, but they will play an important rôle in § 6.

5 . 21. Let us begin with Case 1 and complete the discussion of § 3 . 23. Suppose that we have a real zero or extremum at a point x_1, interior to the interval $(n-\frac{1}{2})\pi, n\pi$. The case in which $n\pi < x_1 < (n+\frac{1}{2})\pi$ can be dealt with in the same manner. If we construct $DI(x_1)$ we find that it contains the strip $(n-\frac{1}{2})\pi \leqslant x \leqslant x_1$, all the strips $(k+\frac{1}{2})\pi \leqslant x \leqslant (k+1)\pi, k < n$, and $k\pi \leqslant x \leqslant (k+\frac{1}{2})\pi$, $k \geqslant n$, further the horizontal strip $-y_0 \leqslant y \leqslant +y_0$ with the restriction that either $x \leqslant x_1$ or $x > (n+\frac{1}{2})\pi$. Hence all strings to the right of the line $x = x_1$ are l-strings which are isolated except S_n^+ and S_n^-, and all the strings to the left are d-strings. The strings S_k^+ and S_k^- are symmetric with respect to the real axis for all values of k.

5 . 22. If $x_1 = n\pi$ the solution is of real character on both r and l_{2n}. The form of $DI(n\pi)$ can be obtained by letting $x_1 \to n\pi$ in the result of the preceding section. The only difference is that the strings S_n^+ and S_n^- are r-strings and there are no other zeros in D_n^+ and D_n^- than those on l_{2n}.

We get the same result if we assume the solution to be of real character on l_{2n} only and not on r.

5 . 23. Let us say a few words about the solution $M^+(z) = e^{-\mu z}F(z)$ where $F(z)$ is a periodic function of period π. We know that $M^+(z) \to 0$ when $z \to \infty$ in a direction parallel to the positive real axis. It is easy to see that we can operate on $M^+(z)$ with the operator $\mathcal{Q}_y$ of § 2 . 31 taking $z_1 = +\infty$. The integrals in $\mathcal{Q}_y\{M^+(z), +\infty\,;\, z\}$ have a meaning. Constructing $DI(+\infty)$ we find that it contains the strip $-y_0 \leqslant y \leqslant +y_0$ and in addition all the strips $(k-\frac{1}{2})\pi \leqslant x \leqslant k\pi$ where k is any positive or

negative integer including zero. Hence all the strings are isolated d-strings congruent modulo π and symmetric to the real axis. This completes the omissions of § 3 . 23.

5 . 31. We do not consider Case 2, but pass directly to the more interesting Case 3. The fundamental result is as follows:—

THEOREM 5 . 311.—*A solution of* (M) *which is of real character on the real axis cannot have any complex zeros or extrema in the part of the plane where* $P(z) = \Re[G(z)] \geqslant 0$ *if* $B \geqslant A$.

If $z_2 = x_2 + iy_2$ is a point in the region indicated by the theorem and x_1 is a real zero then the broken line $x_1 \text{——} x_2 \text{——} z_2$ is an *li*. Thus the strings are confined to the interior of the curves $P = 0$. This does not give us any information about the nature of the strings whether they are d-, r-, or l-strings. Additional information can be had by making additional assumptions. Suppose $(n-\frac{1}{2})\pi < x_1 < n\pi$ and take a solution whose parameter at x_1, $\lambda(x_1)$ in the notation of § 1 . 1 is negative or zero, then the strip $(n-\frac{1}{2})\pi \leqslant x \leqslant x_1$ is found to be free from complex zeros or extrema. If we change the condition on x_1 and make it

$$n\pi < x_1 < (n+\tfrac{1}{2})\pi$$

and assume $\lambda(x_1) \geqslant 0$ instead, we find that the strip $x_1 \leqslant x \leqslant (n+\frac{1}{2})\pi$ is free from complex zeros and extrema. We obtain the most interesting result by taking $x_1 = n\pi$, namely:

THEOREM 5 . 312.—*According as* $\lambda(n\pi)$ *is positive, zero or infinity, or negative, the strings* S_n^+ *and* S_n^- *are l-, r-, or d-strings.*

For the proof of this theorem we use Theorem 2 . 331. Suppose that $\omega = \arg\lambda(n\pi) = -\pi$ and let $z_2 = x_2 + iy_2$ be a point in D_n^+. We use the broken line $n\pi \text{——} n\pi + iy_2 \text{——} z_2$ as our *li*. The angles are found to be: $\omega + \pi = 0$; $\pi - \theta = \pi/2$, 0; $\phi + \theta = \pi/2$ or $3\pi/2$, Θ_1; and $\vartheta = -(\pi/2)$, which shows that the path has a definite vector field. Hence if the parameter is negative the strings are d-strings. The cases when the parameter is zero or positive is treated in similar manner. We remark that the theorem is true also in Cases 1 and 2.

5 . 32. Now let us suppose the solution to be of real character on the line l_{2n}. Then we obtain the following

THEOREM 5 . 32.—*If a solution of* (M) *is of real character on the line* l_{2n}, *the complex zeros and extrema in the strip* $(n-\frac{1}{2})\pi \leqslant x \leqslant (n+\frac{1}{2})\pi$ *which lie in the region where* $P(z) \leqslant 0$ *are on the line* l_{2n} *itself.*

We omit the proof of this theorem, which is a direct application of Theorem 2.331. It holds good in all three cases. There exists a finite upper limit, independent of n, for the ordinate of any zero or extremum in the strip and not on the line.

5.33. According to the investigation in §1.3 the functions of the elliptic cylinder are of real character on the real axis as well as on all the lines l_k. This implies that the parameters $\lambda\{k(\pi/2)\}$ are zero or infinity for all integral values of k. Hence by the theorems of the preceding paragraphs we obtain

THEOREM 5.33.—*The functions of the elliptic cylinder have all their zeros or extrema on the real axis and on the lines l_{2n}. The strings are isolated, congruent modulo π and symmetric with respect to the real axis.*

It does not matter for the proof whether we are in Case 2 or in Case 3, since Theorem 5.32 holds in both. It is essential to remember, however, that we cannot be in Case 1, since that case does not allow of any periodic solutions.

The functions considered by Whittaker have the property that π is either the primitive period or the semi-period. The functions introduced by Poole have the same property with respect to $s\pi$, where s is an integer greater than 1. The functions are further either even or odd. From these two properties it follows that $\lambda\{ks(\pi/2)\}$ is either zero or infinity for all integral values of k. Hence the functions in question will be of real character on the real axis and on all the lines l_{ks}. In particular, if $s = 2$ the corresponding functions will be of real character on all the lines l_{2k}; hence their zeros and extrema will be subjected to the conditions of Theorem 5.33.

The subsequent discussion in §6 allows us to draw certain conclusions with regard to the structure of the strings. We restrict ourselves to solutions of period or semi-period π.

In Case 2 the zeros and extrema alternate on the lines l_{2k}. On the real axis a somewhat more complicated situation holds. Let us denote by ξ_k^+ and ξ_k^-, $\xi_k^- < \xi_k^+$, the two roots of $G(z) = 0$ in the interval $(k-\frac{1}{2})\pi$, $(k+\frac{1}{2})\pi$. There is only one zero or extremum in the interval ξ_k^-, ξ_k^+, namely at $z = k\pi$. In the interior of the interval ξ_{k-1}^+, ξ_k^- we have at least one extremum and, in the general case, p extrema and $p-1$ zeros, where p is some integer independent of k. These points are symmetrically distributed with respect to $z = (k-\frac{1}{2})\pi$.

In Case 3 the zeros and extrema alternate on the real axis. Let iy_0, $y_0 > 0$, be a root of $G(z) = 0$. On the interval $k\pi - iy_0$, $k\pi + iy_0$ of l_{2k} there is only one zero or extremum, namely at $k\pi$. On the remaining intervals of l_{2k} the zeros and extrema alternate and the point nearest to the real axis is always an extremum.

5.34. There is still some doubt whether there can exist real values of A and B other than the trivial $A = 0$, $B = n^2$, for which we have two functions of the elliptic cylinder as linearly independent solutions. This question can be settled.*

THEOREM 5.34.—*If A and B are real and $A \neq 0$, there cannot exist two linearly independent solutions of* (M) *having π either as a period or as a semi-period.*

Suppose two such solutions did exist. Then we can assume one to be an even function $E(z)$ and the other to be an odd function $O(z)$. There are three hypotheses to examine, namely π may be (i) a period of both solutions, (ii) a semi-period of them both, or (iii) a period of one and a semi-period of the other.

If the constants are correctly chosen we can express these solutions in terms of the truncated solution $T(z)$ of § 4 in the following manner

$$E(z) = \tfrac{1}{2}[T(z)+T(-z)], \qquad O(z) = \tfrac{1}{2}[T(z)-T(-z)], \tag{5.341}$$

whence

$$T(z) = E(z)+O(z). \tag{5.342}$$

If π is the period of the two solutions, we have

$$T(z+\pi) = T(z).$$

But this is impossible, as we found in § 4.31. The semi-periodic case can be disposed of in the ssme manner.

If π is a period of one and a semi-period of the other, we can assume that $E(z)$ is the periodic solution and $O(z)$ the semi-periodic one. In view of formula (5.342) we conclude that

$$T(z+\pi) = E(z)-O(z) = T(-z),$$

which is also impossible. We can handle the case in which the odd function is periodic and the even one is semi-periodic in the same manner.

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* I have recently found that E. L. Ince has proved that equation (M) cannot have two independent solutions of period 2π, see *Proc. Camb. Phil. Soc.*, 21 (1922), 117 and *Proc. London Math. Soc.* (2), 23 (1924), 73. This result is equivalent to Theorem 5.34 in view of the discussion in § 1.3. The method used by Mr. Ince is, however, quite different from the methods of the present paper. [Note added June 23rd, 1924.]

We can prove a little more. We can show that there are no non-trivial values of A and B for which (M) can have two linearly independent solutions admitting of $s\pi$ (s integral), one as a period, the other as a semi-period. This depends upon the fact that $T(z+s\pi)$ and $T(-z)$ are linearly independent in the Cases 2 and 3.

6. *Variation of the parameters.*

6.0. The solutions of (M) depend upon three essentially different parameters, namely A, B, and $\lambda(z_1)$, where z_1 is an arbitrary but fixed complex number, zero for instance. If the values of these parameters are fixed, then a definite Mathieu fuuction is determined up to an arbitrary multiplicative constant. If we vary one or several of the parameters we vary the Mathieu function too, and, in general, we displace the zeros and extrema.

The roles played by these parameters are quite different. Roughly speaking, the values of A and B regulate the frequency of the oscillation. A look at formula (3.18) shows that A determines the asymptotic frequency in the vertical strings. The role of B is not quite so clear; it seems to control the frequency in the horizontal string which is fully developed only in Case 3 and is altogether missing in Case 1. The value of λ has very little effect upon the frequency. Almost all solutions of (M) for given values of A and B have the same frequency. Thus, for example, is we consider the region R defined by $-(\pi/2) \leqslant x \leqslant +(\pi/2)$, $-\eta \leqslant y \leqslant +\eta$ and $N(\eta, \lambda)$ the number of zeros in R of the solution of (M) whose parameter at the origin is equal to λ, then

$$\lim_{\eta \to \infty} N(\eta, \lambda)/N(\eta, 0) = 1$$

for all values of λ excepting two. This, again, appears from § 3.1.

Thus if we vary one of the parameters in (M) it makes a good deal of difference which one we vary. If $A (A > 0)$ is increased the points in the several strings move closer together. An increase in the value of B will not affect considerably the distance between zeros far from the real axis, but will increase the frequency in the horizontal string if such a one exists. Finally, a change in the value of λ will cause a distortion in the configuration of the zeros and extrema which, however, is rather slight away from the real axis.

In the following paragraphs we shall concentrate our attention on the changes in the strings imposed by a change in λ. In order to arrive at

a better understanding of Cases 2 and 3 we also investigate the effect of varying B when λ is kept fixed and equal to zero. The value of A will be held constant throughout the investigation.

6.11. We begin with Case 1 and take $z_1 = 0$. The two principal solutions at $z = 0$ we denote by $E(z)$ and $O(z)$; $E(z)$ being an even function of z and $O(z)$ an odd function. The general solution, $M(z, \lambda)$, is given by the combination

$$M(z, \lambda) = c_1 E(z) + c_2 O(z). \tag{6.111}$$

The parameter λ of this solution at the origin is given by

$$\lambda = c_2/c_1. \tag{6.112}$$

Let us start with $\lambda = 0$, or, in other words, with the solution

$$M(z, 0) = c_1 E(z).$$

Then there is an extremum at $z = 0$ and there is no other real zero or extremum. The solution is of real character on the real as well as on the imaginary axis; on the latter the zeros and extrema alternate. We designate the zeros on l_0 (the imaginary axis) according to increasing imaginary parts by z_n (n integral different from zero) and the extrema in the same way by ζ_n (n integral including zero). If we put

$$z_n = iy_n, \qquad \zeta_n = i\eta_n, \tag{6.113}$$

then

$$y_{-n} = -y_n, \qquad \eta_{-n} = -\eta_n, \qquad \eta_0 = 0, \tag{6.114}$$

and

$$y_n < \eta_n < y_{n+1}$$

for all positive values of n.

In view of Theorem 3.23 compared with § 5.22, we know that the strings S_k^+ and S_k^- are d-strings if $k < 0$, r-strings if $k = 0$, and l-strings if $k > 0$. Thus the symbol of the solution is d-r-l. The four strings S_k^+, S_k^-, S_{-k}^+, and S_{-k}^- are of course symmetric with respect to the origin for every value of the subscript k.

6.12. Now let us find out what happens if we let λ increase from zero through real positive values. In view of Theorem 5.312, the strings S_0^+ and S_0^- become l-strings; all other strings are unchanged in character. The extremum originally at $z = 0$ moves to the left to a point $\zeta_0(\lambda)$. Let us denote the zeros and extrema in S_0^+ by $z_n(\lambda)$ and $\zeta_n(\lambda)$ respectively,

where the notation is so chosen that

$$z_n(0) = z_n, \qquad \xi_n(0) = \xi_n, \tag{6.121}$$

and put $$z_n(\lambda) = x_n(\lambda) + iy_n(\lambda), \qquad \xi_n(\lambda) = \xi_n(\lambda) + i\eta_n(\lambda). \tag{6.122}$$

As long as the points remain in D_0^+, we have

$$\xi_n(\lambda) < x_{n+1}(\lambda) < \xi_{n+1}(\lambda) < 0 \quad (n \geqslant 0), \tag{6.123}$$

$$0 < y_n(\lambda) < \eta_n(\lambda) < y_{n+1}(\lambda) \quad (n \geqslant 1). \tag{6.124}$$

Thus the mutual order of the points is preserved. The only way in which the order could be changed would be if for some value of λ there could exist two points in the string having either the same ordinates or the same abscissae, which can not be the case as long as the points remain in the interior of the regions $D_{0,4}^+$ and $D_{0,3}^+$.

For a certain value of $\lambda = \lambda'_{-1}$ we have

$$\xi_0(\lambda'_{-1}) = -\frac{\pi}{2}. \tag{6.125}$$

The corresponding solution is of real character on r and on l_{-1}. The solution in question, which we denote by $e(z)$, is an even function of $z+(\pi/2)$ and the zeros and extrema are symmetrically distributed with respect to the point $z = -(\pi/2)$. Let us denote the zeros of $G(z)$ on l_{-1} by $-(\pi/2)-iy_0$ and $-(\pi/2)+iy_0$. Then $e(z)$ will be oscillatory in the interval $\mathfrak{I}$ on l_{-1}, determined by these two points, provided only the values of A and B are large enough. Suppose that the actual number of zeros of $e(z)$ on l_{-1} is $2m$ (m positive integer); then the number of extrema including the point $\xi_0(\lambda'_{-1}) = -(\pi/2)$ is either $2m-3$ or $2m-1$ or $2m+1$ or, finally, $2m+3$. In order to find out whether or not all these different possibilities can be realized, and, further, to determine the distribution of the points on the different intervals on l_{-1}, we employ the method of varying the parameter B. We shall merely state the results.* It was found that the first hypothesis is the only one which cannot be realized. The third hypothesis requires that $e(z)$ is a doubly truncated solution (see § 4.21); the zeros and extrema on l_{-1} lie in the interior of $\mathfrak{I}$. The fourth hypothesis calls for $2m$ zeros and $2m+1$ extrema in the interior of $\mathfrak{I}$ and one extremum in each of the intervals outside of $\mathfrak{I}$. There is an excep-

* The investigation follows the same lines as the one below in § 6.22, but the mechanism regulating the arrival of zeros and extrema on the line l_{-1} from infinity and from the complex plane is very much simpler than the one operating in the case mentioned.

tional sub-case in which two extrema coalesce at each of the points $-(\pi/2)-iy_0$ and $-(\pi/2)+iy_0$ forming double extrema. This sub-case is an intermediary between the second and the fourth case. In all cases the zeros and extrema on $\mathfrak{F}$ alternate.

Let us denote the zeros and extrema on l_{-1} by a_μ and a_ν respectively, where $\mu = \pm 1, \pm 2, \ldots, \pm m$ and $\nu = 0, \pm 1, \pm 2, \ldots, \pm(m-1), [\pm m, \pm(m+1)]$. Then we have

$$z_\mu(\lambda'_{-1}) = a_\mu, \qquad \zeta_\nu(\lambda'_{-1}) = a_\nu, \tag{6.126}$$

where, however, ν is at most equal to $\pm m$. The extrema a_{m+1} and a_{-m-1} if they exist, are not obtained from the strings S_0^+ and S_0^- respectively, but from S_{-1}^+ and S_{-1}^- respectively. If the extremum a_{m+1} were obtained from S_0^+ there must be a zero a_{m+1} between a_m and a_{m+1}, which is not the case. We notice that when $0 \leqslant \lambda \leqslant \lambda'_{-1}$ the ordinates of the points in S_{-1}^+ exceed y_0.

6.13. What happens to the other points of the strings S_0^+ and S_0^- when $\lambda \to \lambda'_{-1}$? There is a general case and an exceptional case.

In general the points tend toward definite finite limiting positions in the interior of D_0^+ and D_0^- respectively, the ordinates of which exceed in absolute value the largest of the numbers $\mathfrak{Im}(a_\mu)$ and $\mathfrak{Im}(a_\nu)$. At the same time the points of S_{-1}^+ and S_{-1}^- tend towards limiting positions such that in the limit the four strings S_0^+, S_0^-, S_{-1}^+, and S_{-1}^- form a configuration symmetric with respect to r and l_{-1}. From this symmetry we can draw conclusions regarding the structure of the strings S_{-1}^+ and S_{-1}^-, to begin with for $\lambda = \lambda'_{-1}$ and thence for other values of λ. Suppose that the number of extrema of $e(z)$ on l_{-1} is given by $2m-1$. Then the point in S_{-1}^+ whose ordinate is the smallest is an extremum, the next smallest a zero, and so on alternately. This order which holds for $\lambda = \lambda'_{-1}$ cannot have been changed as long as $0 \leqslant \lambda \leqslant \lambda'_{-1}$. Hence the lowest point in the string S_{-1}^+ is an extremum when $\lambda = 0$.

Now the strings S_{-1}^+ and S_{+1}^+ are symmetric with respect to l_0 when $\lambda = 0$; hence the string S_{+1}^+ also starts off with an extremum followed by a zero and so on. Moreover, this order cannot have changed while λ increased from 0 to λ'_{-1}, and it will remain unchanged as long as λ is positive. In a similar manner we can conclude that all the strings S_k^+, $k > 0$, start with an extremum followed by a zero, which order will remain unchanged for all positive values of λ. These strings are isolated as long as $\lambda > 0$; hence they will always consist of the same points grouped in the same relative order.

In view of the symmetry for $\lambda = 0$ we conclude also that the strings S_k^+, $k < 0$ start with an extremum followed by a zero when $\lambda = 0$. These strings are modified, however, in their constitution when λ increases, as we already have seen and shall see further below.

The case when the number of extrema of $e(z)$ on l_{-1} is $2m+3$ is somewhat harder to handle, inasmuch as all four strings in the strip

$$-\pi < x < 0$$

contribute to the building up of the set on the line l_{-1}. It can be shown without difficulties, however, that our conclusions with regard to the structure of the strings hold also in this case. Thus all the strings S_k^+ and S_k^-, $k \neq 0$, start with an extremum followed by a zero and so on alternately, as long as $0 \leqslant \lambda < \lambda'_{-1}$.

Finally, we have the exceptional case when $e(z)$ is the doubly truncated solution and we have $2m+1$ extrema on l_{-1}. We observe that the zeros and extrema of $e(z)$ are analytic functions of the parameter B. The particular value $B = B_0$ for which $e(z)$ becomes equal to the doubly truncated solution is a singular point of the zeros and extrema in the strip $-\pi < x < 0$, but it is a regular point of all the zeros and extrema outside of this strip. Consequently, if we change the value of B very slightly from B_0 to a value B_1, the zeros and extrema outside of the strip mentioned will move very little and, moreover, while they move they will preserve their mutual order. But for $B = B_1$ $e(z)$ will no longer be truncated and our previous conclusions hold. Thus, even in the exceptional case, the strings S_k^+ and S_k^-, $k \neq 0$, will start with an extremum when $0 \leqslant \lambda < \lambda'_{-1}$.

6.14. We can get the rest of the movement of the points by symmetry. Let $f(z)$ be the solution whose real extremum is located at the point x_0, $-(\pi/2) < x_0 < 0$. Then the solution whose real extremum is at the point $-\pi - x_0$ is given by $f(-z-\pi)$. When we let λ increase through λ'_{-1} there is an exchange of points between the strings S_0^+ and S_{-1}^+ and between S_0^- and S_{-1}^-. In fact, the m zeros of least ordinates in S_0^+ pass over to S_{-1}^+, likewise $m-1$ or m extrema. If there is an extremum a_{m+1} on l_{-1} it will pass in the opposite direction, namely from S_{-1}^+ to S_0^+. If we are in the truncated case, the strings S_0^+ and S_{-1}^+ will return from infinity, an extremum forming the vanguard in both strings. After λ has passed the value λ'_{-1} the string S_0^+ is not subjected to any further changes in structure as long as $\lambda > 0$, *i.e.* S_0^+ will be an l-string starting with an extremum when $\lambda > \lambda'_{-1}$. The developments in the lower half-plane are symmetrical.

When λ increases further the points in the d-strings S^+_{-1} and S^-_{-1} are pushed to the left until for a value $\lambda = \lambda'_{-2}$

$$\xi_0(\lambda'_{-2}) = -\pi, \tag{6.141}$$

when the strings in question become r-strings. This solution is simply a constant multiple of $E(z+\pi)$. If we increase λ still further the strings S^+_{-1} and S^-_{-1} are affected in exactly the same manner as S^+_0 and S^-_0 were changed above.

If we keep on increasing λ we finally hit upon a finite value Λ such that

$$\xi_0(\lambda) \to -\infty \quad \text{when} \quad \lambda \to \Lambda. \tag{6.142}$$

The corresponding solution is $M^-(z)$ in the notation of §1.2. Then the strings are l-strings, congruent modulo π. The point of an arbitrary string that is nearest to the real axis will always be an extremum.

6.15. If λ exceeds Λ the solution will no longer admit of a real extremum, but has a real zero instead, the abscissa of which is very large and negative when $\lambda = \Lambda+\epsilon$. This zero $z_0(\lambda)$ moves rapidly to the right when λ increases.

For a certain value $\lambda = \lambda_{-2n-2}$ we have

$$z_0(\lambda_{-2n-2}) = -(n+1)\pi. \tag{6.151}$$

The corresponding solution is a constant multiple of $O[z+(n+1)]$. It is of real character on r and on the line l_{-2n-2}. On the latter the zeros and extrema alternate, symmetrically distributed with respect to the point $z = -(n+1)\pi$, which is the affix of a zero. When λ increases, the two r-strings S^+_{-n-1} and S^-_{-n-1} become d-strings. The structure* of the strings remains unchanged until we hit upon a value $\lambda = \lambda_{-2n-1}$, for which we have

$$z_0(\lambda_{-2n-1}) = -(n+\tfrac{1}{2})\pi. \tag{6.152}$$

The corresponding solution is a constant multiple of $o(z+n\pi)$, where $o(z)$ is an odd function of $z+(\pi/2)$ and is one of the principal solutions at $z = -(\pi/2)$, $e(z)$ being the other. The solution in question has an odd number

* If the points in a string S^+ are ordered according to increasing ordinates the zeros and extrema occur in a certain definite order, *e.g.* $e-z-e-$, etc., or $z-e-z-$, etc., where z stands for zero and e for extremum. This order we call the structure of the string. The reader will notice that it is not a question of the order from a certain point on, but from the very beginning on.

of zeros on the line l_{-2n-1}, which we denote by $2p+1$, and an even number of extrema, namely $2p$ or $2p+2$ or $2p+4$. It seems as if p would have to be equal to either m or $m-1$, but that is of no importance for our present purpose. The distribution of these points on the line l_{-2n-1} is exactly analogous to the distribution of the zeros and extrema of $e(z)$ on l_{-1} described above.

The zeros and extrema of $o(z+n\pi)$ on l_{-2n-1} are furnished from the strings S^+_{-n-1} and S^-_{-n-1} except the two extrema β_{p+2} and β_{-p-2}—if they exist—which come from S^+_{-n} and S^-_{-n} respectively. When λ is made to increase beyond λ_{-2n-1} the points on l_{-2n-1} are turned over to the strings S^+_{-n} and S^-_{-n} except the extrema β_{p+2} and β_{-p-2}—if they exist—which are turned over to S^+_{-n-1} and S^-_{-n-1} respectively. After this reduction the strings S^+_{-n-1} and S^-_{-n-1} are not made to suffer any further changes in their structure when λ increases. If $\lambda_{-2n-1} < \lambda < \lambda_{-2n}$ the strings S^+_k and S^-_k are d-strings when $k < -n$. The point nearest to the real axis is always an extremum. This latter statement holds good also for the strings with subscript $k > -n$.

Finally when $\lambda \to +\infty$ the resulting solution is a constant multiple of $O(z)$. If we let λ tend to infinity through negative values instead, we get the remaining solutions of real character on r. We do not need to investigate these solutions separately, since the solution $f(-z)$ will correspond to the value $-\lambda_0$ of the parameter provided $f(z)$ corresponds to λ_0.

6.16. It remains to say a few words about the solutions whose parameter at $z = 0$ is a non-real number.

Suppose that $\arg\lambda = \Theta_1$—*i.e.* an angle in the first quadrant—and construct $DI(0\,;\,\Theta_1)$. This region contains the whole real axis, the semi-infinite strip $x \geqslant 0$, $-y_0 \leqslant y \leqslant +y_0$ and all the strips

$$k\pi \leqslant x \leqslant (k+\tfrac{1}{2})\pi, \quad k \geqslant 0.$$

Thus the strings S^+_k and S^-_k, $k \geqslant 0$ are isolated l-strings. We do not get any information about the character of the strings whose subscript k is negative.

Let us take $|\lambda|$ to be a very small quantity Then the configuration of the strings differs very little from the known configuration for $\lambda = 0$. The extremum $\xi_0(\lambda)$ is no longer real; using formula (6.111) we conclude that it lies in the third quadrant close to the origin in the region $D^-_{0,2}$. Thus all the strings start with an extremum except S^+_0. When we increase $|\lambda|$ no exchange of points between the different strings can occur

in the right half-plane, whereas such exchanges will occur in the left half-plane, namely between a pair of strings S^+_{-n} and S^-_{-n} on the one hand and the pair S^+_{-n-1} and S^-_{-n-1} on the other. The mechanism of the exchange is the same as described above in § 6.14. A solution which is of real character on the line l_{-2n-1} serves as intermediary in the transaction. There can never be an exchange between strings separated by the real axis. In the contrary case we should have an intermediate solution which has a real zero or extremum, which is impossible as long as $\arg\lambda \neq 0, \pi$.

The case when $\arg\lambda = \Theta_j$, $j \neq 1$, is treated in the same manner. The preceding investigation has shown that the essential effect of varying the parameter λ is to shift a certain definite number of zeros and extrema from one string to another or, rather, from one pair of strings to another pair. There are two types of such movable point-sets; one, the *e*-set, contains an *even* number of zeros, the other, the *o*-set, contains an *odd* number. Typical representatives of these sets are the sets of zeros and extrema on l_{-1} of $e(z)$ and $o(z)$ respectively. The number of points in these two sets are characteristic for the equation (M) in Case 1. Every solution of (M) with the exception of $M^+(z)$ and $M^-(z)$ has either an *e*-set or an *o*-set; all *e*-sets have the same structure, likewise all *o*-sets. If, for a moment, we leave out of consideration this movable set all strings have the same structure. In the strings the zeros and extrema alternate and all strings start with an extremum. The movable set will attach itself to the pair of strings to which it is nearest.

6.21. In order to set forth the great distinction between Case 1 and Case 3, we shall study what takes place when we pass from the one case to the other by a continuous increase of B, keeping A fixed. This will also enable us to study Case 3 with greater ease.

We denote the even solution of (M) by $E(z; B)$ and let B increase from a value slightly less than $-A$ to a value slightly greater than $+A$. The solution $E(z; -A)$ has only one real extremum, namely at the origin, and no real zero; all the strings begin with extrema except S^+_0 and S^-_0. This is our starting point. We end up with $E(z; +A)$; this solution on the other hand has infinitely many zeros and extrema which alternate; at the origin we have a triple extremum. There arises the question of where these zeros and extrema come from. There are only two possible sources: (1) from infinity along the real axis, and (2) from the strings in the complex plane.

Now $E(z; B)$ is real for real values of z and B: thus complex zeros and extrema occur in conjugate pairs. If a non-real zero should become

real, it would first have to coalesce with its conjugate, forming a double zero, which is impossible at a non-singular point. Hence we cannot get real zeros through confluence of complex zeros, but we can get real extrema in that way. Two conjugate extrema can coalesce at a point where $G(z;\ B) \equiv B - A\cos 2z = 0$, forming a double extremum. This later dissolves into two real simple extrema.

Let us denote for the moment a real zero of $E(z;\ B)$ by $z(B)$ and an extremum by $\xi(B)$. These quantities are analytic functions of B, single-valued, real and continuous at least on some range of the real B-axis. By ordinary Sturmian methods we find that

$$[E(z;\ B)]^2 \frac{d}{dB}[z(B)] = -\int_0^z [E(t;\ B)]^2\, dt, \qquad (6.211)$$

$$G(\xi;\ B)\left\{\frac{\partial}{\partial \xi}[E(\xi;\ B)]\right\}^2 \frac{d}{dB}[\xi(B)] = -\int_0^\xi [E(t;\ B)]^2\, dt. \qquad (6.212)$$

The first formula shows that a real zero will move towards the origin when B increases. The direction of motion of a real extremum on the other hand varies, *i.e.* it is towards the origin in an interval where $G(z;\ B) > 0$, from the origin in the opposite case.

This shows that we cannot obtain any new real extrema from infinity, at least as long as $B < +A$. Thus new zeros come from infinity along the real axis and real extrema come from the strings through confluence. But now the difficulties begin. The extrema arrive in pairs which coalesce forming a double extremum and then dissolve. But in the end (for $B > +A$) we always find a pair of consecutive real extrema separated by a real zero. How can the zero get between the extrema? It can neither get round them by moving in from the complex plane, nor can it pass right through an extremum. This dilemma shows that the exchange of extrema between the complex plane and the real axis must work both ways. It must be possible to return extrema to the plane when they are in the way.

6.22. Let us look a little closer at this process. We have one important question, however, to settle first. How can a string S_n^+ belonging to $E(z;\ B)$ change its structure when B increases from $-A$ on? The answer is, only through losses or gains across the boundary of D_n^+. This boundary is made up of l_{2n-1} and l_{2n+1} and r. There can be only one zero or extremum on l_{2n-1} when $B > -A$ which on account of the symmetry

must be real. Hence we have only to consider what happens on the real axis. There is no value of B for which $E(z\,;\,B)$ can be a truncated solution, since $T'(k\pi)$ is different from zero. Consequently the mutual order between the points in the string will be preserved.

We return to the exchange of extrema between the real axis and the complex plane. Let B_n' (n positive integer) be the first value of B greater than $-A$ for which two extrema of S_n^+ and S_n^- respectively coalesce on the real axis. The double extremum dissolves and the two components travel, one to the left, the other to the right. A study of formula (6.212) will show that the confluence has to take place at a point where $G(z\,;\,B)$ passes from plus to minus when z increases. Thus the point in question is the root of the equation $G(z\,;\,B_n') = 0$ lying in the interval $(n-\frac{1}{2})\pi,\ n\pi$. The extremum which turned to the right pursues its course until it meets another extremum travelling in the opposite direction. This will happen for a certain value B_n'' at a point in the interval $n\pi,\ (n+\frac{1}{2})\pi$, the affix of which is a root of the equation $G(z\,;\,B_n'') = 0$. This second extremum is obtained from one of the two strings S_{n+1}. The two extrema coalesce into a double extremum, then dissolve again into two simple conjugate imaginary extrema. This restores the strings S_n^+ and S_n^- to their original structure. So much for the extremum which turned to the right. The one which turned to the left may either stay on the real axis or collide with another extremum and move out into the complex plane. The former hypothesis requires that the extremum in question remain on the range $(n-1)\pi,\ n\pi$ when $B_n' < B < A$. A moment's consideration will show, however, that this assumption will lead to contradictions except when $n = 1$. If $n > 1$ the extremum must get out of the way by confluence with an extremum previously introduced on the real axis from one of the two strings S_{n-1}.

The following description of the mechanism involved in the transition of the zeros and extrema to the real axis which is partly hypothetical seems to cover the facts of the case. We restrict ourselves to the right half-plane. The process of *confluence* of complex extrema starts at $+\infty$ when $B = -A+0$ and progresses from string to string until the pair S_1^+ and S_1^- is affected. It is followed by the first process of *re-confluence* of real extrema, which also progresses towards the origin until the pair S_1^+ and S_1^- is restored to its original structure. The net result of these two processes is that one real extremum has been introduced on the interval $(0,\ \pi)$. This extremum will coalesce with its negative at the origin forming a triple extremum for $B = +A$. When B exceeds $+A$ this triple extremum dissolves and two conjugate purely imaginary extrema are turned over to the strings S_0. When the first re-confluence process has cleared

the road the first real zero of $E(z\,;\ B)$ can proceed from infinity. In the wake of the zero the second confluence process takes place. Suppose that the first positive extremum of $E(z\,;\ +A)$ lies between $j\pi$ and $(j+1)\pi$. Then the second confluence process will proceed until it has affected the strings S^+_{j+1} and S^-_{+1}, when it stops. It is followed by the second re-confluence process which restores all the strings affected to their original structure. The net result of the second confluence process and re-confluence process is to introduce an extremum on the interval $j\pi, (j+1)\pi$. Then the second zero arrives, and so on. If m_n is the number of extrema of $E(z\,;\ A)$ on the open interval $0, n\pi$, then the strings S^+_n and S^-_n will perform the confluence and re-confluence processes, each m_n+1 times.

6.31. Thus we get the following picture* of the distribution of the zeros and extrema of $E(z\,;\ +A)$. There is a triple extremum at the origin and infinitely many real zeros and extrema which alternate. The points nearest to the origin are zeros. The strings S^+_0 and S^-_0 are r-strings which start with zeros; every other string starts with an extremum. When B exceeds A the triple extremum breaks up into three simple extrema, namely one at $z=0$ and two conjugate purely imaginary ones. By this process all the strings obtain the same structure, the point nearest to the real axis is an extremum, the next a zero, and so on.

We take a fixed value of B greater than A and consider solutions which are of real character on r. We can obtain them by giving λ a real value in equations (6.111) and (6.112). Suppose we let λ increase from zero. This will merely shift the real zeros and extrema of $M(z\,;\ \lambda)$ to the right. When λ goes from 0 to infinity a zero z_n of $E(z)$ is shifted to the nearest zero of $O(z)$ which is greater than z_n, and likewise with the extrema. The movement of the strings is controlled by Theorem 5.311, according to which the complex zeros and extrema have to stay in the region of the plane where $P(z) = \Re[G(z)]$ is negative. The string S^+_n may move back and forth as it likes in the regions $D^+_{n,2}$ and $D^+_{n,3}$ but it has to stay in these regions and, moreover, the structure of the string has to be preserved during the motion.

We shall find it convenient to say that the real zeros and extrema form a string S^0. This string is not so intimately attached to the real axis as, for instance, the string S^+_n is to the line l_{2n}, but it is, roughly speaking, parallel to the real axis.

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* The reader will perceive that this picture is not dependent upon the hypothetical element in the description at the end of the preceding paragraph.

6.32. Let us study the solutions which have a complex parameter at the origin. Suppose that λ in equation (6.112) is so chosen that

$$\arg \lambda = \Theta_1,$$

an angle in the first quadrant, and $|\lambda|$ is small.

We approximate $DI(0\,;\Theta_1)$ in our usual manner. We find that the region in question contains the axes, the region in the lower half-plane where $P(z) \geqslant 0$ and in addition the regions $D^+_{0,2}$ and $D^-_{0,3}$. This means that the strings S^-_k are isolated each in its own enclosure where $P(z) < 0$. In particular the string S^-_0 is an l-string. The string S^0 moves up from the real axis and enters the upper half-plane.

Let us look a little closer at the conditions in D^+_0. The extremum $\xi_0(\lambda)$, which for $\lambda = 0$ is at the origin, enters the region $D^+_{0,1}$, and so do the other zeros and extrema which for $\lambda = 0$ are real and on the interval $0 < x < \pi/2$. Those that were on the interval $-(\pi/2) < x < 0$ enter the region $D^+_{0,4}$. Finally, the points on the positive imaginary axis move into $D^+_{0,3}$. Thus, in the semi-strip $-(\pi/2) < x < 0$, $y > 0$, we have the string S^+_0 and a little portion of the string S^0. These two strings join each other in such a manner that if we order the points according to increasing abscissae (or increasing ordinates) we first have to run through the points of S^0, the last of which is a *zero*, and then we encounter the points of S^+_0, the first of which is an *extremum*. Hence we see that zeros and extrema alternate all the way through in the semi-strip.

In $D^+_{0,1}$ we have only a finite number of points belonging to S^0. If we order these according to increasing abscissae, the ordinates decrease. The first point is an extremum, then a zero, and so on alternately. These are the extraneous zeros of §§3.22 and 3.24 which caused us some concern. There we decided not to count them in the string S^+_0. Strictly speaking, we should have excluded the points of S^0 in the left half of D^+_0 as well, but we had no means at our disposal at that stage to do so.

If we vary λ, still keeping $0 < \arg \lambda < (\pi/2)$ and avoiding critical points, we may lose or gain some points across l_{-1} and l_1, one at a time, but they do not affect the validity of our conclusions.

6.33. We can extend our conclusions to the other regions D^+_k by observing that $\arg \lambda(k\pi) = \Theta_1$ or Θ_2 for all values of the integer k, provided the statement is true for $k = 0$. The assumption $\arg \lambda(k\pi) = \Theta_3$ or Θ_4 would imply that the string S^0 lies in the lower half-plane.

The case $\arg \lambda(0) = \Theta_2$ can be treated in the same way as that in which $\arg \lambda(0) = \Theta_1$, but it is easiest settled by observing that if

$w(z;\lambda)=0$ when $z=x+iy$ and $\lambda=\mu+i\nu$ then it will also vanish for $z=-x+iy$ and $\lambda=-\mu+i\nu$. Hence we have only to make a reflection of $DI(0;\Theta_1)$ in the imaginary axis in order to obtain $DI(0;\Theta_2)$.

We find that all the strings S_k^+, S_k^-, and S^0 are made up of zeros and extrema which alternate. Moreover, whenever pieces of different strings join on to each other, the zeros and extrema will alternate in the combined set.

The results of this section will serve to elucidate Problem B proposed in § 3.20. The reader will notice that the movement of the zeros when the parameter is varied has not been studied in Case 2. This can certainly be done by means of the same methods as those used in Cases 1 and 3, but the investigation seems to require the consideration of a great number of particular sub-cases. The patient reader who has followed us so far, will, we trust, cover that point for himself.

THE DISTRIBUTION OF TEMPERATURE IN A COLUMN OF LIQUID FLOWING FROM A COLD SOURCE INTO A RECEIVER MAINTAINED AT A HIGHER TEMPERATURE

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Consider a column of liquid flowing through a tube of uniform cross section, the liquid flowing from a cold source at one end into a receiver maintained at a higher temperature at the other end. The problem arises as to how the distribution of temperature in the column is modified by the flow.

The problem has been the subject of experiment by Nettleton.* A column of mercury was made to move vertically upwards with constant velocity through a pipe by means of a pump. The lower end was maintained at a constant temperature 0°C., and the upper end was maintained at a constant temperature 100°C. Full experimental details are given in the papers mentioned.* An important result emerged from the observations, which showed that in such a column, the isothermals were remarkably horizontal even when the disturbance of temperature gradient by flow was very large.

The result shows that the shape of the isothermals is independent of the character of the flow. It may be mentioned that the apparatus employed was sensitive to differences of temperature of the order 0·02°C.

Assuming this result, consider an element of length δx fixed in space. Let A and p be the area of cross-section and perimeter of the column respectively, v the average velocity over a cross-section, θ the temperature

* "On the thermal conductivity of mercury by the impressed velocity method", *Proc. Phys. Soc. London*, 22 (1910), 26 (1913).

of the element, E the emissivity, k, ρ, and s the thermal conductivity, density, and specific heat of the liquid respectively.

The heat which enters the element is given by

$$A\left\{-k\frac{d\theta}{dx}+\rho sv\theta\right\}.$$

The heat which leaves the element is

$$A\left\{-k\frac{d}{dx}\left(\theta+\frac{d\theta}{dx}\,\delta x\right)+\rho sv\left(\theta+\frac{d\theta}{dx}\,\delta x\right)\right\}+Ep(\theta-\theta_0)\,\delta x,$$

where θ_0 is the temperature of the enclosure which surrounds the pipe.

The total heat entering the element is

$$A\left\{k\frac{d^2\theta}{dx^2}-\rho sv\frac{d\theta}{dx}\right\}\delta x-Ep(\theta-\theta_0)\,\delta x.$$

If the state is steady, this quantity will be equal to zero, if not, then

$$A\left\{k\frac{\partial^2\theta}{\partial x^2}-\rho sv\frac{\partial\theta}{\partial x}\right\}\delta x-Ep(\theta-\theta_0)\,\delta x=A\rho s\frac{\partial\theta}{\partial t}\,\delta x$$

or
$$a^2\frac{\partial^2\theta}{\partial x^2}-v\frac{\partial\theta}{\partial x}-\frac{Ep}{A\rho s}(\theta-\theta_0)=\frac{\partial\theta}{\partial t}, \tag{1}$$

where
$$a^2=k/\rho s.$$

It will be noted that the effects of the walls of the pipe and the variation of density with temperature are neglected.

I. *The source and receiver maintained at constant temperatures. No radiation from the lateral surface.*

In this case the equation (1) reduces to

$$a^2\frac{\partial^2\theta}{\partial x^2}-v\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial t} \tag{2}$$

with the boundary conditions

$$\theta=0, \quad x=0, \tag{3}$$

$$\theta=\theta_1, \quad x=l, \tag{4}$$

$$\theta=0, \quad t=0, \tag{5}$$

where l is the length of the column and t is the time.

A solution of (2) is given by

$$\theta=e^{vx/2a^2}\left\{Ae^{\beta x}+Be^{-\beta x}\right\}e^{-a^2\alpha^2 t},$$

where A, B, and a are constants and

$$\beta = \frac{\sqrt{(v^2 - 4a^4\alpha^2)}}{2a^2}.$$

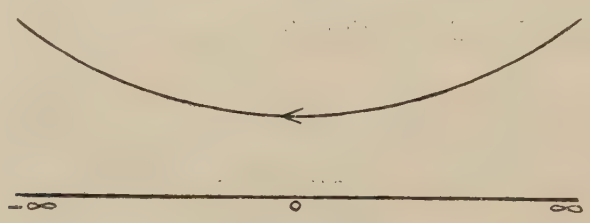
From (3) it follows that

$$\theta = Ae^{vx/2a^2} \sinh \beta x \, . \, e^{-\alpha^2 a^2 t}.$$

Consider the integral

$$\theta = \frac{\theta_1}{\pi i} \int_P \frac{e^{vx/2a^2} \sinh \beta x}{e^{vl/2a^2} \sinh \beta l} \cdot \frac{e^{-\alpha^2 a^2 t}}{\alpha} \, . \, d\alpha \qquad (6)$$

taken over a suitable path P in the α plane. The path chosen is that indicated by Carslaw* as shown in Fig. 1, the argument of α on the right lying between 0 and $\pi/4$, on the left between $3\pi/4$ and π.

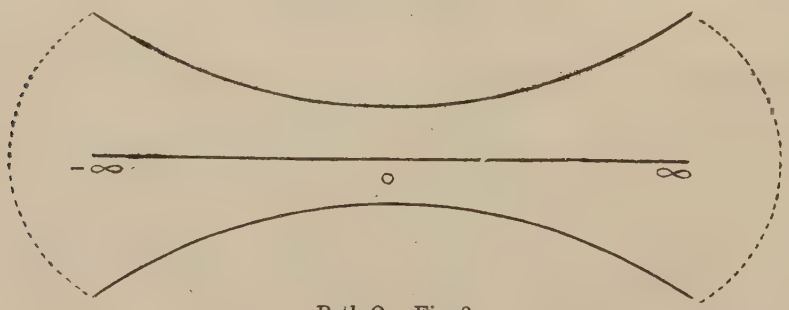


Path P. Fig. 1.

The integral (6) satisfies (2) since each element does so.

When $x = l$ $\qquad \theta = \frac{\theta_1}{\pi i} \int_P \frac{e^{-\alpha^2 a^2 t}}{\alpha} \, . \, d\alpha.$

Since the integrand is odd, the path P may be replaced by the path Q (Fig. 2), consisting of the path P, its image in the real axis, and two circular arcs having their centres at the origin.



Path Q. Fig. 2.

* *Mathematical theory of the conduction of heat*, Ch. XI; *Phil. Mag.* (1920), 604.

Then, at $x = l$, $$\theta = \frac{\theta_1}{2\pi i} \int_Q \frac{e^{-\alpha^2 a^2 t}}{\alpha} \cdot d\alpha,$$

since, in the limit, the value of this integral over the circular arcs vanishes.

The only pole within the contour is at $\alpha = 0$, and the residue there is unity.

Hence, by Cauchy's theorem,

$$\theta = \theta_1$$

and the condition (4) is satisfied.

When $t = 0$, $$\theta = \frac{\theta_1}{\pi i} \int_P \frac{e^{vx/2a^2} \sinh \beta x}{e^{vl/2a^2} \sinh \beta l} \cdot \frac{d\alpha}{\alpha}.$$

There are no poles of the integrand above P, and thus, if the path P be completed by the arc of a circle as shown in Fig. 3, the centre of which is at the origin, the value of the integral over this completed path vanishes. In the limit the value over the circular arc vanishes, hence when $t = 0$, $\theta = 0$.

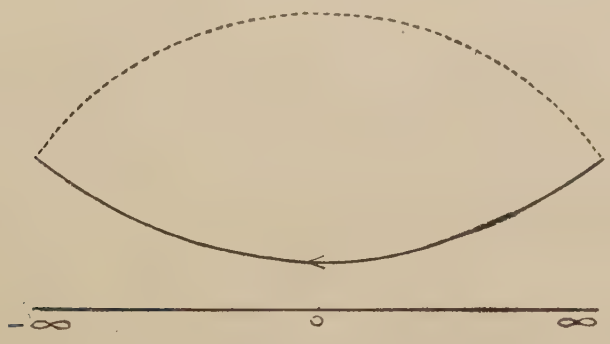


Fig. 3.

Thus the integral (6) satisfies all the boundary conditions.

Since the integrand of (6) is odd, the path P may be replaced again by the path Q. Then

$$\theta = \frac{\theta_1}{2\pi i} \int_Q \frac{e^{vx/2a^2} \sinh \beta x}{e^{vl/2a^2} \sinh \beta l} \cdot \frac{e^{-\alpha^2 a^2 t}}{\alpha} \cdot d\alpha,$$

since the value over the circular arcs vanishes in the limit.

The poles of the integrand are at

$$\alpha = 0 \quad \text{and} \quad \beta_n = n\pi i/l,$$

or $$\alpha_n = \pm\sqrt{\{(n\pi/l)^2 + (v^2/4a^4)\}},$$

where n has the values 1, 2, 3, ...

The residue at $\alpha = 0$ is

$$(e^{vx/a^2}-1)/(e^{vl/a^2}-1).$$

The residue at $\alpha = \alpha_n$ is

$$\frac{e^{vx/2a^2}}{e^{vl/2a^2}} \cdot \frac{\sinh \beta_n x}{\cosh \beta_n l \,.\, l(\partial\beta_n/\partial\alpha_n)} \cdot \frac{e^{-a^2\alpha_n^2 t}}{\alpha_n},$$

where

$$\beta_n = \frac{\sqrt{(v^2-4a^4\alpha_n^2)}}{2a^2}.$$

Applying Cauchy's theorem, we find, after reduction, that

$$\theta = \theta_1\left[\frac{e^{v\rho sx/k}-1}{e^{v\rho sl/k}-1} + \frac{2}{l^2}\sum_{n=1}^{\infty}(-1)^n \frac{e^{v\rho sx/2k}}{e^{v\rho sl/2k}} \times \frac{n\pi}{\{(n\pi/l)^2+(v\rho s/2k)^2\}} \cdot \sin\frac{n\pi x}{l} \cdot e^{-\{(n^2\pi^2 k/l^2\rho s)+(v^2\rho s/4k)\}t}\right].$$

This result agrees with that deduced by Somers.*

II. *Same as* I, *with radiation from the lateral surface.*

In the experiments quoted, the pipe conveying the liquid was surrounded by a vacuum jacket, the outer coat of which was kept at a constant temperature. In the following, this temperature is taken as zero, the temperature of the cold source.

The equation to be satisfied is

$$a^2\frac{\partial^2\theta}{\partial x^2} - v\frac{\partial\theta}{\partial x} - \frac{Ep}{A\rho s}\theta = \frac{\partial\theta}{\partial t}, \tag{7}$$

subject to the boundary conditions (3), (4), and (5).

A solution of this equation satisfying (3) is

$$\theta = \theta_1 \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l} \cdot e^{-a^2\alpha^2 t}$$

where

$$\beta = \frac{\sqrt{\{v^2-4a^4(\alpha^2-Ep/Ak)\}}}{2a^2}.$$

As in I, consider the integral

$$\theta = \frac{\theta_1}{\pi i}\int_P \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l} \cdot \frac{e^{-a^2\alpha^2 t}}{\alpha} \cdot d\alpha.$$

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* *Proc. Phys. Soc. London*, 25 (1912), 74.

It is easy to see, by the same analysis, that it satisfies all the boundary conditions.

Since the integrand is odd and the value over the circular arcs vanishes in the limit,

$$\theta = \frac{\theta_1}{2\pi i}\int_Q \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l}\cdot\frac{e^{-a^2a^2t}}{a}\,.\,da.$$

The poles of the integrand occur at $a = 0$ and when $\beta l = n\pi i$ or when

$$a_n = \pm\sqrt{\{(n\pi/l)^2+(v^2/4a^4)+(Ep/Ak)\}}.$$

Since the roots occur in pairs, and are non-repeated, the value of θ may be written

$$\theta = \theta_1\left[\frac{e^{vx/2a^2}\sinh\{\sqrt{(v^2+4a^4Ep/Ak)}x/2a^2\}}{e^{vl/2a^2}\sinh\{\sqrt{(v^2+4a^4Ep/Ak)}l/2a^2\}}\right]$$

$$+\frac{2\theta_1}{l^2}\sum_{n-1}^{\infty}(-1)^n\frac{e^{vx/2a^2}}{e^{vl/2a^2}}\cdot\frac{n\pi}{\{(n\pi/l)^2+(v\rho s/2k)^2+(Ep/Ak)\}}\cdot\sin\frac{n\pi x}{l}$$

$$\times e^{-\{(n^2\pi^2k/l^2\rho s)+(v^2\rho s/4k)+(Ep/A\rho s)\}t}.$$

III. *The same as* I, *with the receiver maintained at a temperature which is a periodic function of the time.*

This problem is suggested by Ångström's* method for the determination of the conductivity of solid bars of small cross-section.

In this case θ must satisfy (2), (3), and (5), together with

$$\theta = b\cos wt, \qquad x = l. \tag{8}$$

As in I,
$$\theta = Ae^{vx/2a^2}\sinh\beta x\,.\,e^{-a^2a^2t},$$

where
$$\beta = \frac{\sqrt{(v^2-4a^4a^2)}}{2a^2}.$$

Consider the integral

$$\theta = \frac{b}{\pi i}\int_P \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l}\cdot\frac{a^3}{a^4+(w^2/a^4)}\,.\,e^{-a^2a^2t}\,da. \tag{9}$$

The integral clearly satisfies (2).

When $t = 0$,

$$\theta = \frac{b}{\pi i}\int_P \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l}\cdot\frac{a^3}{a^4+(w^2/a^4)}\,.\,da.$$

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* *Ann. Phys. Leipzig*, 114 (1861), 513; 123 (1864), 628.

If we take the value of this integral over the path shown in Fig. 3, it follows, as before, that it is equal to zero.

Hence $\theta = 0$ when $t = 0$.

When $x = l$, $$\theta = \frac{b}{\pi i}\int_P \frac{a^3}{a^4+(w^2/a^4)}\cdot e^{-a^2\alpha^2 t}\,da.$$

Since the integrand is odd, replace the path P by the path Q, then

$$\theta = \frac{b}{2\pi i}\int_Q \frac{a^3}{a^4+(w^2/a^4)}\cdot e^{-a^2\alpha^2 t}\,da.$$

The poles of the integrand are at $(\sqrt{w}/a)e^{\pm\pi i/4}$ and $(\sqrt{w}/a)e^{\pm 3\pi i/4}$.

It follows, by Cauchy's theorem, that $\theta = b\cos wt$ when $x = l$. Thus the integral (9) satisfies all the boundary conditions.

Since the integrand of (9) is odd, replace the path P by the path Q. The value of the integral over the circular arcs vanishes in the limit, hence

$$\theta = \frac{b}{2\pi i}\int_Q \frac{e^{vx/2a^2}\sinh\beta x}{e^{vl/2a^2}\sinh\beta l}\cdot\frac{a^3}{a^4+(w^2/a^4)}\cdot e^{-a^2\alpha^2 t}\,da.$$

The poles of the integrand are at

$$(\sqrt{w}/a)\,e^{\pm\pi i/4},\quad (\sqrt{w}/a)\,e^{\pm 3\pi i/4},\quad \text{and}\quad \pm\sqrt{\{(n\pi/l)^2+(v^2/4a^4)\}},$$

where $n = 1, 2, 3, \ldots$.

The sum of the residues at the roots of $a^4+(w^2/a^4) = 0$ is given by

$$\tfrac{1}{2}\frac{e^{vx/2a^2}}{e^{vl/2a^2}}\left\{\frac{\sinh\{\sqrt{(v^2-4wa^2i)}\,x/2a^2\}}{\sinh\{\sqrt{(v^2-4wa^2i)}\,l/2a^2\}}\cdot e^{-iwt}\right.$$
$$\left.+\frac{\sinh\{\sqrt{(v^2+4wa^2i)}\,x/2a^2\}}{\sinh\{\sqrt{(v^2+4wa^2i)}\,l/2a^2\}}\cdot e^{iwt}\right\}. \quad (10)$$

This may be written

$$\tfrac{1}{2}\frac{e^{vx/2a^2}}{e^{vl/2a^2}}\left\{\frac{\sinh\sqrt{(p-iq)}\,x}{\sinh\sqrt{(p-iq)}\,l}\cdot e^{-iwt}+\frac{\sinh\sqrt{(p+iq)}\,x}{\sinh\sqrt{(p+iq)}\,l}\cdot e^{iwt}\right\},$$

where $p = v^2/4a^4$ and $q = w/a^2$.

Put $p\pm iq = re^{\pm i\theta}$ where $r = \sqrt{(p^2+q^2)}$ and $\tan\theta = q/p$.

Then $$\sqrt{(p\pm iq)} = r^{\frac{1}{2}}e^{\pm i\theta/2} = s\pm im,$$

where $$s = r^{\frac{1}{2}}\cos(\theta/2) = (p^2+q^2)^{\frac{1}{4}}\cos\{\tfrac{1}{2}\tan^{-1}(q/p)\}$$
$$= \{(v/2a^2)^4+(w/a^2)^2\}^{\frac{1}{4}}\cos\{\tfrac{1}{2}\tan^{-1}(4a^2w/v^2)\}.$$

Similarly $$m = \{(v/2a^2)^4+(w/a^2)^2\}^{\frac{1}{4}}\sin\{\tfrac{1}{2}\tan^{-1}(4a^2w/v^2)\}.$$

Hence

$$\frac{\sinh\sqrt{(p-iq)}\,x}{\sinh\sqrt{(p-iq)}\,l} = \frac{e^{(s-im)x}-e^{-(s-im)x}}{e^{(s-im)l}-e^{-(s-im)l}}$$

$$= \frac{\cos mx \sinh sx - i \sin mx \cosh sx}{\cos ml \sinh sl - i \sin ml \cosh sl}$$

$$= \frac{2(\cos mx \sinh sx - i \sin mx \cosh sx)(\cos ml \sinh sl + i \sin ml \cosh sl)}{\cosh 2sl - \cos 2ml}.$$

Similarly

$$\frac{\sinh\sqrt{(p+iq)}\,x}{\sinh\sqrt{(p+iq)}\,l}$$

$$= \frac{2(\cos mx \sinh sx + i \sin mx \cosh sx)(\cos ml \sinh sl - i \sin ml \cosh sl)}{\cosh 2sl - \cos 2ml}.$$

Substituting these values in (10) and rejecting the imaginary part, the sum of the residues at the roots of $a^4+(w^2/a^4) = 0$ is obtained.

The sum of the residues at the poles

$$a_n = \sqrt{\{(n\pi/l)^2+(v^2/4a^4)\}} \quad (n = 1, 2, 3, \ldots)$$

can be written down as in I.

Thus

$$\theta = \frac{b\, e^{vx/2a^2}}{(\cosh 2sl - \cos 2ml)\, e^{vl/2a^2}}$$

$$\times[\{\cos m(x-l) \cosh s(x+l) - \cos m(x+l) \cosh s(x-l)\} \cos wt$$

$$- \{\sin m(x-l) \sinh s(x+l) - \sin m(x+l) \sinh s(x-l)\} \sin wt]$$

$$+ \frac{2b}{l^2} \sum_{n=1}^{\infty} (-1)^n \frac{e^{vx/2a^2}}{e^{vl/2a^2}} \cdot n\pi \sin \frac{n\pi x}{l} \frac{a_n^2}{a_n^4+(w^2/a^4)} \cdot e^{-a^2 a_n^2 t},$$

where $$a_n = \sqrt{\{(n\pi/l)^2+(v^2/4a^4)\}} \quad (n = 1, 2, 3, \ldots)$$

IV. *The source, receiver, and lateral surface maintained at constant temperatures.*

If the vacuum jacket surrounding the pipe were replaced by a steam jacket the outer lateral surface would be maintained at a constant temperature. The problem, thus stated, approximates to the conditions in a steam condenser, where waste steam is passed over the pipes conveying water to the boilers.

The fundamental equation to be satisfied by θ is

$$a^2 \frac{\partial^2 \theta}{\partial x^2} + a^2 \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta}{\partial r} \right) - v \frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial t}, \tag{11}$$

where x is measured along the length and r along the radius of the cross-section. In this case the isothermals will not correspond with the normal cross-sections.

Since θ depends on r as well as on x and t, the method of the previous sections is inapplicable.

The simplest boundary conditions are taken, viz.:

$$\theta = 0 \quad \text{when} \quad x = 0, \tag{12}$$

$$\theta = \theta_1 \quad \text{when} \quad x = l, \tag{13}$$

$$\theta = \theta_1 \quad \text{when} \quad r = c, \text{ the radius}, \tag{14}$$

$$\theta = 0 \quad \text{when} \quad t = 0. \tag{15}$$

The solution of (11) satisfying (12) is

$$\theta = A e^{vx/2a^2} \sinh \beta x \, J_0(\mu r) e^{-a^2(\alpha^2+\mu^2)t}, \tag{16}$$

where $$\beta = \frac{\sqrt{(v^2 - 4a^4\alpha^2)}}{2a^2},$$

A, α, and μ being constants, this value of θ being finite when $r = 0$.

Assume $\theta = u_1 + u_2 + u_3 + u_4$ where u_1 and u_2 are functions of r and x only, u_3 and u_4 are functions of r, x, and t.

Let the u's satisfy the following conditions:—

$$u_1 = 0, \quad x = 0; \quad u_1 = \theta_1, \quad x = l; \quad u_1 = 0, \quad r = c;$$

$$u_2 = 0, \quad x = 0; \quad u_2 = 0, \quad x = l; \quad u_2 = \theta_1, \quad r = c;$$

$$u_3 = u_4 = 0, \quad x = 0, \quad x = l, \quad \text{and} \quad r = c;$$

$$u_3 = -u_1, \quad t = 0; \quad u_4 = -u_2, \quad t = 0.$$

The conditions (12), (13), (14), and (15) will be satisfied.

Also u_1 and u_2 must satisfy

$$a^2 \frac{\partial^2 \theta}{\partial x^2} + a^2 \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta}{\partial r} \right) - v \frac{\partial \theta}{\partial x} = 0,$$

a solution of which is

$$\theta = Ae^{vx/2a^2} \sinh \gamma x J_0(ar),$$

where

$$\gamma = \frac{\sqrt{(v^2+4a^4a^2)}}{2a^2}.$$

The solution for u_1 is

$$u_1 = \sum_{m=1}^{\infty} A_m \frac{e^{vx/2a^2} \sinh \eta_m x}{e^{vl/2a^2} \sinh \eta_m l} J_0(\mu_m r),$$

where

$$\eta_m = \frac{\sqrt{(v^2+4a^4\mu_m^2)}}{2a^2},$$

and μ_m is the m-th positive root of $J_0(\mu c) = 0$.

Since $u_1 = \theta_1$ when $x = l$,

$$\theta_1 = \sum_{m=1}^{\infty} A_m J_0(\mu_m r).$$

By the usual methods it is found that

$$A_m = 2\theta_1 / \{\mu_m c J_1(\mu_m c)\}.$$

Hence

$$u_1 = \frac{2\theta_1}{c} \sum_{m=1}^{\infty} \frac{e^{vx/2a^2} \sinh \eta_m x}{e^{vl/2a^2} \sinh \eta_m l} \cdot \frac{J_0(\mu_m r)}{\mu_m J_1(\mu_m c)}.$$

The solution for u_2 is

$$u_2 = \sum_{n=1}^{\infty} B_n e^{vx/2a^2} \sin \frac{n\pi x}{l} \cdot \frac{J_0(\alpha_n r)}{J_0(\alpha_n c)},$$

where

$$\alpha_n = i\sqrt{\{(n\pi/l)^2+(v^2/4a^4)\}}.$$

Since $u_2 = \theta_1$ when $r = c$,

$$\theta_1 = \sum_{n=1}^{\infty} B_n e^{vx/2a^2} \sin (n\pi x/l)$$

or

$$e^{-vx/2a^2} . \theta = \sum_{n=1}^{\infty} B_n \sin (n\pi x/l).$$

Multiply both sides by $\sin (n\pi x/l)$ and integrate from 0 to l

$$\theta_1 \int_0^l e^{-vx/2a^2} \sin \frac{n\pi x}{l} \, dx = B_n \int_0^l \sin^2 \frac{n\pi x}{l} \, dx,$$

since $$\int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0 \quad (m \neq n).$$

$$B_n = \frac{2\theta_1}{l}\cdot\frac{n\pi}{l}\left\{\frac{1-(-)^n e^{-vl/2a^2}}{(n\pi/l)^2+(v^2/4a^4)}\right\}.$$

Since u_3 and u_4 satisfy (11) and vanish at $x = 0$, $x = l$, and $r = c$, it follows from (16) that they have the form

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\, e^{vx/2a^2}\sin(n\pi x/l)\,.\,J_0(\mu_m r)\, e^{-a^2\{\mu_m^2+(n\pi/l)^2+(v^2/4a^4\}t},$$

where μ_m is, as before, the m-th positive root of $J_0(\mu c) = 0$.

When $t = 0$, $u_3 = -u_1$, hence the coefficient of $J_0(\mu_m r)$ in u_1 must be expressed as a sine series in the range $0 < x < l$.

Therefore

$$u_1 = \frac{2\theta_1}{c}\frac{e^{vx/2a^2}}{e^{vl/2a^2}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_0(\mu_m r)}{\mu_m J_1(\mu_m c)}\sin\frac{n\pi x}{l}\cdot\frac{2}{l}\int_0^l\frac{\sinh\eta_m x}{\sinh\eta_m l}\sin\frac{n\pi x}{l}\,.\,dx.$$

Now $$\int_0^l \sinh\eta_m x\sin\frac{n\pi x}{l}\,.\,dx = (-1)^{n+1}\frac{n\pi}{l}\cdot\frac{\sinh\eta_m l}{\eta_m^2+(n\pi/l)^2},$$

hence

$$u_1 = \frac{4\theta_1}{cl}\frac{e^{vx/2a^2}}{e^{vl/2a^2}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n\pi}{l}\cdot\frac{1}{\eta_m^2+(n\pi/l)^2}\sin\frac{n\pi x}{l}\cdot\frac{J_0(\mu_m r)}{\mu_m J_1(\mu_m c)}.$$

When $t = 0$, $u_4 = -u_2$, hence the coefficient of $\sin(n\pi x/l)$ in u_2 must be expressed as a series of Bessel Functions of zero order, of the form $J_0(\mu_m r)$, where μ_m is the m-th positive root of $J_0(\mu_n c) = 0$.

Therefore

$$u_2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n e^{vx/2a^2}\sin\frac{n\pi x}{l}\cdot\frac{J_0(\mu_m r)}{J_0(a_n c)}$$

$$\times\left[\int_0^c r\,J_0(a_n r)\,J_0(\mu_m r)\,dr\Big/\int_0^c r\{J_0(\mu_m r)\}^2\,dr.\right]$$

Now

$$\int_0^c r\,J_0(a_n r)\,J_0(\mu_m r)\,dr = \frac{c}{\mu_m^2-a_n^2}\{a_n J_0(\mu_m c)\,J_0'(a_n c) - \mu_m J_0'(\mu_m c)\,J_0(a_n c)\}$$

$$= \frac{c\,\mu_m}{\mu_m^2-a_n^2}\,J_0(a_n c)\,J_1(\mu_m c),$$

since $$J_0(\mu_m c) = 0.$$

Also $$\int_0^c r\{J_0(\mu_m r)\}^2 dr = \frac{c^2}{2}\left[\{J_0'(\mu_m c)\}^2 + \{J_0(\mu_m c)\}^2\right]$$

$$= \frac{c^2}{2} J_1^2(\mu_m c).$$

Hence

$$u_2 = \frac{4\theta_1}{cl} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n\pi}{l} \left\{\frac{1-(-1)^n e^{-vl/2a^2}}{(n\pi/l)^2+(v^2/4a^4)}\right\} e^{vx/2a^2} \cdot \sin\frac{n\pi x}{l} \cdot \frac{\mu_m}{\mu_m^2-a_n^2} \frac{J_0(\mu_m r)}{J_1(\mu_m c)}.$$

Thus the complete solution may be written

$$\theta = \frac{4\theta_1}{cl} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n\pi}{l} \cdot \left[\frac{e^{vx/2a^2}}{e^{vl/2a^2}} \cdot \frac{(-1)^{n+1}}{\eta_m^2+(n\pi/l)^2} \cdot \frac{1}{\mu_m J_1(\mu_m c)}\right.$$

$$\left. + e^{vx/2a^2} \frac{1-(-1)^n e^{-vl/2a^2}}{(n\pi/l)^2+(v^2/4a^4)} \cdot \frac{\mu_m}{(\mu_m^2-a_n^2) J_1(\mu_m c)}\right]$$

$$\times \sin\frac{n\pi x}{l} \cdot J_0(\mu_m r) \cdot \left[1-e^{-a^2\{\mu_m^2+(n\pi/l)^2+v^2/4a^4\}t}\right].$$

THE INFLEXIONAL TANGENTS OF PLANE QUARTIC CURVES

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I.

1. I first prove the following theorem for the non-singular quartic.

The twenty-four inflexional tangents and the twelve bitangents of a Steiner's complex touch a class-sextic.

The method of proof is seen to be applicable to any quartic and I have investigated the form assumed by the theorem when we consider quartics with ordinary singularities. I have not considered any cases where an inflexional tangent coincides with a nodal tangent or is a tangent at a point of undulation. I have not given complete details in all cases, as the algebra is very tedious and is of no interest in itself.

2. If $x = 0$, $y = 0$ are a pair of bitangents of a quartic, the equation of the quartic may be written in the form $xyS = S'^2$, where $S = 0$ and $S' = 0$ are conics.

This equation can be written in the form

$$xy(S+2\lambda S'+\lambda^2 xy) = (S'+\lambda xy)^2.$$

There are, in general, five values of λ, for which $S+2\lambda S'+\lambda^2 xy = 0$ is a line-pair distinct from $xy = 0$. The two members of such a line-pair are obviously bitangents, and these five pairs of bitangents, together with $xy = 0$, are said to form a Steiner's complex (*vide* Hilton, *Plane Algebraic Curves*, 337).

3. If we consider the general form

$$U \equiv kS_1S_2+S_3^2 = 0, \quad \text{where} \quad S_1 \equiv (a_1, b_1, c_1, f_1, g_1, h_1)(x, y, z)^2,$$

with similar meanings for S_2 and S_3, then the condition that

$$lx+my+nz = 0$$

should be an inflexional tangent of $U = 0$ is the vanishing of the invariants I and J of the biquadratic in x and y obtained by eliminating z between $lx+my+nz = 0$ and $U = 0$.

The biquadratic is

$$k(a_1x^2+2\beta_1xy+\gamma_1y^2)(a_2x^2+2\beta_2xy+\gamma_2y^2)+(a_3x^2+2\beta_3xy+\gamma_3y^2)^2 = 0,$$

where α, β, γ denote respectively

$$cl^2-2gnl+an^2, \quad clm-fnl-gmn+hn^2, \quad cm^2-2fmn+bn^2.$$

It is obvious both analytically and geometrically that I and J calculated from this biquadratic contain as factors powers of n. When these factors have been removed, $I = 0$ is the envelope of straight lines on which $U = 0$ determines an equianharmonic range, and $J = 0$ is the envelope of straight lines on which $U = 0$ determines an harmonic range.

If a homogeneous biquadratic in y and z is formed by eliminating x, its two invariants will be the two contravariants of $U = 0$, which, equated to zero, give these two envelopes, multiplied by irrelevant factors which are powers of l. Now it is easily verified that the terms in I, which are free from k, are in the two cases $\frac{4}{3}n^4\Sigma_3^2$ and $\frac{4}{3}l^4\Sigma_3^2$ respectively, where Σ_3 is the tangential form corresponding to S_3. Hence the irrelevant factors are n^4 and l^4, and the proper envelope is

$$\tfrac{4}{3}\Sigma_3^2+kU_4+k^2V_4 = 0,$$

where U_4 and V_4 are tangential forms of the fourth degree.

Similarly J, calculated from the first equation, will be seen to have an irrelevant factor n^6, and the proper envelope is

$$\tfrac{8}{27}\Sigma_3^3+kU_6+k^2V_6+k^3W_6 = 0,$$

where U_6, V_6, W_6 are tangential forms of the sixth degree.

The actual forms, with the irrelevant factors omitted, are given by

$$\begin{aligned} 12I &= (4\Sigma_3-k\Phi_{12})^2+12k(\Phi_{13}\Phi_{23}+k\Sigma_1\Sigma_2), \\ 216J &= (4\Sigma_3-k\Phi_{12})^3+18k(\Phi_{13}\Phi_{23}+k\Sigma_1\Sigma_2)(4\Sigma_3-k\Phi_{12}) \\ &\quad +54k^2(\Phi_{23}\Phi_{31}\Phi_{12}-\Psi^2-\Phi_{12}^2\Sigma_3) \\ &\quad +54k^3\Phi_{12}\Sigma_1\Sigma_2, \end{aligned}$$

where Σ_1, Σ_2, Σ_3 are the tangential forms corresponding to S_1, S_2, S_3,

$$\phi_{12} = (b_1c_2+b_2c_1-2f_1f_2)l^2+\ldots+\ldots$$

$$+2(g_1h_2+g_2h_1-a_1f_2-a_2f_1)mn+\ldots+\ldots$$

is the quadratic contravariant of S_1 and S_2, with similar meanings for Φ_{23} and Φ_{31}, and

$$n^3\Psi = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

so that $\Psi = 0$ is the equation of the class-cubic which is the envelope of straight lines met by $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ in involution. This class-cubic touches the twelve bitangents of the Steiner's complex derived from this form of $U = 0$. (*Vide* Hilton, *loc. cit.*)

$I = 0$ touches the inflexional tangents and they form the complete system of common tangents of $I = 0$ and the quartic $U = 0$.

$J = 0$ touches the inflexional tangents and moreover touches them at their points of contact, since the tangential equation of $U = 0$ is $I^3-27J^2 = 0$. $J = 0$ and $U = 0$ have therefore no other common tangents.

That the inflexional tangents all touch a class-quartic is stated in a foot-note on page 280 of Clebsch-Lindeman (Ed. 1876). Salmon (*Higher Plane Curves*, §§ 92, 292) states that the contravariants I and J are of degrees four and six respectively for the general quartic, and (§ 250) that the inflexional tangents touch $I = 0$ and $J = 0$.

4. If $S_1 = 0$ is a line-pair, the coordinates of either line satisfy the equations $\Sigma_1 = 0$, $\Phi_{12} = 0$, $\Phi_{13} = 0$, $\Psi = 0$, so that the coefficients of the various powers of k in the expressions for I and J vanish when the coordinates of either line are substituted in them.

This is obvious analytically from the fact that these coordinates will make the terms in k in the biquadratic above disappear, and geometrically from the fact that, if one of these lines is a tangent to $S = 0$, it will meet $U = 0$ in four coincident points for all values of k.

Now if $S_1 = 0$ and $S_2 = 0$ are both line-pairs, the four lines all touch $\Phi_{12} = 0$, and hence these four lines and the inflexional tangents all touch the class-sextic $(2\Sigma_3+k\Phi_{12})I = 9J$.

The theorem enunciated at the commencement will be proved, if we can show that $2\Sigma_3+k\Phi_{12}$ is unaltered, when any other pair of the complex is substituted for S_2.

If we write the equation $U=0$ in the form $kS_1S_2'+S_3'^2=0$, where $S_2'=S_2-2\lambda S_3-\lambda^2 kS_1$ and $S_3'=S_3+\lambda kS_1$, we have

$$\Phi_{12}'=\Phi_{12}-2\lambda\Phi_{13}-2\lambda^2 k\Sigma_1$$

and

$$\Sigma_3'=\Sigma_3+\lambda k\Phi_{13}+\lambda^2 k^2\Sigma_1,$$

so that

$$2\Sigma_3'+k\Phi_{12}'=2\Sigma_3+k\Phi_{12}.$$

Hence the theorem is proved.

5. When the equation of any quartic is written in the form

$$kS_1S_2+S_3^2=0,$$

then evidently we shall obtain a class-sextic touching the inflexional tangents and all the line-pairs of the system of conics

$$kS_2-2\lambda S_3-\lambda^2 S_1=0.$$

These lines will be bitangents of the quartic in some sense, but when the quartic has one or more double points they will not all be proper bitangents. Some of them will be tangents from the double points or lines joining two double points. There will not always be six distinct pairs in such a complex, and, as may be expected, the class-sextic will sometimes degenerate into one or more singular points and a proper curve of lower class.

In what follows I shall use K to denote the form $2\Sigma_3+k\Phi_{13}$ associated with a complex. I shall also suppose that I and J have irrelevant factors removed.

II.

Quartics with one double point.

1. If

$$(z+p_1x+q_1y)(z+p_2x+q_2y)(z+p_3x+q_3y)(z+p_4x+q_4y)$$
$$=(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)^2$$

is the equation of a quartic with a double point at $x=0$, $y=0$, the

equation contains no terms in z^4 or z^3, so that $c^2 = 1$, $4cg = \Sigma p$, $4cf = \Sigma q$.

If we take $c = 1$, the conic

$$(z+p_1x+q_1y)(z+p_2x+q_2y)+2\lambda(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)$$
$$+\lambda^2(z+p_3x+q_3y)(z+p_4x+q_4y)$$

becomes, when $\lambda = -1$, a pair of straight lines through $x = 0$, $y = 0$. These straight lines are therefore a pair of tangents from the double point. If we take this pair to be $x = 0$, $y = 0$, we can take the equation of the quartic in the form

$$kxyz(x+y+z)+(ax^2+by^2+2fyz+2gzx+2hxy)^2 = 0,$$

where $z = 0$ and $x+y+z = 0$ are a pair of bitangents.

The tangents at the double point are $kxy+4(gx+fy)^2 = 0$, so that the double point is a cusp, if $k+16gf = 0$.

2. To form a complex of which two pairs are $xy = 0$ and $z(x+y+z) = 0$, we choose λ so that

$$kxy-2\lambda(ax^2+by^2+2fyz+2gzx+2hxy)-\lambda^2z(x+y+z) = 0$$

is a line-pair.

This gives us

$$\tfrac{1}{4}\lambda^2(\lambda+4f)(\lambda+4g)(k-4\lambda h)-4\lambda^4ab$$
$$+\tfrac{1}{2}\lambda^3a(\lambda+4f)^2+\tfrac{1}{2}\lambda^3b(\lambda+4g)^2+\tfrac{1}{4}\lambda^2(k-4\lambda h)^2 = 0,$$

Two values of λ are zero, and a third value is zero if $k+16fg = 0$, that is, if the double point is a cusp.

We can form a complex of another type by taking $xz = 0$ as one pair and $y(x+y+z) = 0$ as another pair.

$$kxz-2\lambda(ax^2+by^2+2fyz+2gzx+2hxy)-\lambda^2y(x+y+z) = 0$$

is a line-pair, if $\lambda = 0$ or

$$\lambda(\lambda+4f)(\lambda+4h)(4g\lambda-k)-2\lambda^2a(\lambda+4f)^2-2(\lambda+2b)(4g\lambda-k)^2 = 0.$$

If $k = -16gf$, then $4g\lambda-k = 4g(\lambda+4f)$, and $\lambda+4f = 0$ is a repeated root of this equation. In this case the line-pair corresponding to the repeated root is a pair of lines through the cusp.

Since there is no term in z^2 in the equation of any of these line-pairs, such a line-pair in general consists of a proper double tangent and a tangent drawn from the double point.

We have then for the quartic with one node, two types of complex.

Type I.—A pair of tangents from the node repeated and four pairs of bitangents.

Type II.—Six line-pairs, each consisting of a proper bitangent and a tangent from the node.

For a quartic with a cusp we have two types.

Type I.—A pair of tangents from the cusp taken three times and three pairs of bitangents.

Type II.—A pair of tangents from the cusp repeated and four line-pairs, each consisting of a proper bitangent and a tangent from the cusp.

3. If we take

$$S_1 \equiv xy, \quad S_2 \equiv z(x+y+z), \quad S_3 \equiv ax^2+by^2+2fgz+2gzx+2hxy,$$

$$\Sigma_1 = -\tfrac{1}{4}n^2, \quad \Sigma_2 = -\tfrac{1}{4}(l-m)^2,$$

$$\Sigma_3 = -(fl-gm)^2+2n(Fm+Gl)+Cn^2,$$

$$\Phi_{13} = n(gm+fl-hn),$$

$$\Phi_{12} = \tfrac{1}{2}n(l+m)-lm,$$

$$\Phi_{23} = (b-f)l^2+(a-g)m^2+(f+g-2h)lm+n\{(h-b)l+(h-a)m\},$$

$$\Psi = \tfrac{1}{2}n(am^2-bl^2)-\tfrac{1}{2}n^2(am-bl)+lmn(g-f)-lm(gm-fl).$$

It may be verified without difficulty that $\Phi_{23}\Phi_{31}\Phi_{12}-\Psi^2-\Phi_{12}^2\Sigma_3$ contains n^2 as a factor, so that, when only the first powers of n are retained,

$$12I = (4\Sigma_3-k\Phi_{12})^2+12k\Phi_{13}\Phi_{23},$$

$$216J = (4\Sigma_3-k\Phi_{12})^3+18k\Phi_{13}\Phi_{23}(4\Sigma_3-k\Phi_{12}),$$

so that, if V denotes $4(fl-gm)^2-klm$, we have

$$12I = V^2+12nU_3,$$

$$-216J = V^3+18nU_3V,$$

where U_3 is a tangential form of the third degree.

Also

$$2K = -4(fl-gm)^2+8n(Fm+Gl)+4Cn^2+kn(l+m)-2klm$$
$$= -(V+3klm)+nU_1,$$

where U_1 is a tangential form of the first degree. Hence $9J-IK=0$ is a proper class-sextic, in which the terms free from n are a multiple of V^2lm, so that it has a node at the node of the quartic with the same nodal tangents and touches the pair of tangents from the node. Since these tangents are repeated in the complex, it may be anticipated that they touch the sextic at the same points as the quartic, which will be proved in the next section.

If we consider the complex of the second type, we shall have the same forms for I and J, but the Φ contravariant of $xz=0$ and $y(x+y+z)=0$ is $\frac{1}{2}m(n+l)-nl$, so that $2K$ is now

$$-4(fl-gm)^2+8n(Fm+Gl)+4Cn^2+kmn+klm-2knl$$

or

$$-(V+nU_1).$$

The class-sextic, $9J-IK=0$, then degenerates into the node and a class-quintic, in which the terms free from n contain V as a factor, not in general repeated. Hence the class-quintic touches the nodal tangents but not at the node.

4. If we take the points of contact of $x=0$, $y=0$ to lie on $z=0$, we can take the equation of our quartic as

$$kxy(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)+z^2(x+y)^2=0.$$

The point $x=0$, $y=0$ will be a cusp instead of a node, if $ck+4=0$ or $c=0$.

With $ck+4=0$ the complex containing $xy=0$ as a pair is of Type I, with $c=0$ it is of Type II.

For a node, the complex is of Type I.

With $S_1=xy$, $S_2=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, $S_3=z(x+y)$, we have

$$\Sigma_1=-\tfrac{1}{4}n^2, \qquad \Sigma_2=Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm,$$
$$\Sigma_3=-\tfrac{1}{4}(l-m)^2, \quad \Phi_{12}=-hn^2+gmn+fnl-clm,$$
$$\Phi_{13}=\tfrac{1}{2}n(l+m), \qquad \Phi_{23}=-(fl-gm)(l-m)+(h-a)mn+(h-b)nl,$$
$$\Psi=\tfrac{1}{2}n^2(am-bl)+\tfrac{1}{2}clm(l-m)+(f-g)lmn.$$

Omitting terms which contain n^2 or the product lm, we may write

$$\Sigma_1 = 0, \quad \Sigma_3 = -\tfrac{1}{4}(l-m)^2, \quad \Phi_{12} = n(gm+fl),$$

$$\Phi_{13} = \tfrac{1}{2}n(l+m), \quad \Phi_{13}\Phi_{23} = -\tfrac{1}{2}n(l-m)^2(fl+gm), \quad \Psi = 0,$$

so that

$$12I = (l-m)^4 - 4kn(gm+fl)(l-m)^2,$$

$$-216J = (l-m)^6 - 6kn(gm+fl)(l-m)^4,$$

$$2K = -\{(l-m)^2 - 2kn(gm+fl)\}.$$

The equation $9J-IK = 0$ is of the form $lmU_4+n^2V_4 = 0$, and therefore the class-sextic touches $x = 0$, $y = 0$ at the points where they meet $z = 0$.

By putting $n = 0$ we can verify that the terms independent of n are a multiple of $lm\{(l-m)^2-cklm\}^2$, so that, as previously found, the class-sextic has a node at $x = 0$, $y = 0$ with the same nodal tangents as the quartic.

5. When the double point is a cusp we have seen that $xy = 0$ is repeated three times in the complex and we may anticipate that the quartic and the class-sextic have three-point contact at the point of contact of each of the tangents $x = 0$, $y = 0$.

To verify this take the equation of the quartic as

$$kxy(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)+z^2(x+y)^2 = 0,$$

with $kc+4 = 0$. We may notice that the equation of the class-sextic may be taken to be

$$(4\Sigma_3-k\Phi_{13})^2\Phi_{12}-2(4\Sigma_3+2k\Phi_{12})(\Phi_{13}\Phi_{23}+k\Sigma_1\Sigma_2)+18k(\Psi^2+\Phi_{12}^2\Sigma_3) = 0.$$

To find the equation of a conic having three-point contact with this sextic at the point where it touches $y = 0$, we may write in the equation of the sextic $m = 1$, and retain only terms of the first degree in l and of the second degree in n. The equation will be found to reduce to

$$-cl-\tfrac{1}{2}kg^2n^2-an^2-(k/2)Bn^2 = 0,$$

or with $kc+4 = 0$, $cl = an^2$; or, if we restore m, $clm = an^2$.

A conic having three-point contact with the quartic at the point where

$y = 0$ touches it is $kaxy + z^2 = 0$ or $4axy = cz^2$, the tangential equation of which is $clm = an^2$.

Hence the sextic and quartic have three-point contact at the point where the quartic is touched by $y = 0$, and similarly at the point where it is touched by $x = 0$.

The cuspidal tangent is $y - x = 0$ or $n = 0$, $l + m = 0$.

To ascertain its nature on the sextic, put

$$l + m = L, \quad m = 1, \quad l = L - 1.$$

It will be readily found that the only terms of the equation, which are not divisible by other terms, are L^4, nL, and n^2. Hence the sextic has a cusp at the cusp of the quartic with the same cuspidal tangent, which also touches the sextic at a further point.

For $c = 0$, which gives for the cuspidal quartic a complex of the second type, it will easily be found that the class-sextic degenerates into the cusp and a class-quintic, in which the terms free from n are a multiple of $(l - m)^3 lm$, so that the quintic has a cusp at the cusp of the quartic with the same cuspidal tangent.

Hence we obtain the following theorems.

6. *A quartic with a node.*

Plücker's numbers are $m = 10$, $i = 18$, $\tau = 16$.

(1) *There are proper class-sextics, each of which touches the eighteen inflexional tangents, eight bitangents, the nodal tangents at the node, and a pair of tangents from the node at their points of contact.*

(2) *There are class-quintics, each of which touches the eighteen inflexional tangents, six bitangents, and the nodal tangents, but not at the node.*

The six bitangents which touch such a class-quintic must touch a conic, for six of the twelve lines of the complex pass through the node, so that the class-cubic which touches all twelve (*vide* Hilton, *loc. cit.*) must degenerate into the node and a conic.

The class-quartic $I = 0$, which touches the eighteen inflexional tangents, will easily be seen to touch the nodal tangents at the node.

All the common tangents of the quartic and the class-quartic $I = 0$ and the class-sextics (1) and the class-quintics (2) are accounted for.

7. *A quartic with a cusp.*

$$m = 9, \quad i = 16, \quad \tau = 10.$$

(1) *There are class-sextics, which touch the sixteen inflexional tangents and six bitangents, have three-point contact with the quartic at the points of contact of a pair of tangents from the cusp, and have a cusp at the cusp of the quartic with the same cuspidal tangent, which also touches the sextic at a further point.*

The fifty-four common tangents are all accounted for.

(2) *There are class-quintics, which touch the sixteen inflexional tangents, four bitangents, and a pair of tangents from the cusp, and have a cusp at the cusp of the quartic with the same cuspidal tangent.*

The forty-five common tangents are all accounted for.

$I = 0$ has a cusp at the cusp of the quartic with the same cuspidal tangent, which it touches again at a further point. This tangent counted four times and the inflexional tangents each counted twice make up the thirty-six common tangents of $I = 0$ and the quartic.

III.

Quartics with two double points.

1. If the quartic $T_1T_2T_3T_4+S^2 = 0$ has double points at the points $x = 0$, $z = 0$ and $y = 0$, $z = 0$, it is verified, without difficulty, that any complex formed from the four bitangents $T_1 = 0$, $T_2 = 0$, $T_3 = 0$, $T_4 = 0$ contains a pair of tangents from either double point. If we take the first pair to be $x = 0$, $x+z = 0$ and the second pair to be $y = 0$, $y+z = 0$, we can write the equation of the quartic in the form

$$kxy(x+z)(y+z)+(cz^2+2fyz+2gzx+2hxy)^2 = 0.$$

For a general value of k, so that both double points are ordinary nodes and Plücker's numbers for the quartic are $m = 8$, $i = 12$, $\tau = 8$, we obtain four distinct types of complex.

Type I.—This is formed by taking $x(x+z) = 0$ and $y(y+z) = 0$ as two pairs of a complex. The complex consists of these two pairs, each repeated, and two pairs of bitangents. The four points of contact of the bitangents of either pair lie on a conic with the nodes.

Type II.—This is formed by taking as two pairs $xy = 0$ and $(x+z)(y+z) = 0$. Two of the remaining pairs each consist of a tangent from either node, the remaining two each consist of the line joining the nodes and a bitangent.

We can therefore write the equation of the quartic in either of the following forms

(i) $$kxyz(x+y+z)+(cz^2+2fyz+2gzx+2hxy)^2 = 0,$$

where $x+y+z = 0$ is a bitangent,

(ii) $$kz^2(a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy) + (cz^2+2fyz+2gzx+2hxy)^2 = 0,$$

where $a_1x^2+\ldots \equiv UV = 0$ is the equation of a pair of bitangents.

Type III.—This is formed from the equation (ii) above by taking $z^2 = 0$ and $UV = 0$ as two pairs of the complex. The line-pairs of such a complex consist of $z^2 = 0$ repeated and four pairs of bitangents, so that all the bitangents appear in this complex.

Type IV.—This is obtained from equation (i) above by taking as two pairs of the complex $xz = 0$ and $y(x+y+z) = 0$. Such a complex consists of $xz = 0$ repeated and four other pairs, each consisting of a bitangent and a tangent from the node $y = 0,\ z = 0$.

3. Corresponding to Type I we obtain the following result :—

There are proper class-sextics, which touch the twelve inflexional tangents, four bitangents and a pair of tangents from each node at their points of contact, and have nodes at the nodes of the quartic with the same nodal tangents.

4. When the equation of the quartic is taken to be

$$kz^2(a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy) + (cz^2+2fyz+2gzx+2hxy)^2 = 0$$

or $$kz^2S_1+S_2 = 0,$$

the forms of I and J will be found to be given by the equations

$$12I = (4\Sigma_2-k\Phi)^2-24khlm\Phi_{12},$$

$$216J = (4\Sigma_2-k\Phi)^3-36khlm\Phi_{12}(4\Sigma_2-k\Phi)+216h^2k^2l^2m^2\Sigma_1,$$

where Σ_1 and Σ_2 are the tangential forms corresponding to S_1 and S_2, Φ_{12} is the quadratic contravariant of S_1 and S_2, and Φ is the quadratic contravariant of z^2 and S_1, that is $b_1l^2-2h_1lm+a_1m^2$.

For the complex of Type II, in which we take $zU=0$ and $zV=0$ to be two members of the complex, $2K=4\Sigma_2-k\Phi$.

The class-sextic then degenerates into the two nodes and a class-quartic

$$(4\Sigma_2-k\Phi)\Phi_{12}=18hklm\Sigma_1.$$

This class-quartic touches the nodal tangents but not at the nodes. Since $\Sigma_1=0$ is the square of the equation of the point of intersection of $U=0$, $V=0$, each of which touches $\Phi_{12}=0$, this class-quartic touches $U=0$, $V=0$ at their points of contact with $\Phi_{12}=0$, which are the harmonic conjugates of their point of intersection with the two pairs of points in which they touch the quartic.

We have then the result :—

There are class-quartics which touch the twelve inflexional tangents, two bitangents, and the four nodal tangents, not at the nodes.

5. $I=0$ is a class-quartic touching the twelve inflexional tangents and having nodes at the nodes of the quartic with the same nodal tangents. The thirty-two common tangents of $I=0$ and the quartic are all accounted for.

From $I=0$ and the equation of the class-quartic of (4) we can obtain the equation of another class-quartic touched by the inflexions, viz.

$$3\Sigma_1(4\Sigma_2-k\Phi)=4\Phi_{12}^2.$$

This class-quartic does not touch the nodal tangents. It has each of the bitangents given by $S_1=0$ as a bitangent. The points of contact are not the same.

We have then the result :—

There are class-quartics which touch the twelve inflexional tangents and have as bitangents a pair of bitangents of the quartic, such a pair being such that their points of contact and the two nodes lie on a conic.

6. For a complex of Type III, K is $2\Sigma_2+k\Phi$, so that the class-sextic does not degenerate, and has nodes at the nodes of the quartic with the same nodal tangents.

We thus obtain the following result :—

The twelve inflexional tangents and the eight bitangents touch a class-sextic which has nodes at the nodes of the quartic with the same nodal tangents.

The line joining the nodes is a bitangent of this sextic.

7. For a complex of Type IV we can identify $kxyz(x+y+z)+S^2=0$ with an equation of the form $kz^2(x+y+z)(px+qy+rz)+S'^2=0$ so as to use the values of I and J, obtained from the second form, and we shall find

There are class-quintics which touch the twelve inflexional tangents, four bitangents, and one pair of nodal tangents, not at the node, and have a node at the second node with the same nodal tangents as the quartic and touch a tangent from this node at its point of contact.

8. Since for the quartic with two nodes $m=8$ and $i=12$, it is not impossible at first sight that the inflexional tangents may touch a class-cubic. However, they do not. If twelve of the common tangents of two class-quartics touch a class-cubic, the remaining four must meet at a point. Now, in addition to the class-quartic $I=0$, we have the class-quartic of (4), each of which touches the inflexional tangents and the nodal tangents, and the nodal tangents do not meet at a point, so that the twelve inflexional tangents do not touch a class-cubic.

It is not necessarily true that, if four of the common tangents of two class-quartics meet at a point, then the other twelve touch a class-cubic, unless the four which meet at a point are ordinary distinct tangents, and the twelve inflexional tangents of a quartic with a tac-node, a case which will be briefly noticed presently, do not touch a class-cubic.

9. If $y=0$, $z=0$ is a cusp and $z=0$, $x=0$ is a node, then k has the value $16g^2-16gh$. Plücker's numbers for a quartic with a node and a cusp are $m=7$, $i=10$, $\tau=4$.

There are five types of complex, three of which correspond to Types I, II, and III for the binodal quartic, with modifications which may be anticipated by comparing the cases of the quartics with one node and

with one cusp respectively. Corresponding to Type IV for the binodal quartic there are two distinct types, according as the tangent associated with the bitangent is drawn from the cusp or from the node. Three tangents can be drawn from the node, distinct from the nodal tangents, and four from the cusp, distinct from the cuspidal tangent.

Commencing with the equation of the quartic in the form

$$(16g^2-16gh)xy(x+z)(y+z)+(cz^2+2fyz+2gzx+2hxy)^2 = 0,$$

we have

Type I.—This consists of $x(x+z) = 0$ taken twice, $y(y+z) = 0$ taken three times, and a pair of bitangents.

Type II.—This consists of $xy = 0$, $(x+z)(y+z) = 0$, $z = 0$ with a bitangent, $z = 0$ with another tangent from the cusp, this pair being repeated, and the remaining tangent from the node associated with the remaining tangent from the cusp.

We can write the equation of the quartic in either of the forms

(i) $$-16ghxyz(x+y+z)+(cz^2+2fyz+2gzx+2hxy)^2 = 0,$$

where $x+y+z = 0$ is a bitangent.

(ii) $$kz^2S_1+S_2^2 = 0,$$

where $S_1 \equiv b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy = 0$ consists of a tangent from the node and a bitangent, and S_2 is of the form

$$cz^2+2fyz+2gzx+2hxy.$$

From the form (ii) we obtain :—

Type III.—This consists of $z^2 = 0$ repeated, $S_1 = 0$, and three other pairs each consisting of a tangent from the cusp and a bitangent.

From the form (i) we obtain :—

Type IV.—This consists of $xz = 0$ repeated, $y(x+y+z) = 0$, a pair of tangents from the cusp, repeated, and the remaining tangent from the cusp with a bitangent.

Type V.—This consists of $yz = 0$ taken three times, $x(x+y+z) = 0$, and two other pairs, each consisting of another tangent from the node with a bitangent.

10. From Type I we obtain the result :—

There are class-sextics which touch the ten inflexional tangents and two bitangents, touch the quartic at the points of contact of two tangents from the node and have three-point contact with the quartic at the points of contact of a pair of tangents from the cusp, have a node at the node of the quartic with the same nodal tangents, and a cusp at the cusp of the quartic with the same cuspidal tangent, which touches the class-sextic again at a further point.

11. Corresponding to Type II we have as in (4) the class-quartic

$$(4\Sigma_2 - k\Phi)\Phi_{12} = 18hklm\Sigma_1.$$

This will be found to be of the form

$$(fl - gm)^3(f_1 l - g_1 m) + lU_3 = 0.$$

This class-quartic then touches the tangent from the cusp which is a factor of $S_1 = 0$, and has a cusp at the cusp of the quartic with the same cuspidal tangent. It will be found also to touch the nodal tangents. We have then :—

There are class-quartics which touch the ten inflexional tangents, one bitangent, the nodal tangents, not at the node, and one tangent from the cusp, and have a cusp at the cusp of the quartic with the same cuspidal tangent.

The class-quartic $4\Phi_{12}^2 = 3(4\Sigma_2 - k\Phi)\Sigma_1$ touches the cuspidal tangent at the cusp, but has not a cusp there. It has the tangent from the cusp and the bitangent, which are given by the equation $S_1 = 0$ as bitangents.

12. For a complex of Type III it will be found that, owing to the absence of the term x^2 from the form S_1, the class-sextic of 6 degenerates into the cusp and a class-quintic, and we have :—

There is a class-quintic which touches the ten inflexional tangents and the four bitangents, and has a node at the node of the quartic and a cusp at the cusp of the quartic, the nodal and cuspidal tangents being the same.

13. From Type IV we obtain the result :—

There are class-quintics which touch the ten inflexional tangents and two bitangents, a pair of tangents from the cusp, and a tangent from the node at its point of contact, and have a node at the node of the quartic and a cusp at the cusp of the quartic with the same nodal and cuspidal tangents.

14. From Type V we obtain the result :—

There are class-quintics which touch the ten inflexional tangents, three bitangents, and the nodal tangents, not at the node, have three-point contact with the quartic at the point of contact of a tangent from the cusp, and have a cusp at the cusp of the quartic with the same cuspidal tangent, which touches the class-quintic again at a further point.

15. For the quartic with two cusps, Plücker's numbers are $m = 6$, $i = 8$, $\tau = 1$.

If the cusps are $x = 0$, $z = 0$ and $y = 0$, $z = 0$, the cuspidal tangents being $x = 0$, $y = 0$, and the equation of the bitangent is $x+y+z = 0$, the equation of the quartic may be written in the form

$$kz^3(x+y+z)+(cz^2+2hxy)^2 = 0.$$

Since this equation is unaltered by interchanging x and y, there is symmetry, and many results for this curve follow from symmetry alone.

Another form of the equation is

$$kz^2xy+(cz^2+2fyz+2fzx+2hxy)^2 = 0,$$

where $x = 0$, $y = 0$ are tangents from the cusps instead of the cuspidal tangents.

There are four types of complex, as in the case of two nodes, but except for the fourth type the results obtained directly from the complex are obvious from symmetry. There are, however, results not entirely obvious from symmetry obtained by combining the equations derived from the complex with $I = 0$.

The only results that appear to me to be worth recording are the following :—

(i) *There are class-quintics which touch the eight inflexional tangents, the bitangent, and two tangents from one cusp, have three-point contact*

with the quartic at the point of contact of a tangent from the second cusp, and have cusps at the cusps of the quartic with the same cuspidal tangents, that at the second cusp touching the class-quintic again at a further point.

(ii) *There are class-quartics which touch the eight inflexional tangents and touch the cuspidal tangents at the cusps, without having cusps there, and have a tangent from each cusp as a bitangent.*

When the equation of the quartic is taken to be $kz^2xy+S^2=0$, S being of the form $cz^2+2fyz+2fzx+2hxy$, the equation of such a class-quartic is

$$3n^2(4\Sigma+klm)+4\Phi^2=0,$$

Σ being the tangential form corresponding to S, and Φ being the quadratic contravariant of S and xy.

(iii) *There is a class-quartic which touches the cuspidal tangents at the cusps, without having cusps there, touches the eight inflexional tangents, and has the line joining the cusps and the bitangent as bitangents. It passes through the points of contact of the bitangent with the quartic, but these are not its points of contact with the bitangent.*

When the equation of the quartic is taken to be

$$kz^3(x+y+z)+(cz^2+2hxy)^2=0,$$

the equation of this class-quartic is

$$4h(mn+nl-2lm)^2=3(hn^2+2clm)(l-m)^2.$$

Since this equation does not involve k, this class-quartic is the envelope of the inflexional tangents of quartics which have two cusps with the corresponding cuspidal tangents given, and also have a given bitangent with given points of contact.

I have discussed this class-quartic in a communication made to the Society at the meeting of December 1922. If $c=0$, that is, if the cuspidal tangents pass through the points of contact of the bitangents, it breaks up into two conics. In this case the eight points of inflexion lie on two straight lines.

It is obvious from symmetry that the inflexional tangents and the cuspidal tangents touch a class-cubic. The equation of this class-cubic is not given directly by any complex, but when the first form of the equation of the quartic is used, its equation can be found without difficulty to be

$$(32ch+24k)lm(l+m)+32chlmn-3kn(l^2+12lm+m^2)$$
$$+16ch(l+m)n^2+16h^2n^3=0.$$

16. For a quartic with a tac-node Plücker's numbers are the same as for a quartic with two nodes, namely $m=8,\ i=12,\ \tau=8$; but of the eight double-tangents, two coincide with the tangent at the tac-node and there are only six proper bitangents.

If $x=0,\ y=0$ is the tac-node, the tangent there being $y=0$, we can write the equation of the quartic in either of the forms

$$k(x+\alpha y)(x+\beta y)(x+\gamma y)(x+\delta y)=(x^2+2yz)^2$$

or $$ky^2(a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy)$$
$$+(ax^2+by^2+2fyz+2hxy)^2=0$$

or $ky^2S_1+S_2^2=0$, where $S_1\equiv UV=0$ is the equation of a pair of bitangents.

From the second form we obtain

$$12I=(4\Sigma_2-k\Phi)^2+12akn^2\Phi_{12},$$

$$216J=(4\Sigma_2-k\Phi)^3+18akn^2(4\Sigma_2-k\Phi)\Phi_{12}+54a^2k^2n^4\Sigma_1,$$

where Σ_1 and Σ_2 are the tangential forms corresponding to S_1 and S_2, Φ_{12} is the quadratic contravariant of S_1 and S_2, and Φ is the quadratic contravariant of y^2 and S_1, that is $c_1l^2-2g_1ln+a_1n^2$.

The following is a summary of the results obtained :—

(i) *There are class-quartics which touch the twelve inflexional tangents and two bitangents, and the tac-nodal tangent at the tac-node, having a singularity at the tac-node, which is the reciprocal of a point of undulation.*

The equation of such a quartic is

$$(4\Sigma_2-k\Phi)\Phi_{12}+9akn^2\Sigma_1=0.$$

(ii) *There are class-quartics which touch the twelve inflexional tangents and have a pair of bitangents as bitangents, the bitangents being such that the conic through their points of contact and the tac-node touches the tac-nodal tangent.*

The equation of such a quartic is

$$4\Psi_{12}^2 = 3\Sigma_1(4\Sigma_2 - k\Phi)$$

and is obtained by combining the equation of the class-quartic (i) with $I = 0$.

(iii) *There is a class-sextic which touches the twelve inflexional tangents and the six proper bitangents, and has a triple point at the tac-node with three distinct branches, each touching the tac-nodal tangent.*

(iv) *There are class-quintics which touch the twelve inflexional tangents, three bitangents, and a tangent from the tac-node at its point of contact, and have a tac-node at the tac-node of the quartic with the same tac-nodal tangent.*

IV.

Quartics with three double points.

1. The only case that I propose to discuss in any detail is the quartic with three nodes.

Plücker's numbers are $m = 6$, $i = 6$, $\tau = 4$.

The standard equation of such a quartic, when the nodes are taken at the vertices of the triangle of reference, is

$$U \equiv ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz(fx + gy + hz) = 0.$$

If $lx + my + nz = 0$ is an inflexional tangent, then the conics

$$lyz + mzx + nxy = 0 \quad \text{and} \quad ax^2 + bz^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

have three-point contact, so that

$$\Theta^2 - 3\Delta\Theta' = 0, \quad \Theta\Theta' - 9\Delta\Delta = 0, \quad \Theta'^2 = 3\Delta'\Theta,$$

where $\Delta \equiv abc+2fgh-af^2-bg^2-ch^2,$

$\Theta \equiv 2(Fl+Gm+Hn),$

$\Theta' \equiv -(al^2+bm^2+cn^2-2fmn-2gnl-2hlm),$

$\Delta' = 2lmn.$

The tangential equation of the quartic is

$$4(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta) = (\Theta\Theta'-9\Delta\Delta')^2.$$

$\Theta' = 0$ is the equation of the conic which touches the nodal tangents.

$\Theta^2-4\Delta\Theta' = 0$ is the equation of the conic which touches the three pairs of tangents drawn from the nodes.

The six inflexional tangents touch the conic $\Theta^2-3\Delta\Theta' = 0$, a well known result.

These three conics have double contact at the same two points, the pole of the chord of contact being the point F, G, H.

Evidently many properties of the inflexional tangents may be obtained without the use of the complex method, which, however, gives some properties involving the bitangents as well as the inflexional tangents.

2. Results which may be obtained without the use of the complex method are the following :—

(i) *The six inflexional tangents touch the conic* $\Theta^2 = 3\Delta\Theta'$.

(ii) *The six inflexional tangents, the six nodal tangents, and the straight lines joining* F, G, H *to the nodes touch the class-cubic*

$$\Theta\Theta' = 9\Delta\Delta'.$$

(iii) *The six inflexional tangents, the three pairs of tangents from the nodes, and the straight lines joining the nodes to the point* F, G, H *touch the class-cubic* $\Theta^3-4\Delta\Theta\Theta'+9\Delta^2\Delta' = 0.$

This equation may be written in the form

$$\Theta(\Theta^2-3\Delta\Theta')-\Delta(\Theta\Theta'-9\Delta\Delta') = 0.$$

(iv) *The six inflexional tangents, the six nodal tangents, and the three pairs of tangents from the nodes touch the class-quartic*

$$\Theta'(\Theta^2-4\Delta\Theta')+3\Delta\Delta'\Theta = 0,$$

which has double contact with the conics touching the six inflexional tangents and the six nodal tangents, at the points where they have double contact.

This equation may be written in the form

$$\Theta'(\Theta^2-3\Delta\Theta') = \Delta(\Theta'^2-3\Delta'\Theta).$$

3. There are five types of complex.

Type I.—We write the equation of the quartic in the form

$$2xyz[\{f-\sqrt{(bc)}\}x+\{g-\sqrt{(ca)}\}y+\{h-\sqrt{(ab)}\}z] + \{\sqrt{(ayz)}+\sqrt{(bzx)}+\sqrt{(cxy)}\}^2 = 0,$$

and take as two pairs $xy = 0$ and $z[\{f-\sqrt{(bc)}\}x+\ldots] = 0$.

The other four pairs are $xy = 0$ again,

$$z[\{f+\sqrt{(bc)}\}x+\{g+\sqrt{(ca)}\}y+\{h-\sqrt{(ab)}\} = 0,$$

a tangent from the node A associated with a tangent from the node B, the other tangent from A with the other tangent from B.

Type II.—We write the equation of the quartic in the form

$$-(z^2/c)[\{f-\sqrt{(bc)}\}x+\{g-\sqrt{(ca)}\}y + \{(h-\sqrt{(ab)}\}z][\{f+\sqrt{(bc)}\}x+\{g+\sqrt{(ca)}\}y+\{h-\sqrt{(ab)}\}z] + (1/c)[z^2\{h-\sqrt{(ab)}\}+gyz+fzx+cxy]^2 = 0,$$

and take $z^2 = 0$ as one pair and the two tangents as another pair. The remaining pairs are the tangents from the node C taken twice, the other pair of double tangents and $z^2 = 0$ again.

Type III.—We write the equation of the quartic in the form

$$(1/c)z^2(Ax^2-2Hxy+By^2)+(1/c)(gyz+fzx+cxy)^2 = 0,$$

and take as two pairs $z = 0$ with one of the straight lines,

$$Ax^2-2Hxy+By^2 = 0$$

and $z = 0$ with the other straight line. The remaining pairs consist each of a line joining two nodes together with a tangent from the third node.

Type IV.—We write the equation of the quartic in the form

$$\Delta^{-1}(Cz^2-2Gzx+Ax^2)(Cz^2-2Fyz+By^2) - \Delta^{-1}(Cz^2-Gzx-Fyz+Hxy)^2 = 0,$$

and take $Cz^2-2Gzx+Ax^2=0$ and $Cz^2-2Fyz+By^2=0$ as two pairs. The other pairs are these pairs again with the pair of tangents from the remaining node, taken twice.

Type V.—We can write the equation in the form

$$kxy(x+z)(y+z)+(pyz+qzx+rxy)^2=0,$$

where $x+z=0$ is a tangent from the node B and $y+z=0$ is a tangent from the node A.

If we take $x(x+z)=0$ and $y(y+z)=0$ as two pairs, the remaining pairs of the complex are these pairs again, with two more pairs, each consisting of a tangent from C with a bitangent.

4. From the form of the equation of the quartic used to obtain Type III we easily find that

$$12I=\Theta'^2-3\Delta'\Theta,$$

$$432J=2\Theta'^3-9\Delta'\Theta\Theta'+27\Delta'^2\Delta=2\Theta'(\Theta'^2-3\Delta'\Theta)-3\Delta'(\Theta\Theta'-9\Delta\Delta').$$

Any equation of a class-sextic of the form $J+U_2I=0$, derived from a complex, can be written in the form

$$\Delta'(\Theta\Theta'-9\Delta\Delta')+V_2I=0,$$

and, by combining this with the tangential equation of the quartic, namely

$$4(\Theta^2-3\Delta\Theta')I=(\Theta\Theta'-9\Delta\Delta'),$$

we can obtain another curve touching the inflexional tangents and the lines of the complex tangents, namely

$$4\Delta'(\Theta^2-3\Delta\Theta')+V_2(\Theta\Theta'-9\Delta\Delta')=0.$$

The class-sextic corresponding to Type I, for which $4K$ is

$$\Theta'-6\{h-\sqrt{(ab)}\}lm,$$

degenerates into the nodes A, B and the class-quartic

$$n(\Theta\Theta'-9\Delta\Delta')=2\{h+\sqrt{(ab)}\}(\Theta'^2-3\Delta'\Theta).$$

This class-quartic has a node at C, with the same nodal tangents as the quartic, and touches the nodal tangents from A and B, but has not nodes there.

We have then the following result :—

There are class-quartics which touch the six inflexional tangents, two bitangents, two pairs of nodal tangents, and have a node at the third node with the same nodal tangents as the quartic.

By combining the equation of this class-quartic with that of the quartic, we obtain the equation of a class-cubic

$$2n(\Theta^2-3\Delta\Theta') = \{h-\sqrt{(ab)}\}(\Theta\Theta'-9\Delta\Delta'),$$

and obtain the result :—

There are class-cubics which touch the six inflexional tangents, two bitangents, a pair of nodal tangents, and the straight line joining the node to the point F, G, H.

Corresponding to the class-quartic of III. 5, we obtain from this type of complex the following result :—

There are six class-quartics, each of which touches the six inflexional tangents, has a node at a node of the quartic with the same nodal tangents, and has as bitangents a pair of bitangents of the quartic.

The equation of one of these class-quartics is

$$\{h-\sqrt{(ab)}\}^2(\Theta'^2-3\Delta'\Theta)-\{h-\sqrt{(ab)}\}n(\Theta\Theta'-9\Delta\Delta')+n^2(\Theta^2-3\Delta\Theta') = 0.$$

5. For the complex of Type II, the form $2K$ is

$$\Theta'+3c^{-1}(Bl^2+2Hlm+Am^2).$$

The class-sextic does not degenerate. Its equation is

$$c\Delta'(\Theta\Theta'-9\Delta\Delta')+2(\Theta'^2-3\Delta'\Theta)(Bl^2+2Hlm+Am^2) = 0.$$

The class-sextic evidently has the line joining the nodes A and B as a bitangent, and it can be easily verified that it touches the tangents from C at their points of contact. The points of contact with the bitangent $z = 0$ are given by $Bl^2+Am^2 = 0$.

We have then the following result :—

There are class-sextics which touch the six inflexional tangents and the four bitangents, have nodes at the nodes of the quartic with the same nodal tangents, have the line joining two of the nodes as a bitangent, and touch the tangents from the remaining node at their points of contact.

By combining the equation of this class-sextic with that of the quartic, we obtain a class-quintic

$$2c\Delta'(\Theta^2-3\Delta\Theta')+(\Theta\Theta'-9\Delta\Delta')(Bl^2+2Hlm+Am^2)=0.$$

We have then :—

There are class-quintics which touch the six inflexional tangents, the four bitangents, the six nodal tangents and a pair of tangents from a node at their points of contact, and have the line joining the other two nodes as a bitangent. They also touch the lines joining the point F, G, H to the nodes.

6. The sextic corresponding to Type III degenerates into the three nodes and the class-cubic $\Theta\Theta'-9\Delta\Delta'=0$, which has been discussed already in (2).

The class-quartic for this type, which corresponds to the class-quartic of III, 5, degenerates into a node, taken twice over, together with the conic which touches the six inflexional tangents.

7. Corresponding to Type IV, the form $2K$ is $\Theta'-3\Theta^2/8\Delta$. The class-sextic does not degenerate, and its equation is

$$(\Theta^2-4\Delta\Theta')(\Theta'^2-3\Delta'\Theta)+2\Delta'\Delta(\Theta\Theta'-9\Delta\Delta')=0.$$

Whence we have :—

There is a class-sextic which touches the six inflexional tangents, the six tangents from the node at their points of contact, has nodes at the nodes of the quartic with the same nodal tangents, and touches the common tangents of the cubic $\Theta\Theta'=9\Delta\Delta'$ *and the conic* $\Theta^2=4\Delta\Theta'$, *which touches the three pairs of tangents drawn from the nodes.*

By combining the equation of this class-sextic with that of the quartic, we obtain the class-quintic

$$2(\Theta^2-4\Delta\Theta')(\Theta\Theta'-9\Delta\Delta')+\Delta\Delta'(\Theta^2-3\Delta\Theta')=0,$$

whence we have :—

There is a class-quintic which touches the six inflexional tangents, the six nodal tangents, the three pairs of tangents from the nodes at their points of contact, and the straight lines joining the nodes to the

point F, G, H; and has double contact with the three conics, which touch respectively the inflexional tangents, the nodal tangents, and the pairs of tangents from the nodes, at the points where these three conics have double contact.

8. For the complex of Type V the equations become very complicated with surds, when the standard form of the equation of the quartic is used. The results, however, are obtained without difficulty when we use the form

$$kxy(x+z)(y+z)+(pyz+qzx+rxy)^2 = 0.$$

The class-sextic corresponding to this complex degenerates into the node C and a class-quintic, whose equation is

$$lm(\Theta\Theta'-9\Delta\Delta')+2k(\Theta'^2-3\Delta'\Theta)(n-m-l) = 0,$$

where Θ, Θ', Δ, Δ' are calculated from the form written above.

We have then :—

There are class-quintics which touch the six inflexional tangents and two bitangents, have nodes at two of the nodes of the quartic with the same nodal tangents as the quartic, touch a tangent from each of these nodes at its point of contact, and touch the tangents from the remaining node.

By combining the equation of this class-quintic with that of the quartic, we obtain the equation of a class-quartic

$$2lm(\Theta^2-3\Delta\Theta')+k(\Theta\Theta'-9\Delta\Delta')(n-m-l) = 0.$$

We have then :—

There are class-quartics which touch the six inflexional tangents, two bitangents, the nodal tangents at two of the nodes, a tangent from each of these nodes at its point of contact, and the lines joining the point F, G, H to these two nodes.

8. For a quartic with two nodes and a cusp, Plücker's numbers are $m = 5$, $i = 4$, $\tau = 2$.

The standard equation can be taken to be that of the trinodal quartic with the condition $h = \sqrt{(ab)}$.

We can draw one tangent to the quartic from each node, their equations being $By-2Fx=0$ and $Ax-2Gz=0$.

We can draw two tangents from the cusp, their equations being

$$\{f\pm\sqrt{(bc)}\}x+\{g\pm\sqrt{(ca)}\}y=0.$$

$\Theta = 2(g\sqrt{b}-f\sqrt{a})(l\sqrt{a}-m\sqrt{b})+2Hn,$

$\Theta' = -(l\sqrt{a}-m\sqrt{b})^2-cn^2+2fmn+2gnl,$

$\Delta' = -(g\sqrt{b}-f\sqrt{a})^2,$

$\Theta=0$, the point F, G, H, lies on the cuspidal tangent,

$\Theta'=0$ touches the nodal tangents and the cuspidal tangent at the cusp.

$\Theta^2-4\Delta\Theta'=0$ breaks up into the cusp and the point of intersection of the tangents from the nodes.

The results given in (2) become in this case :—

(i) *A conic can be drawn to touch the four inflexional tangents and to touch the cuspidal tangent at the cusp.*

Its equation is $\Theta^2-3\Delta\Theta'=0$.

Since we may write this in the form $\Theta^2+3(\Theta^2-4\Delta\Theta')=0$, we see that this conic passes through the point of intersection of the nodal tangents, the tangent there being the straight line joining this point to the point F, G, H.

(ii) *There is a class-cubic which touches the four inflexional tangents and the two pairs of nodal tangents, has a cusp at the cusp of the quartic with the same cuspidal tangent as the quartic, and touches also the lines joining the nodes to the point F, G, H.*

Its equation is $\Theta\Theta'=9\Delta\Delta'$.

(iii) *The four inflexional tangents, the tangents from the nodes, and the lines joining the nodes to the point F, G, H touch a conic.*

For the class-cubic $\Theta^3-4\Delta\Theta\Theta'+9\Delta^2\Delta'=0$ now breaks up into the cusp and a conic.

We can find the equation of the inflexional tangents as the common tangents of this conic and the conic of (i).

A single conic cannot be drawn to touch the inflexional tangents and the tangents from the cusp, and it is clear from Plücker's numbers that a single conic cannot be drawn to touch the four inflexional tangents and the two bitangents.

(iv) *A class-cubic can be drawn to touch the four inflexional tangents, the four nodal tangents, the cuspidal tangent, and the tangents from the nodes.*

For the class-quartic $\Theta'(\Theta^2-4\Delta\Theta')+3\Delta\Delta'\Theta=0$ breaks up into the cusp and a class-cubic.

9. Some of the results obtained by complex method are of little interest, since the curves obtained do not obviously satisfy more conditions than are required to specify a curve of their class. The results generally correspond to those found for the trinodal quartic, a single result for the trinodal quartic sometimes giving two results according as we are dealing with two nodes or one node and a cusp. I give only the following.

Corresponding to (4) we have :—

There are class-quartics which touch the four inflexional tangents, one bitangent and a tangent from the cusp, have a cusp at the cusp with the same cuspidal tangent, and a node at one of the nodes with the same nodal tangents, and touch also the remaining pair of nodal tangents.

The equation of such a class-quartic is

$$2\{\sqrt{(bc)}-f\}(\Theta'^2-3\Delta'\Theta)+l(\Theta\Theta'-9\Delta\Delta')=0.$$

Also :—

There are class-cubics which touch the four inflexional tangents, one bitangent, a tangent from the cusp, the cuspidal tangent at the cusp, a pair of nodal tangents, and the line joining this node to the point F, G, H.

One equation of such a cubic is

$$\{\sqrt{(bc)}-f\}(\Theta\Theta'-9\Delta\Delta')+l(\Theta^2-3\Delta\Theta')=0.$$

These cubics do not have cusps at the cusp of the quartic.

Corresponding to the last part of (5), we have the result :—

There is a class-quintic which touches the four inflexional tangents, the two bitangents, the two pairs of nodal tangents, has a cusp at the cusp of the quartic with the same cuspidal tangent, has three-point contact with the quartic at each of the points where it is touched by the tangents from the cusp, and has the line joining the nodes as a bitangent.

The equation of this class-quintic is

$$2c\Delta'(\Theta^2-3\Delta\Theta')+(\Theta\Theta'-9\Delta\Delta')(Bl^2+2Hlm+Am^2) = 0.$$

We have also :—

There are two class-quintics which touch the four inflexional tangents, the two bitangents, have a cusp at the cusp of the quartic with the same cuspidal tangent, have nodes at the nodes of the quartic with the same nodal tangents, touch the tangent from one node at its point of contact, and touch at the cusp the line joining the other node to the cusp.

The equation of one of these quintics is

$$alm(\Theta\Theta'-9\Delta\Delta')+2(\Theta'^2-3\Delta'\Theta)(Bn+2Fm) = 0,$$

and in addition :—

There are two class-quartics which touch the four inflexional tangents, the two bitangents, the four nodal tangents, the cuspidal tangent at the cusp, the tangent from one node at its point of contact, the line joining the other node to the cusp, and the lines joining F, G, H to the nodes.

The equation of one of these quartics is

$$2alm(\Theta^2-3\Delta\Theta')+(\Theta\Theta'-9\Delta\Delta')(Bn+2Fm) = 0.$$

Corresponding to (7), we obtain :—

There is a class-quintic which touches the four inflexional tangents, the tangents from the cusp, the tangents from the nodes at their points of contact, has nodes at the nodes with the same nodal tangents, and a cusp at the cusp with the same cuspidal tangent.

The equation of this class-quintic is

$$(2BGl+2AFm+ABn)(\Theta'^2-3\Delta'\Theta)+\Delta lm(\Theta\Theta'-9\Delta\Delta') = 0.$$

We have also :—

There is a class-quartic which touches the four inflexional tangents, the four nodal tangents, the cuspidal tangent at the cusp, the tangents from the nodes at their points of contact, and the lines joining the point F, G, H to the nodes. It also passes through the point of intersection of the nodal tangents, touching there the line joining this point to the point F, G, H.

The equation of this class-quartic is

$$(2BGl+2AFm+ABn)(\Theta\Theta'-9\Delta\Delta')+4\Delta lm(\Theta^2-3\Delta\Theta') = 0$$

or
$$(2BGl+2AFm+ABn)(\Theta\Theta'-3\Delta\Delta')+\Delta lm\Theta^2 = 0.$$

10. For the quartic with two cusps and a node the results generally are obvious from symmetry, or lead to curves to which more obvious conditions cannot be assigned than are sufficient to determine curves of their class. Plücker's numbers for this curve are $m = 4$, $i = 2$, $\tau = 1$, and I mention one result only, which it may be noticed is unsymmetrical :—

There are two class-quartics, each of which touches the two inflexional tangents, the bitangent, the nodal tangents, one cuspidal tangent at its cusp, has a cusp at the other cusp with the same cuspidal tangent, has three-point contact with the quartic at the point of contact of the tangent drawn from this cusp, touches the line joining the node to the other cusp, the line joining the cusps, and the line joining the node to the point of intersection of the cuspidal tangents.

11. The quartic with three cusps has no inflexions, and any results that can be obtained by the complex method are trivial.

A CONTRIBUTION TO THE THEORY OF FOURIER TRANSFORMS

By E. C. TITCHMARSH.

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1. If two functions $f(x)$, $F(x)$ are connected by the relation

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xt\, F(t)\, dt, \tag{1.1}$$

then $f(x)$ is said to be the Fourier cosine transform of $F(x)$. Since by Fourier's integral theorem

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos xu\, du \int_0^\infty \cos ut f(t)\, dt, \tag{1.2}$$

we have the reciprocal formula

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xt f(t)\, dt. \tag{1.3}$$

Thus $F(x)$ is the transform of $f(x)$.

The representation of a function $f(x)$ in the above way has been studied for two particular classes of functions. The first kind are those which are integrable* over the range $(0, \infty)$. These are the functions for which the formula (1.2) is ordinarily proved. The transform of such a function is clearly continuous, and tends to zero at infinity. Even when (1.3) also holds, the apparent reciprocity expressed by (1.1), (1.3) is by no means complete, and the properties of $f(x)$ and $F(x)$ may be very different.

The other class of functions, which have been the real subject of the theory of transforms, are those whose squares are integrable over $(0, \infty)$. The theory of transforms in this case is due to Plancherel.†

* *I.e.*, in the sense of Lebesgue, and so absolutely.

† M. Plancherel, *Rend. di Palermo*, 30 (1910), 289–335; *Math. Annalen*, 76 (1915), 315–326. See also E. C. Titchmarsh, *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473.

It may be observed that, in the former case, the definition of the transform of $f(x)$ was effected immediately by the formula (1.3). In the present case it is not so easy, because the integral in (1.3) may be divergent. It is in fact effected by the formula

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xt}{t} f(t)\, dt. \tag{1.4}$$

This has been shown to have a meaning for almost all values of x. The same is true of the reciprocal formula

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xt}{t} F(t)\, dt. \tag{1.5}$$

These are the generalized forms of (1.3), (1.1), to which they reduce if differentiation under the integral sign is permissible. Finally, it can be proved that the square of $F(x)$ is also integrable over $(0, \infty)$. There is thus a complete reciprocity between the two transforms.

2. The object of the present paper is to define Fourier transforms for other classes of functions, and to deduce some of the properties of the transform from those of the original function. Before stating the precise results, it may be well to point out what are the corresponding theorems in the theory of Fourier series. Here, instead of two functions defined over the range $(0, \infty)$, we have a periodic function $f(x)$, and corresponding to it the sequence of its Fourier coefficients a_n, b_n. We have the two following theorems :—

THE GENERALIZED RIESZ-FISCHER THEOREM.—*If the series*

$$\sum_{n=1}^{\infty} (a_n^2+b_n^2)^{\frac{1}{2}p} \quad (1 < p \leqslant 2) \tag{2.1}$$

is convergent, then the numbers a_n, b_n are the Fourier coefficients of a function whose $p/(p-1)$-th power is integrable.

THE GENERALIZED PARSEVAL THEOREM.—*If the p-th power of $f(x)$ is integrable $(1 < p \leqslant 2)$, then the series*

$$\sum_{n=1}^{\infty} (a_n^2+b_n^2)^{\frac{1}{2}(p/p-1)} \tag{2.2}$$

is convergent.

The former theorem was proved by Riesz* and Fischer† in the case $p=2$. It was extended by Young‡ to values of p such that $p/(p-1)$ is an even integer. The latter theorem was proved by Young§ for values of p such that $p/(p-1)$ is an integer. Finally, both theorems were proved by Hausdorff‖ for all values of p in the range stated.

In the present paper I prove the following analogous theorems in the theory of transforms :—

ANALOGUE OF THE RIESZ-FISCHER THEOREM.—*If the function* $f(x)$ *has its* p*-th power* $(1<p\leqslant 2)$ *integrable over* $(0, \infty)$, *then there is a function* $F(x)$ *whose* $p/(p-1)$*-th power is integrable over* $(0, \infty)$, *such that*

$$f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xt}{t}F(t)\,dt. \tag{2·3}$$

ANALOGUE OF THE PARSEVAL THEOREM.—*If the function* $f(x)$ *has its* p-th power $(1<p\leqslant 2)$ *integrable over* $(0, \infty)$, *then the function*

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xt}{t}f(t)\,dt \tag{2·4}$$

has its $p/(p-1)$*-th power integrable over* $(0, \infty)$.

The function $F(x)$ is, as will appear in the proof, the same in the two theorems, and is called the Fourier cosine transform, or simply the transform, of $f(x)$; $f(x)$ may also be called the transform of $F(x)$, but the reciprocity between the two functions is by no means complete, except in the case $p=2$. It is shown by means of an example that the theorems are false if $p>2$. Thus if $1<p\leqslant 2$, a function whose p-th power is integrable is necessarily the transform of a function whose $p/(p-1)$-th power is integrable; but if $p>2$ this may or may not be the case.

It will be seen that the two theorems appear almost together as results of the same argument; there is, of course, not the formal

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* F. Riesz, *Göttinger Nachrichten*, March 1907.

† E. Fischer, *Comptes Rendus*, 144 (1907), 1022–1024. For an account of these and other proofs in the case $p=2$ see W. H. and G. C. Young, *Quart. J. of Math.*, 44 (1913), 49–88.

‡ W. H. Young, *Proc. London Math. Soc.* (2), 12 (1913), 71–88.

§ W. H. Young, *Comptes Rendus*, 155 (1912), 30–33; *Proc. Royal Soc.* (A), 87 (1912), 331–339.

‖ F. Hausdorff, *Math. Zeitschrift*, 16 (1923), 163–169.

asymmetry between a pair of transforms that there is between the function and its Fourier coefficients in the series theorems.

One difference between the Riesz-Fischer theorem and its analogue may be noticed. Suppose that the Riesz-Fischer theorem has been proved for $p=2$. Then in proving it for $p<2$ we are assured of the first part of the result, that the numbers a_n, b_n are the Fourier coefficients of a function whose square at least is integrable; for if the series (2.1) is convergent for $p<2$, it is also convergent for $p=2$. There is nothing corresponding to this in the case of integrals, since a function may have a particular power integrable over $(0, \infty)$ but no higher or lower power. Thus each case of the analogue of the Riesz-Fischer theorem stands completely by itself. It is shown by means of an example that, however small ϵ is, we may have the p-th power of $f(x)$ integrable, and the integrals

$$\int_0^\infty |F(x)|^{p/(p+1)-\epsilon}\,dx, \qquad \int_0^\infty |F(x)|^{p/(p-1)+\epsilon}\,dx$$

both divergent.

The whole theory applies equally well to Fourier sine transforms, the cosine in (1.1) being replaced by the sine. The general formula for a sine transform is

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{1-\cos xt}{t}f(t)\,dt.$$

3. The starting point of our argument is an inequality proved by Hausdorff in the paper referred to, as an extension of the earlier work of Young. Let $f(t)$ be a trigonometrical polynomial

$$f(t)=\Sigma a_k e^{ikt}, \tag{3.1}$$

only a finite number of the coefficients being not zero. Then it is proved by Hausdorff that

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{p/(p-1)}\,dt\right)^{(p-1)/p}\leqslant(\Sigma|a_k|^p)^{1/p} \quad (1<p<2). \tag{3.2}$$

Let $a_k=0$ ($|k|\leqslant m$ or $|k|>n$), and elsewhere $a_k=a_{-k}$. Then Hausdorff's inequality becomes

$$\frac{1}{2\pi}\int_0^{2\pi}\left|2\sum_{k=m+1}^{n}a_k\cos kt\right|^{p/(p-1)}dt\leqslant\left(2\sum_{k=m+1}^{n}|a_k|^p\right)^{1/(p-1)},$$

i.e.

$$\int_0^\pi\left|\sum_{k=m+1}^{n}a_k\cos kt\right|^{p/(p-1)}dt\leqslant\tfrac{1}{2}\pi\left(\sum_{k=m+1}^{n}|a_k|^p\right)^{1/(p-1)}. \tag{3.3}$$

4. *The existence of transforms.*—We now follow a procedure used in my paper already referred to. Let $f(x)$ be a function defined for all positive values of x, and such that the integral

$$\int_0^\infty |f(x)|^p\,dx \tag{4.1}$$

is convergent for a given value of p between 1 and 2. Let

$$h_n = \int_{n/\lambda}^{(n+1)/\lambda} f(x)\,dx. \tag{4.2}$$

Then, by the generalized inequality of Schwarz,

$$|h_n| \leqslant \left[\int_{n/\lambda}^{(n+1)/\lambda} |f(x)|^p\,dx\right]^{1/p}\left[\int_{n/\lambda}^{(n+1)/\lambda} dx\right]^{(p-1)/p},$$

and so

$$|h_n|^p \leqslant \lambda^{1-p}\int_{n/\lambda}^{(n+1)/\lambda} |f(x)|^p\,dx. \tag{4.21}$$

Let

$$\Phi_n = \sum_{k=1}^{n} h_k \cos(kx/\lambda). \tag{4.3}$$

Let a and b be constants $(0 < a < b)$, and let m and n be defined as functions of λ by the inequalities

$$\frac{m}{\lambda} < a \leqslant \frac{m+1}{\lambda}, \quad \frac{n+1}{\lambda} \leqslant b < \frac{n+2}{\lambda}.$$

Then, as I proved in my paper referred to,

$$\lim_{\lambda\to\infty}(\Phi_n - \Phi_m) = \int_a^b \cos ux\, f(u)\,du \tag{4.4}$$

uniformly for $0 < x_1 \leqslant x \leqslant x_2$. Also

$$\begin{aligned}\int_0^{\pi\lambda} |\Phi_n - \Phi_m|^{p/(p-1)}\,dx &= \int_0^{\pi\lambda} \left|\sum_{k=m+1}^{n} h_k \cos(kx/\lambda)\right|^{p/(p-1)} dx \\ &= \lambda\int_0^{\pi} \left|\sum_{k=m+1}^{n} h_k \cos kx\right|^{p/(p-1)} dx \\ &\leqslant \tfrac{1}{2}\pi\lambda\left(\sum_{k=m+1}^{n} |h_k|^p\right)^{1/(p-1)}\end{aligned}$$

by (3.3); and so by (4.21)

$$\int_0^{\pi\lambda} |\Phi_n - \Phi_m|^{p/(p-1)}\,dx \leqslant \tfrac{1}{2}\pi\left[\int_{(m+1)/\lambda}^{(n+1)/\lambda} |f(x)|^p\,dx\right]^{1/(p-1)}$$

$$\leqslant \tfrac{1}{2}\pi\left[\int_a^b |f(x)|^p\,dx\right]^{1/(p-1)}.$$

Hence, if $0 < x_1 < x_2 < \pi\lambda$, we have

$$\int_{x_1}^{x_2} |\Phi_n - \Phi_m|^{p/(p-1)}\,dx \leqslant \tfrac{1}{2}\pi\left[\int_a^b |f(x)|^p\,dx\right]^{1/(p-1)}. \tag{4.5}$$

Making $\lambda \to \infty$, the integral on the left may be replaced by

$$\int_{x_1}^{x_2} \left|\int_a^b \cos ux\, f(u)\,du\right|^{p/(p-1)} dx \tag{4.51}$$

since the right-hand side is independent of λ. Also clearly x_1 may be as small and x_2 as large as we please; making $x_1 \to 0$, $x_2 \to \infty$, we have

$$\int_0^{\infty} \left|\int_a^b \cos ux\, f(u)\,du\right|^{p/(p-1)} dx \leqslant \tfrac{1}{2}\pi\left[\int_a^b |f(x)|^p\,dx\right]^{1/(p-1)}. \tag{4.52}$$

It follows that when $a \to \infty$, $b \to \infty$, in any manner

$$\lim \int_0^{\infty} \left|\int_a^b \cos ux\, f(u)\,du\right|^{p/(p-1)} dx = 0. \tag{4.6}$$

Writing

$$F_a(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a \cos ux\, f(u)\,du, \tag{4.7}$$

we have

$$\lim_{a\to\infty,\, b\to\infty} \int_0^{\infty} |F_b(x) - F_a(x)|^{p/(p-1)}\,dx = 0. \tag{4.8}$$

It follows then, from a theorem of Riesz,* that there is a function $F(x)$, defined uniquely almost everywhere, whose $p/(p-1)$-th power is integrable over $(0, \infty)$ such that

$$\lim_{a\to\infty} \int_0^{\infty} |F(x) - F_a(x)|^{p/(p-1)}\,dx = 0. \tag{4.9}$$

This function $F(x)$ is the transform of $f(x)$.

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* F. Riesz, *Math. Annalen*, 69 (1910), 449-497 (464-468).

5. *Proof of the theorems stated in* §2.—It follows from (4.9) that if $g(x)$ has its p-th power integrable over $(0, \infty)$, then

$$\lim_{a\to\infty} \int_0^\infty g(x)\{F(x)-F_a(x)\}\,dx = 0. \tag{5.1}$$

In particular, for any finite value of x

$$\lim_{a\to\infty} \int_0^x \{F(t)-F_a(t)\}\,dt = 0.$$

But

$$\int_0^x F_a(t)\,dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x dt \int_0^a \cos ut\, f(u)\,du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a \frac{\sin ux}{u} f(u)\,du.$$

Hence

$$\int_0^x F(t)\,dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin ux}{u} f(u)\,du,$$

and, differentiating, we have almost everywhere

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin ux}{u} f(u)\,du.$$

This formula, together with the result of the preceding section, completes the proof of the analogue of Parseval's theorem.

Again, since $f(x)$ is integrable over any finite interval, we have the integrated form of Fourier's formula*

$$\int_0^x f(t)\,dt = \frac{2}{\pi} \int_0^\infty \frac{\sin xu}{u}\,du \int_0^a \cos ut\, f(t)\,dt,$$

which holds if $x < a$. Hence

$$\int_0^x f(t)\,dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F_a(u)\,du,$$

if $x < a$. But by (5.1)

$$\lim_{a\to\infty} \int_0^\infty \frac{\sin xu}{u}\{F(u)-F_a(u)\}\,du = 0.$$

Hence

$$\int_0^x f(t)\,dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F(u)\,du,$$

* This may be proved by inverting the order of integration "by bounded convergence." See G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 23 (1924), 1–26 (10).

and, differentiating, we have almost everywhere

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} F(u)\, du.$$

This completes the proof of the analogue of the Riesz-Fischer theorem.

6. It must now be shown that the theorems of §2 are false if $p > 2$. Let

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 t^{-1-a} \cos t^{-b} \cos xt\, dt, \tag{6.1}$$

where $0 < a < 1$, $b > a$. Then

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xt}{t} f(t)\, dt = x^{-1-a} \cos x^{-b}\, (0 < x < 1),\ 0 (x > 1),$$

provided we may invert the order of integration of

$$\int_0^\infty \frac{\sin xt}{t}\, dt \int_0^1 u^{-1-a} \cos u^{-b} \cos ut\, du.$$

We may do this if

$$\lim_{\lambda \to \infty} \int_0^1 u^{-1-a} \cos u^{-b}\, du \int_\lambda^\infty \frac{\sin xt \cos ut}{t}\, dt = 0$$

or if

$$\lim_{\lambda \to \infty} \int_0^1 u^{-1-a} \cos u^{-b}\, du \int_\lambda^\infty \frac{\sin xt(1-\cos ut)}{t}\, dt = 0.$$

But

$$\left| \int_\lambda^\infty \frac{\sin xt(1-\cos ut)}{t}\, dt \right| = \left| \int_{\lambda x}^{\lambda(x+u)} \frac{\sin t}{t}\, dt - \int_{\lambda(x-u)}^{\lambda x} \frac{\sin t}{t}\, dt \right|$$

$$\leqslant \log \frac{x+u}{x-u},$$

and the result follows from Lebesgue's convergence theorem, since

$$u^{-1-a} \cos u^{-b} \log \frac{x+u}{x-u}$$

is integrable over (0, 1).

The behaviour of $f(x)$ as $x \to \infty$ can be discussed on the lines of Hardy's "Oscillating Dirichlet Integrals."* We have

$$f(x) = O(x^{-(b-2a)/2(b+1)}). \tag{6.3}$$

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* G. H. Hardy, *Quart. J. of Math.*, 44 (1913), 251–263. See also M. Kuniyeda, *ibid.*, 48 (1920), 113–135; Kuniyeda discusses certain cases left aside by Hardy, and it is these which would be relevant here.

By taking b sufficiently large we can make $f^{2+\epsilon}$ integrable over $(0, \infty)$ for any given positive ϵ. But $F(x)$ is not (absolutely) integrable, since $a > 0$. Hence the analogue of Parseval's theorem is not true if $p > 2$. Again, if the analogue of the Riesz-Fischer theorem were true for $p > 2$, then (2.3) would be true with $p/(p-1) < 2$, and so the general theory would apply to $F(x)$, and, in particular (2.4). Hence

$$F(x) = x^{-1-a} \cos x^{-b} \quad (0 < x < 1),$$

which again gives a contradiction.

Again, consider the function

$$f(x) = x^{-\alpha} \quad (0 < x < 1), \qquad x^{-\beta} \quad (1 < x) \tag{6.4}$$

where $\alpha < \beta < 1$. The p-th power of this function is integrable over $(0, \infty)$ if $p\alpha < 1$, $p\beta > 1$, *i.e.*

$$1/\alpha > p > 1/\beta. \tag{6.5}$$

Now

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 \frac{\cos xt}{t^\alpha} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_1^\infty \frac{\cos xt}{t^\beta}\, dt \tag{6.6}$$

and it is easily seen that

$$F(x) \sim Ax^{\alpha-1} \quad (x \to \infty),$$

$$F(x) \sim Ax^{\beta-1} \quad (x \to 0).$$

Hence the q-th power of $F(x)$ is only integrable if

$$1/(1-\alpha) < q < 1/(1-\beta). \tag{6.7}$$

Now we may have, consistently with (6.5),

$$\alpha = \frac{1-(p-1)\epsilon}{p-(p-1)\epsilon}, \qquad \beta = \frac{1+(p-1)\epsilon}{p+(p-1)\epsilon},$$

if $p > 1$, $\epsilon > 0$; and then (6.7) becomes

$$\frac{p}{p-1} - \epsilon < q < \frac{p}{p-1} + \epsilon.$$

This proves the statement at the end of §2.

7. *Further properties of transforms.*—Let $F(x)$, $G(x)$ be the transforms of $f(x)$, $g(x)$. If the squares of $f(x)$ and $g(x)$ are integrable over $(0, \infty)$ we have the relation

$$\int_0^\infty f(x)\, G(x)\, dx = \int_0^\infty g(x)\, F(x)\, dx, \tag{7.1}$$

and, putting $g(x) = F(x)$, the particular case

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{F(x)\}^2\,dx. \tag{7.2}$$

The equation (7.1) is still true if $f(x)$ and $g(x)$ have their p-th powers integrable. For

$$\begin{aligned}\int_0^b g(x)\,F_a(x)\,dx &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^b g(x)\,dx \int_0^a \cos xt\, f(t)\,dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a f(t)\,dt \int_0^b \cos xt\, g(x)\,dx \\ &= \int_0^a f(t)\,G_b(t)\,dt.\end{aligned}$$

But

$$\lim_{a\to\infty} \int_0^b g(x)\{F(x)-F_a(x)\}\,dx = 0,$$

$$\lim_{b\to\infty} \int_0^\infty f(x)\{G(x)-G_b(x)\}\,dx = 0.$$

Hence on making $a\to\infty$, $b\to\infty$, we obtain (7.1).

The equation (7.2) is clearly not true in the general case, for the integrals do not exist. We have instead the inequality

$$\int_0^\infty |F(x)|^{p/(p-1)}\,dx \leqslant \tfrac{1}{2}\pi \left[\int_0^\infty |f(x)|^p\,dx\right]^{1/(p-1)}. \tag{7.3}$$

This is an immediate deduction from the work of §4; for (4.52) gives

$$\int_0^\infty |F_a(x)|^{p/(p-1)}\,dx \leqslant \tfrac{1}{2}\pi \left[\int_0^a |f(x)|^p\,dx\right]^{1/(p-1)}. \tag{7.4}$$

This is true for all values of a. *A fortiori*

$$\int_0^\infty |F_a(x)|^{p/(p-1)}\,dx \leqslant \tfrac{1}{2}\pi \left[\int_0^\infty |f(x)|^p\,dx\right]^{1/(p-1)} \tag{7.5}$$

for all values of a. But

$$\left|\left[\int_0^\infty |F(x)|^q\,dx\right]^{1/q} - \left[\int_0^\infty |F_a(x)|^q\,dx\right]^{1/q}\right| \leqslant \left[\int_0^\infty |F(x)-F_a(x)|^q\,dx\right]^{1/q},$$

where $q = p/(p-1)$. Hence

$$\lim_{a\to\infty} \int_0^\infty |F_a(x)|^{p/(p-1)}\,dx = \int_0^\infty |F(x)|^{p/(p-1)}\,dx. \tag{7.6}$$

The result (7.3) now follows from (7.5) and (7.6).

8. The results of Plancherel's paper in *Mathematische Annalen* already referred to can now be extended to the classes of transforms considered here. They depend on the formula (7.1), which is still valid in the general case. Plancherel's proofs require very little alteration, and it will be sufficient to state the results. Let $f(x)$ be a function whose p-th power $(1 < p \leqslant 2)$ is integrable over $(0, \infty)$, and let $F(x)$ be its transform. Then

(i) for almost all values of x

$$\int_0^z f(t)\cos xt\,dt = o(\log z);$$

(ii) the integral $$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(t)\cos xt\,dt$$

is summable (C, 1) to the sum $f(x)$ for almost all values of x;

(iii) we have almost everywhere

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos xu\,du \int_0^\infty \cos ut\, f(t)\,dt$$

if both integrals are taken in the (C, 1) sense.

If, further, the integral

$$\int_a^\infty |f(x)\log x|^p\,dx$$

is convergent, then

$$\lim_{z\to\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^z f(t)\cos xt\,dt = F(x)$$

for almost all values of x.

PROPERTIES OF PRIME NUMBERS DEDUCED FROM THE CALCULUS OF SYMMETRIC FUNCTIONS

By Major P. A. MacMahon.

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It is the purpose of the present investigation to regard the subject of the prime numbers from a standpoint which differs from that which it has been usual to select. The original idea was to see if any light could be thrown upon the celebrated Euler-Goldbach conjecture concerning the expression of any even number as the sum of two primes. It soon appeared that the number of such representations of an even number $2n$, denoted throughout by $N_{2n,2}$, presents itself in a natural but exceptional manner in the investigation. The question that arises at once is the ascertaining of the number of representations of a number as the sum of m prime numbers, where m is any integer. The case $m = 2$ thus appears in a natural way, but it is also to be regarded as exceptional because it is adapted to examination by means of symmetric function analysis in a manner which does not exist for any higher value of m. I commence by defining a symmetric function which is derived from the complete succession of prime numbers.

1. Any integer n is expressed as a product of prime numbers

$$p_1 p_2 p_3 \cdots$$

where $p_1, p_2, p_3, \ldots$ are primes, the same or different.

We may derive, from this expression, a symmetric function in regard to arbitrary numerical quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ the number of which is assumed to be infinite.

This is $$\Sigma \alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \ldots$$

denoted in the usual partition notation by

$$(p_1 p_2 p_3 \ldots).$$

In order to bring into evidence repetitions of prime numbers, which occur in the notation, we denote the integer n by the prime number expression

$$p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots,$$

where now $p_1, p_2, p_3, \ldots$ are different primes, and we derive the symmetric function

$$\Sigma a_1^{p_1} a_2^{p_2} \ldots a_{\pi_1}^{p_1} a_{\pi_1+1}^{p_2} a_{\pi_1+2}^{p_2} \ldots a_{\pi_1+\pi_2}^{p_2} a_{\pi_1+\pi_2+1}^{p_3} a_{\pi_1+\pi_2+2}^{p_3} \ldots a_{\pi_1+\pi_2+\pi_3}^{p_3} \ldots,$$

or in partition notation $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$,

wherein the numbers $\pi_1, \pi_2, \pi_3, \ldots$ are repetitional indices which denote the numbers of times the parts $p_1, p_2, p_3, \ldots$ respectively occur in the partition.

The number partitioned is $\Sigma \pi p$,

an integer which, viewed in connexion with the integer n, may be conveniently termed the potency of n.

We have thus a one-to-one correspondence between an integer n and a symmetric function which is denoted by a definite partition of the potency of n.

I now construct an infinite series of symmetric functions

$$W = 1 + \sum_n (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

the summation being in regard to every integer n, unity excepted. The exception arises because the composite number unity has no partition into prime parts.

It is to be remarked that the partitions involve prime parts exclusively. Written at length for a few terms

$$W = 1 + (2) + (3) + (2^2) + (5) + (32) + (7) + (2^3) + (3^2) + (52) + (11) + \ldots,$$

in which, it will be noted, the numbers partitioned (the potencies) are not in ascending order of magnitude.

More than one integer has a given potency—say μ—and μ has in general more than one partition into prime parts.

The numbers partitioned are not all different, and, moreover, the partitions, of a potency μ, do not usually occur consecutively in the function W.

E.g. the terms (7), (52) partitions, of the potency 7 are derived from the values 7, 10 of n, and have two terms between them which are associated with the potency 6 of the two n values 8, 9. In other words, the series W is in ascending order *quâ* the integer n, and not in ascending order *quâ* the potencies of n. I next describe briefly the differential operators that will be utilized.

I write

$$(1-a_1x)(1-a_2x)(1-a_3x)\ldots = 1-a_1x+a_2x^2-a_3x^3+\ldots,$$

and denote $(r) = \Sigma a^r$ by s_r where it is convenient.

The partial differential operators are

$$d_r = \frac{d}{da_r}+a_1\frac{d}{da_{r+1}}+a_2\frac{d}{da_{r+2}}+\ldots,$$

$$D_r = \frac{1}{r!}\left(\frac{d}{da_1}+a_1\frac{d}{da_2}+a_2\frac{d}{da_3}+\ldots\right)^r,$$

where D_r is an operator, of order r, formed by symbolic multiplication as in Taylor's theorem.

We write $D_r = \frac{1}{r!}(d_1^r)$ *inner multiplication*

when we wish to put d_1 in evidence.

When r *successive* operations of d_1 are before us we write

$(d_1)^r$ *outer multiplication,*

and more generally $(d_k)^r$

to denote r successive operations of d_k.

Successive operations of

$$D_{r_1},\ D_{r_2},\ D_{r_3},\ \ldots$$

are represented by $D_{r_1}D_{r_2}D_{r_3}\ldots.$

1.1. It was established long ago by Hammond* that relations between these operators are derivable from symmetric function formulae.

Thus from the formula which expresses a monomial symmetric function in terms of the elementary functions $a_1, a_2, a_3, \ldots$, viz :

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots) = \Sigma C_{m_1^{\mu_1}m_2^{\mu_2}\ldots}a_{m_1}^{\mu_1}a_{m_2}^{\mu_2}a_{m_3}^{\mu_3}\ldots, \qquad (A)$$

where the summation is controlled by the equation in integers

$$\mu_1m_1+\mu_2m_2+\mu_3m_3+\ldots = \Sigma\pi p,$$

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* *Proc. London Math. Soc.*, 14 (1882), 119.

we derive

$$\frac{1}{\pi_1!\,\pi_2!\,\pi_3!\ldots}(d_{p_1}^{\pi_1}d_{p_2}^{\pi_2}d_{p_3}^{\pi_3}\ldots)=\Sigma C_{m_1^{\mu_1}m_2^{\mu_2}m_3^{\mu_3}\ldots}D_{m_1}^{\mu_1}D_{m_2}^{\mu_2}D_{m_3}^{\mu_3}\ldots, \qquad (A')$$

wherein, on the left-hand side, we have *inner* multiplication of linear operators, and on the right-hand side successive operations of operators D_{m_1}, μ_1 times; D_{m_2}, μ_2 times; D_{m_3}, μ_3 times; ... which are of orders $m_1, m_2, m_3, \ldots$ respectively.

It will be observed that the particular case

$$(1^r)=a_r \text{ of (A)}$$

leads to

$$\frac{1}{r!}(d_1^r)=D_r \text{ of (A')},$$

so that the notation above is consistent with the theorem.

Again, the formula

$$a_{p_1}^{\pi_1}a_{p_2}^{\pi_2}a_{p_3}^{\pi_3}\ldots=\Sigma C'_{m_1^{\mu_1}m_2^{\mu_2}\ldots}(m_1^{\mu_1}m_2^{\mu_2}\ldots) \qquad (B)$$

leads to

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots=\Sigma C'_{m_1^{\mu_1}m_2^{\mu_2}\ldots}\frac{(d_{m_1}^{\mu_1}d_{m_2}^{\mu_2}\ldots)}{\mu_1!\,\mu_2!\ldots} \qquad (B')$$

These results, and others which follow, are valid whether or no the integers $p_1, p_2, p_3, \ldots$ be primes.

$$s_n=\Sigma(-)^{n+\Sigma\mu}\frac{n\,.\,(\Sigma\mu-1)!}{\mu_1!\,\mu_2!\ldots}a_{m_1}^{\mu_1}a_{m_2}^{\mu_2}, \quad \Sigma\mu m=n, \qquad (C)$$

leads to

$$d_n=\Sigma(-)^{n+\Sigma\mu}\frac{n\,.\,(\Sigma\mu-1)!}{\mu_1!\,\mu_2!\ldots}D_{m_1}^{\mu_1}D_{m_2}^{\mu_2}\ldots. \qquad (C')$$

$$a_n=\Sigma(-)^{n+\Sigma\mu}\frac{s_{m_1}^{\mu_1}s_{m_2}^{\mu_2}\ldots}{m_1^{\mu_1}m_2^{\mu_2}\ldots\mu_1!\,\mu_2!\ldots}, \quad \Sigma\mu m=n, \qquad (D)$$

leads to

$$D_n=\Sigma(-)^{n+\Sigma\mu}\frac{(d_{m_1})^{\mu_1}(d_{m_2})^{\mu_2}\ldots}{m_1^{\mu_1}m_2^{\mu_2}\ldots\mu_1!\,\mu_2!\ldots}; \qquad (D')$$

where, on the right-hand side, we have the *outer* multiplication of linear operators. The simplest cases of (C') and (D') are

$$d_1=D_1, \qquad D_1=d_1,$$

$$d_2=D_1^2-2D_2, \qquad 2D_2=(d_1)^2-d_2,$$

$$d_3=D_1^3-3D_1D_2+3D_3, \qquad 6D_3=(d_1)^3-3(d_1)(d_2)+2d_3,$$

... ...

Other symmetric function formulae, some of them involving the homogeneous product sum h_n, will be added later.

1.2. When the operator D_n is performed upon a symmetric function which, in the partition notation, involves one or more parts equal to n, it has the effect of striking out one such part.

If no part n be present it causes the function to vanish.

I now take for operand the series of functions W, and evolve successively a number of theorems.

THEOREM I.— $D_n W = W$ *or zero,*

according as n is or is not a prime.

This result is due to the construction of W, and is evident by mere inspection.

COROLLARY (*a*).—*If* $p_1, p_2, p_3, \ldots$ *be primes,*

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \ldots W = W.$$

COROLLARY (*b*).— $\left\{\sum_1^n m D_m\right\} W = P_n W,$

where P_n *enumerates the primes which occur in the first n natural numbers.*

It is clear, from Theorem I, that for the operand W it is only necessary to take account of operators of form D_n when n is a prime. Exceptionally D_0 must be regarded as leaving W unaltered and, to that extent, zero plays the part of a prime. This is important in view of the operand W^m (m a positive integer usually), to which we now proceed.

D_n operates upon W^m either by making use of the operator formula (D') or by a well known rule due to the circumstance that D_n is a numerical multiple of the symbolic power (inner multiplication) of a linear operator (viz., d_1).

The rule makes use of the compositions (partitions in which the order of the parts is taken into account) of n into m parts, zero being included as a possible part; and the only non-zero parts that need consideration are (from Theorem I) primes. If for present purposes we regard zero as a prime, we may say simply that the compositions are composed of prime parts exclusively.

E.g. to operate with D_4 upon W^3 we take the compositions of 4

$$220, \quad 202, \quad 022,$$

and proceed in the manner

$$D_4 W^3 = D_2 W . D_2 W . D_0 W + D_2 W . D_0 W . D_2 W + D_0 W . D_2 W . D_2 W$$
$$= 3 W^3.$$

For D_n we utilize the whole of the prime-part compositions, and add the results.

Each prime-part composition of n gives a *representation* of n as a sum of primes.

1.3. THEOREM II.—

$$D_n W^m = C(n, m) W^m = N_{n,m} W^m$$

where $C(n, m)$ is the number of compositions of n having m prime parts, and $N_{n,m}$ is the number of representations of n as the sum of m primes.

COROLLARY (*a*).— $W^{-2} D_{2n} W^2 = N_{2n, 2}$

denotes the number of representations of the even number $2n$ as the sum of two primes.

It follows from this corollary that the Euler-Goldbach conjecture depends upon establishing that, for every value of n,

$$D_{2n} W^2 \neq 0.$$

1.4. The formula (C') leads to an important result.

THEOREM III.—

$$d_n W = \Sigma(-)^{n+\Sigma\lambda} \frac{n . (\Sigma\lambda - 1)!}{\lambda_2 ! \, \lambda_3 ! \, \lambda_5 ! \ldots} D_2^{\lambda_2} D_3^{\lambda_3} D_5^{\lambda_5} \ldots W,$$

the summation being for every prime-part partition of n

$$(2^{\lambda_2} 3^{\lambda_3} 5^{\lambda_5} \ldots),$$

and thence

$$d_n W = \Sigma(-)^{n+\Sigma\lambda} \frac{n . (\Sigma\lambda - 1)!}{\lambda_2 ! \, \lambda_3 ! \, \lambda_5 ! \ldots} W.$$

I write

$$W^{-1} d_n W = e_n.$$

This number e_n is an integer because it is observed to be equal to the algebraic sum of the coefficients, in the symmetric function formula

$$s_n = \Sigma(-)^{n+\Sigma\lambda} \frac{n . (\Sigma\lambda - 1)!}{\lambda_1 ! \, \lambda_2 ! \, \lambda_3 ! \ldots} a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3} \ldots,$$

for which the attached literal products involve only suffixes which are prime numbers.

From the particular cases of (C') given above we find

$$e_1 = 0, \quad e_2 = -2, \quad e_3 = 3, \quad e_4 = 2, \quad \ldots .$$

When we proceed to evaluate e_n, for values of $n > 4$, we meet at once with a curious result; for

$$d_5 W = (\ldots - 5D_2 D_3 + 5D_5) W = 0,$$

so that

$$e_5 = 0;$$

and we have established that W satisfies the partial differential equation

$$d_5 W = 0.$$

Thereafter e_n does not vanish until we come to $n = 13$.

$$e_{13} = 0,$$

and

$$d_{13} W = 0.$$

When n has the values 14 to 89 inclusive, no other vanishing of e_n is disclosed.

Recalling Brioschi's theorem

$$(-)^{n+1} n \frac{d}{ds_n} = d_n,$$

we enunciate

THEOREM III.—*When the symmetric function W is expressed in terms of $s_2, s_3, s_4, \ldots$ the functions*

$$s_5, \quad s_{13}$$

do not present themselves.

This non-appearance of s_5 and s_{13} appears to be a noteworthy property of the series of prime numbers.

1.5. To evaluate e_n we may proceed as follows:—

The symmetric function relation

$$s_n - a_1 s_{n-1} + a_2 s_{n-2} - \ldots + (-)^n n a_n = 0$$

yields the operator relation

$$d_n - D_1 d_{n-1} + D_2 d_{n-2} - \ldots + (-)^{n-1} D_{n-1} d_1 + (-)^n n D_n = 0;$$

and thence, for the operand W,

$$e_n+e_{n-2}-e_{n-3}-e_{n-5}-\ldots-e_{n-p}-\ldots = (-)^{n+1}n \text{ or zero}$$

according as n is or is not a prime number.

We can now readily calculate e_n and a short table is appended.

It is clear, from the expression of d_n in terms of D_2, D_3, D_5, ..., that n is a factor of e_n when n is prime. This property can be observed in the table. It gives one verification of e_n when n is prime.

It is not true, conversely, that if n be a factor of e_n, n is a prime.

When $n = 5^2$, $e_n = 6.5^2$; when $n = 5.13$, $e_n = 240.5.13$.

It is noticeable that these exceptions (the only ones so far) arise when n is of the form $5^\alpha.13^\beta$; 5 and 13 being the values of n which give $e_n = 0$.

TABLE OF e_n.

n	e_n	n	e_n	n	e_n	n	e^n	n	e_n	n	e_n	n	e_n	n	e_n	n	e_n
1	0	11	22	21	94	31	403	41	656	51	−1986	61	−6405	71	157904	81	1966647
2	−2	12	−7	22	42	32	498	42	3396	52	13678	62	25697	72	−40810	82	−21732
3	+3	13	0	23	−23	33	−140	43	1161	53	17278	63	115123	73	466762	83	971432
4	+2	14	33	24	113	34	219	44	−790	54	−2492	64	46386	74	602395	84	3910956
5	0	15	3	25	150	35	1057	45	4269	55	7447	65	−15600	75	−26547	85	1814461
6	1	16	−14	26	−54	36	326	46	5357	56	35569	66	143562	76	280087	86	−204897
7	7	17	51	27	48	37	−259	47	−987	57	13778	67	184250	77	1206542	87	4925450
8	2	18	46	28	345	38	1271	48	2257	58	−5860	68	−13921	78	535107	88	
9	−6	19	−19	29	116	39	1641	49	10934	59	44368	69	85885	79	−96775	89	+134301
10	8	20	12	30	−109	40	−308	50	3958	60	56403	70	372513	80	1515756	90	

For the first eighty-nine values of n, twenty-one values of n give e_n negative. Numerically these values of e_n are considerably less than when e_n is positive. The value of e_n for $n = 88$ has not been calculated.

The theory of the numerical magnitude also arises from the factorization of W which, in view of the zeta function of Riemann, it is natural to consider.

1.6. We write

$$W = A_0+A_1s_n+A_2s_n^2+A_3s_n^3+\ldots.$$

The theorem

$$d_n = (-)^{n+1}n\frac{d}{ds_n}$$

then leads to

$$e_nA_0+e_nA_1s_n+e_nA_2s_n^2+\ldots = (-)^{n+1}(nA_1+2nA_2s_n+3nA_3s_n^2+\ldots);$$

and comparison of coefficients, of powers of s_n, leads to the relation

$$A_m = (-)^{m(n+1)} \frac{1}{m!} \left(\frac{e_n}{n}\right)^m A_0;$$

so that $$W = A_0 \exp (-)^{n+1} \frac{1}{n} e_n s_n;$$

and we gather that $$\exp (-)^{n+1} \frac{1}{n} e_n s_n$$

is a factor of W.

Thence it is clear that

$$W = \prod_{n=2}^{\infty} \exp (-)^{n+1} \frac{1}{n} e_n s_n;$$

the expression of W as an infinite product.

Also $$W = \exp \sum_{n=2}^{\infty} (-)^{n+1} \frac{1}{n} e_n s_n,$$

$$\log W = \sum_{n=2}^{\infty} (-)^{n+1} \frac{1}{n} e_n s_n.$$

It will be noted that, in these results, the factors or terms which involve s_5 or s_{13} do not exist.

In the expression of W as a symmetric function of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$, put

$$\alpha_1 = \alpha, \quad \alpha_2 = \alpha_3 = \alpha_4 = \ldots = 0,$$

and let W become $W_{(1)}$.

Then $$W_{(1)} = 1 + \sum_p \alpha^p,$$

the summation being for every prime p

$$\sum_{n=2}^{\infty} (-)^{n+1} \frac{1}{n} e_n \alpha^n = \log (1 + \sum_p \alpha^p).$$

Differentiating and comparing powers of α, we obtain at once the difference equation satisfied by e_n. (1.5 *ante*.)

THEOREM V.— $$W = \prod_{n=2}^{\infty} \exp (-)^{n+1} \frac{1}{n} e_n s_n$$

where $$\sum_{n=2}^{\infty} (-)^{n+1} \frac{1}{n} e_n s_n = \log (1 + \sum_p \alpha^p),$$

p denoting a prime number.

1.7. From Brioschi's relation

$$(-)^{m+1} m \partial_{s_m} = d_m$$

we can proceed from the laws of operation that have been developed by Cayley,* Sylvester,† and others,‡ to the formula

$$(d_m^2) = m^2 \partial_{s_m}^2 + 2m \partial_{s_{2m}},$$

$$(d_m^3) = (-)^{m+1}(m^3 \partial_{s_m}^3 + 6m^2 \partial_{s_m} \partial_{s_{2m}} + 6m \partial_{s_{3m}})$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\frac{(d_m^n)}{n!} = (-)^{n(m+1)} \Sigma \frac{m^{\Sigma\mu}}{\mu_1!\,\mu_2!\,\mu_3!\dots} \partial_{s_m}^{\mu_1} \partial_{s_{2m}}^{\mu_2} \partial_{s_{3m}}^{\mu_3} \dots,$$

where $$\mu_1 + 2\mu_2 + 3\mu_3 + \dots = n.$$

Putting $m = 1$,

$$D_n = \Sigma \frac{\partial_{s_1}^{\mu_1} \partial_{s_2}^{\mu_2} \partial_{s_3}^{\mu_3} \dots}{\mu_1!\,\mu_2!\,\mu_3!\dots}, \quad \mu_1 + 2\mu_2 + 3\mu_3 + \dots = n.$$

We now consider an operand W, and write

$$W = \sum_0^\infty W_n,$$

where $W_0 = 1$ and W_n is the sum of those symmetric functions, which occur in W, which are of weight n.

Thence

$$D_n W_n = \left(\Sigma \frac{\partial_{s_1}^{\mu_1} \partial_{s_2}^{\mu_2} \partial_{s_3}^{\mu_3} \dots}{\mu_1!\,\mu_2!\,\mu_3!\dots} \right) W_n = W_n \text{ or zero},$$

according as n is or is not a prime.

Moreover, $$\frac{\partial_{s_1}^{\mu_1} \partial_{s_2}^{\mu_2} \partial_{s_3}^{\mu_3} \dots}{\mu_1!\,\mu_2!\,\mu_3!\dots} s_1^{\mu_1} s_2^{\mu_2} s_3^{\mu_3} \dots = 1.$$

Thence

THEOREM VI.—*If we take the sum of the symmetric functions of weight n, which correspond to partitions into prime parts, and express it in terms of $s_2, s_3, s_4, \dots$ the sum of the numerical coefficients is unity or zero according as n is or is not a prime.*

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* *Phil. Mag.* (4), 32 (1866), 471; *Amer. J. of Math.*, 15 (1893), 60; *Collected Math. Papers*, 7 (1894), 8; 13 (1897), 322.

† *Phil. Mag.* (4), 32 (1866), 461; (4), 33 (1867), 48; *Math. Papers*, 2 (1908), 567, 608.

‡ Hammond (*loc. cit. supra*).

Putting $m = 2$,

$$\frac{(d_2^n)}{n!} = (-)^n \Sigma \frac{2^{\Sigma\mu}}{\mu_1!\,\mu_2!\,\mu_3!\ldots}\, \partial_{s_2}^{\mu_1} \partial_{s_4}^{\mu_2} \partial_{s_6}^{\mu_3} \ldots,$$

$$\mu_1 + 2\mu_2 + 3\mu_3 + \ldots = n,$$

a formula related to the representations of an integer n by means of two primes (or zeros) (*cf.* 2.11 *post*).

2. The study of the representations of an integer as the sum of m primes depends, as above stated, upon a discussion of the operand W^m.

Since $$d_n W^m = m e_n W^m$$

the formula (D') yields

$$D_n W^m = \Sigma(-)^{n+\Sigma\lambda} \frac{(me_2)^{\lambda_2}(me_3)^{\lambda_3}(me_4)^{\lambda_4}\ldots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\ldots\lambda_2!\,\lambda_3!\,\lambda_4!\ldots}\, W^m;$$

and, thence, the number of representations is

$$N_{n,m} = \Sigma(-)^{n+\Sigma\lambda} \frac{(me_2)^{\lambda_2}(me_3)^{\lambda_3}(me_4)^{\lambda_4}\ldots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\ldots\lambda_2!\,\lambda_3!\,\lambda_4!\ldots},$$

$$2\lambda_2 + 3\lambda_3 + 4\lambda_4 + \ldots = n.$$

In particular

$$N_{n,1} = \Sigma(-)^{n+\Sigma\lambda} \frac{e_2^{\lambda_2} e_3^{\lambda_3} e_4^{\lambda_4}\ldots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\ldots\lambda_2!\,\lambda_3!\,\lambda_4!\ldots} = \text{unity or zero,}$$

according as n is or is not a prime.

Again, as an example,

$$\begin{aligned} N_{6,m} &= -m^3 \tfrac{1}{48} e_2^3 + m^2 \tfrac{1}{18} e_3^2 + m^2 \tfrac{1}{8} e_2 e_4 - m \tfrac{1}{6} e_6 \\ &= \tfrac{1}{6}m^3 + \tfrac{1}{2}m^2 - \tfrac{1}{2}m^2 - \tfrac{1}{6}m \\ &= \binom{m+1}{3}. \end{aligned}$$

The verification is

$$6 = 3+3+(m-2)0 \qquad \text{giving } \binom{m}{2} \text{ representations}$$

$$= 2+2+2+(m-3)0 \qquad \text{,,} \quad \binom{m}{3} \qquad \text{,,}$$

and $\binom{m}{2} + \binom{m}{3} = \binom{m+1}{3}$ as required.

For all values of n, $N_{n,m}$ involves the factor m (obviously). Moreover, when n is composite, Theorem I establishes that $N_{n,m}$ involves a factor $m-1$.

Some results are

n	$W^{-m} D_n W^m$
2	m
3	m
4	$\binom{m}{2}$
5	m^2
6	$\binom{m+1}{2}$
7	$\frac{1}{2}m(m^2-m+2)$
8	$\frac{1}{4}\binom{m+1}{3}(m+6)$
9	$\frac{1}{3}\binom{m}{2}(m^2-m+4)$
10	$\frac{1}{60}\binom{m}{2}(m^3+21m^2-4m+96)$.

The Euler-Goldbach conjecture is that there cannot be a factor $m-2$ for any even value of n.

Otherwise, since

$$N_{2n,2} = \Sigma(-)^{\Sigma\lambda} \frac{(2e_2)^{\lambda_2}(2e_3)^{\lambda_3}(2e_4)^{\lambda_4}\dots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\dots\lambda_2!\,\lambda_3!\,\lambda_4!\dots},$$

$$2\lambda_2+3\lambda_3+4\lambda_4+\dots = 2n,$$

the conjecture is that $N_{2n,2}$, which we know to be either zero or a positive integer, is in fact always greater than zero.

2.1. We can obtain another expression for $N_{2n,2}$ by a method which is not available when $m > 2$.

We have the symmetric function relation

$$(2^n) = a_n^2 - 2a_{n-1}a_{n+1} + 2a_{n-2}a_{n+2} - \ldots + (-)^n 2a_{2n},$$

from which we derive the operator relation

$$\frac{1}{n!}(d_2^n) = D_n^2 - 2D_{n-1}D_{n+1} + 2D_{n-2}D_{n+2} - \ldots + (-)^n 2D_{2n}.$$

If the operand be W, we observe that

$$D_{n-t}D_{n+t}W = W \text{ or zero}$$

according as $n-t$ and $n+t$ are or are not both prime numbers.

Thence $$\frac{1}{n!}(d_2^n)W = (-)^{n+1}N_{2n,2} \text{ or } (-)^n N_{2n,2}$$

according as n is or is not greater than 2.

This form of statement arises from the circumstance that 2 is the only even prime.

2.11. To evaluate $$\frac{1}{n!}(d_2^n)W,$$

we note that the symmetric function formula

$$(1^n) = a_n = \Sigma(-)^{n+\Sigma\lambda}\frac{s_1^{\lambda_1}s_2^{\lambda_2}s_3^{\lambda_3}}{1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}\ldots\lambda_1!\,\lambda_2!\,\lambda_3!\ldots},$$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots = n,$$

leads, by squaring the quantities $a_1, a_2, a_3, \ldots$, to the formula

$$(2^n) = \Sigma(-)^{n+\Sigma\lambda}\frac{s_2^{\lambda_1}s_4^{\lambda_2}s_6^{\lambda_3}}{1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}\ldots\lambda_1!\,\lambda_2!\,\lambda_3!\ldots}$$

where $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots = n.$$

Thence we obtain the operator formula

$$\frac{1}{n!}(d_2^n) = \Sigma(-)^{n+\Sigma\lambda}\frac{(d_2)^{\lambda_1}(d_4)^{\lambda_2}(d_6)^{\lambda_3}\ldots}{1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}\ldots\lambda_1!\,\lambda_2!\,\lambda_3!\ldots},$$

where $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots = n,$$

and thence, with an operand W,

$$(-)^{n+1} \text{ or } (-)^n \times N_{2n,2} = \Sigma(-)^{n+\Sigma\lambda}\frac{e_2^{\lambda_1}e_4^{\lambda_2}e_6^{\lambda_3}\ldots}{1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}\ldots\lambda_1!\,\lambda_2!\,\lambda_3!\ldots}$$

according as n is or is not greater than 2.

This result may be written

$$N_{2n,2} = - \text{ or } + \Sigma(-)^{\Sigma\lambda} \frac{e_2^{\lambda_1} e_4^{\lambda_2} e_6^{\lambda_3} \dots}{1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots \lambda_1 ! \, \lambda_2 ! \, \lambda_3 ! \dots},$$

where $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots = n,$$

according as n is or is not greater than 2.

The method, that has just been used, is not available when $m > 2$ because it is only when $m = 2$ that a suitable symmetric function formula exists for the purpose.

2.12. When we come to compare this new expression of $N_{2n,2}$ with the one first obtained, we obtain a third expression of $N_{2n,2}$.

Consider the result

$$N_{2n,2} = \Sigma(-)^{\Sigma\lambda} \frac{(2e_2)^{\lambda_2} (2e_3)^{\lambda_3} (2e_4)^{\lambda_4} \dots}{2^{\lambda_2} 3^{\lambda_3} 4^{\lambda_4} \dots \lambda_2 ! \, \lambda_3 ! \, \lambda_4 ! \dots},$$

where $$2\lambda_2 + 3\lambda_3 + 4\lambda_4 + \dots = 2n.$$

The terms on the right-hand side, which involve e_1, have been omitted because $e_1 = 0$.

We can restore these terms and write, equally,

$$N_{2n,2} = \Sigma(-)^{\Sigma\lambda} \frac{(2e_1)^{\lambda_1} (2e_2)^{\lambda_2} (2e_3)^{\lambda_3} \dots}{1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots \lambda_1 ! \, \lambda_2 ! \, \lambda_3 ! \dots},$$

remembering that e_1 and λ_1 are both zero.

Some terms on the right-hand side only involve the even suffixed e's, e_2, e_4, e_6, ... Such terms are

$$\Sigma(-)^{\Sigma\lambda} \frac{(2e_2)^{\lambda_2} (2e_4)^{\lambda_4} (2e_6)^{\lambda_6} \dots}{2^{\lambda_2} 4^{\lambda_4} 6^{\lambda_6} \dots \lambda_2 ! \, \lambda_4 ! \, \lambda_6 ! \dots},$$

which is $$\Sigma(-)^{\Sigma\lambda} \frac{e_2^{\lambda_2} e_4^{\lambda_4} e_6^{\lambda_6}}{1^{\lambda_2} 2^{\lambda_4} 3^{\lambda_6} \dots \lambda_2 ! \, \lambda_4 ! \, \lambda_6 ! \dots},$$

where $$2\lambda_2 + 4\lambda_4 + 6\lambda_6 + \dots = 2n,$$

or $$\Sigma(-)^{\Sigma\lambda} \frac{e_2^{\lambda_1} e_4^{\lambda_2} e_6^{\lambda_3}}{1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots \lambda_1 ! \, \lambda_2 ! \, \lambda_3 ! \dots},$$

where $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots = n.$$

Now this expression has been shown to be equal to $-N_{2n,2}$ or $+N_{2n,2}$ according as n is or is not greater than 2.

THEOREM VII.—*In the result*

$$N_{2n,\,2} = \Sigma(-)^{\Sigma\lambda}\,\frac{(2e_2)^{\lambda_2}(2e_3)^{\lambda_3}(2e_4)^{\lambda_4}\ldots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\ldots\lambda_2!\,\lambda_3!\,\lambda_4!\ldots},$$

$$2\lambda_2+3\lambda_3+\ldots = 2n,$$

that portion of the right-hand side which involves evenly suffixed c's exclusively is equal to $-N_{2n,\,2}$ *or* $+N_{2n,\,2}$ *according as n is or is not greater than* 2.

2.13. Hence also

THEOREM VIII.—*In the result*

$$N_{2n,\,2} = \Sigma(-)^{\Sigma\lambda}\,\frac{(2e_2)^{\lambda_2}(2e_3)^{\lambda_3}(2e_4)^{\lambda_4}\ldots}{2^{\lambda_2}3^{\lambda_3}4^{\lambda_4}\ldots\lambda_2!\,\lambda_3!\,\lambda_4!\ldots},$$

that portion of the right-hand side, whose terms do not contain evenly suffixed e's exclusively, is equal to

$$+2N_{2n,\,2} \text{ or zero}$$

according as n is or is not greater than 2.

This statement involves a third expression for $N_{2n,\,2}$.

Its verification, for $n = 1$ and $n = 2$, is immediate because in neither case are unevenly suffixed e's involved at all.

For $n = 3, 4, 5, 6$ we have

$$n=3,\quad N_{6,\,2} = \left(-2\,\frac{e_6}{6}+2^2\,\frac{e_2e_4}{8}-2^3\,\frac{e_2^3}{48}\right)+2^2\,\frac{e_3^2}{18}$$

$$= (-\tfrac{1}{3}-2+\tfrac{4}{3})+2 = -1+2 = -N_{6,\,2}+2N_{6,\,2};$$

$$n=4,\quad N_{8,\,2} = -2\,\frac{e_8}{8}+2^2\,\frac{e_2e_6}{12}+2^2\,\frac{e_4^2}{32}-2^3\,\frac{e_2^2e_4}{32}+2^4\,\frac{e_2^4}{384}-2^3\,\frac{e_2e_3^2}{36}$$

$$= (-\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{2}-2+\tfrac{2}{3})+4 = -2+4 = -N_{8,\,2}+2N_{8,\,2};$$

$$n=5,\quad N_{10,\,2} = -2\,\frac{e_{10}}{10}+2^2\,\frac{e_2e_8}{16}+2^2\,\frac{e_4e_6}{24}-2^3\,\frac{e_2^2e_6}{48}-2^3\,\frac{e_2e_4^2}{64}$$

$$+2^4\,\frac{e_2^3e_4}{192}-2^5\,\frac{e_2^5}{3840}+2^2\,\frac{e_3e_7}{21}-2^3\,\frac{e_3^2e_4}{72}+2^4\,\frac{e_2^2e_3^2}{144}$$

$$= (-\tfrac{8}{5}-1+\tfrac{1}{3}-\tfrac{2}{3}+1-\tfrac{4}{3}+\tfrac{4}{15})+(4-2+4)$$

$$= -3+6 = -N_{10,\,2}+2N_{10,\,2}.$$

In the case last written the representations of 10 which are enumerated are

$$7+3, \quad 3+7, \quad 5+5,$$

and so on in all cases.

3. We have next to derive operator relations from symmetric function identities which involve the homogeneous product sums $h_1, h_2, h_3, \ldots$.

The operator corresponding to h_n will be written δ_n.

3.1. Since

$$h_n = \Sigma(m_1^{\mu_1} m_2^{\mu_2} \ldots), \qquad \text{where} \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n,$$

we derive

$$\delta_n = \Sigma \frac{(d_{m_1}^{\mu_1} d_{m_2}^{\mu_2} \ldots)}{\mu_1 ! \, \mu_2 ! \ldots} \qquad ,, \qquad ,,$$

3.2. Since

$$h_n = \Sigma \frac{s_{m_1}^{\mu_1} s_{m_2}^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1 ! \, \mu_2 ! \ldots} \qquad ,, \qquad ,,$$

we derive

$$\delta_n = \Sigma \frac{(d_{m_1})^{\mu_1} (d_{m_2})^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1 ! \, \mu_2 ! \ldots} \qquad ,, \qquad ,,$$

3.3. Since

$$s_n = \Sigma(-)^{\Sigma\mu+1} \frac{(\Sigma\mu - 1) ! \, n}{\mu_1 ! \, \mu_2 ! \ldots} h_{m_1}^{\mu_1} h_{m_2}^{\mu_2} \ldots \qquad ,, \qquad ,,$$

we derive

$$d_n = \Sigma(-)^{\Sigma\mu+1} \frac{(\Sigma\mu - 1) ! \, n}{\mu_1 ! \, \mu_2 ! \ldots} (\delta_{m_1})^{\mu_1} (\delta_{m_2})^{\mu_2} \ldots \qquad ,, \qquad ,,$$

3.4. Since

$$h_n = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu) !}{\mu_1 ! \, \mu_2 ! \ldots} a_{m_1}^{\mu_1} a_{m_2}^{\mu_2} \ldots, \qquad ,, \qquad ,,$$

we derive

$$\delta_n = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu) !}{\mu_1 ! \, \mu_2 ! \ldots} D_{m_1}^{\mu_1} D_{m_2}^{\mu_2} \ldots \qquad ,, \qquad ,,$$

Moreover, the letters a, h may be interchanged so that

(3.5)

$$D_n = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu) !}{\mu_1 ! \, \mu_2 ! \ldots} (\delta_{m_1})^{\mu_1} (\delta_{m_2})^{\mu_2} \ldots \qquad \text{where} \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n.$$

3.6. Since

$$nh_n = s_n + h_1 s_{n-1} + h_2 s_{n-2} + \ldots + h_{n-1} s_1,$$

$$n\delta_n = d_n + (\delta_1)(d_{n-1}) + (\delta_2)(d_{n-2}) + \ldots + (\delta_{n-1})(d_1).$$

3.7. Since $h_n - a_1 h_{n-1} + a_2 h_{n-2} - \ldots + (-)^n a_n = 0,$

$$\delta_n - D_1(\delta_{n-1}) + D_2(\delta_{n-2}) - \ldots + (-)^n D_n = 0.$$

3.8. From the expressions of s_n and h_n in terms of $a_1, a_2, a_3, \ldots$ we see at once that

$$\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + a_3 \frac{d}{da_3} + \ldots\right) s_n = nh_n.$$

Whence, applying this linear operator to the relation

$$s_n - a_1 s_{n-1} + a_2 s_{n-2} - \ldots + (-)^n n a_n = 0,$$

we obtain

$$nh_n - (n-1)\, a_1 h_{n-1} + (n-2)\, a_2 h_{n-2} - \ldots + (-)^{n-1} a_{n-1} h_1 = s_n$$

and the corresponding operator relation

$$n\delta_n - (n-1)\, D_1 \delta_{n-1} + (n-1)\, D_2 \delta_{n-2} - \ldots + (-)^{n-1} D_{n-1} \delta_1 = d_n.$$

We now take the operand to be W, and obtain

(3.21) $$\delta_n W = \Sigma \frac{e_{m_1}^{\mu_1} e_{m_2}^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1!\, \mu_2! \ldots} W, \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n.$$

Writing $W^{-1}\delta_n W = g_n$ where g_n is an integer, positive, zero, or negative.

(3.211) $$g_n = \Sigma \frac{e_{m_1}^{\mu_1} e_{m_2}^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1!\, \mu_2! \ldots}, \qquad \mu_1 m_1 + \mu_2 m_2 + \ldots = n,$$

(3.31) $$e_n = \Sigma(-)^{\Sigma\mu+1} \frac{(\Sigma\mu - 1)!\, n}{\mu_1!\, \mu_2! \ldots} g_{m_1}^{\mu_1} g_{m_2}^{\mu_2} \ldots, \qquad ,,$$

(3.41) $$g_n = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\, \mu_2! \ldots} \qquad ,,$$

where in the last formula $m_1, m_2, \ldots$ are restricted to be prime numbers.

(3.51) $$\Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\, \mu_2! \ldots} g_{m_1}^{\mu_1} g_{m_2}^{\mu_2} \ldots = \text{unity or zero},$$

according as n is or is not a prime; the summation being controlled by

$$\mu_1 m_1 + \mu_2 m_2 + \ldots = n.$$

(3.61) $$n g_n = e_n + e_{n-1} g_1 + e_{n-2} g_2 + \ldots + e_1 g_{n-1},$$

(3.71) $$g_n + g_{n-2} - g_{n-3} - g_{n-5} - \ldots - g_{n-p} - \ldots = 0,$$

where, in the last written difference equation, g_0 is to be regarded as having the value unity and p is an uneven prime.

(3.81) $$n g_n + (n-2) g_{n-2} - (n-3) g_{n-3} - \ldots - (n-p) g_{n-p} - \ldots = e_n,$$

where p is an uneven prime.

3.2. We, in the next place, take the operand to be W^2,

$$\delta_n W^2 = \Sigma \frac{(2e_{m_1})^{\mu_1} (2e_{m_2})^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1 ! \, \mu_2 ! \ldots} W^2, \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n.$$

Moreover $$\delta_n W^2 = \sum_{t=0}^{n} \delta_t W . \delta_{n-t} W = \sum_{t=0}^{n} g_t g_{n-t} W^2,$$

3.21. So that

$$\sum_{t=0}^{n} g_t g_{n-t} = \Sigma \frac{(2e_{m_1})^{\mu_1} (2e_{m_2})^{\mu_2} \ldots}{m_1^{\mu_1} m_2^{\mu_2} \ldots \mu_1 ! \, \mu_2 ! \ldots}, \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n.$$

I write $$\sum_{t=0}^{n} g_t g_{n-t} = G_n,$$

then

(3.22) $$2e_n = \Sigma(-)^{\Sigma\mu+1} \frac{(\Sigma\mu - 1) ! \, n}{\mu_1 ! \, \mu_2 ! \ldots} G_{m_1}^{\mu_1} G_{m_2}^{\mu_2} \ldots, \quad \mu_1 m_1 + \mu_2 m_2 + \ldots = n,$$

(3.23) $$G_n = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu) !}{\mu_1 ! \, \mu_2 ! \ldots} N_{m_1, 2}^{\mu_1} N_{m_2, 2}^{\mu_2} \ldots \quad ,,$$

(3.24) $$N_{n,2} = \Sigma(-)^{n+\Sigma\mu} \frac{(\Sigma\mu) !}{\mu_1 ! \, \mu_2 ! \ldots} G_{m_1}^{\mu_1} G_{m_2}^{\mu_2} \ldots \quad ,,$$

(3.25) $$n G_n = 2e_n + 2e_{n-1} G_1 + 2e_{n-2} G_2 + \ldots + 2e_2 G_{n-2}.$$

(3.26) $$G_n + N_{2,2} G_{n-2} - N_{3,2} G_{n-3} + N_{4,2} G_{n-4} - \ldots + (-)^{n-1} N_{n-1,2} G_1 + (-)^n N_{n,2} G_0 = 0,$$

where in the last written formula G_0 is to be taken as equal to unity.

The last formula but one establishes that n and G_n cannot both be uneven.

$$(3.27)\quad nG_n+(n-2)N_{2,2}G_{n-2}-(n-3)N_{3,2}G_{n-3}$$
$$+(n-4)N_{4,2}G_{n-4}-\ldots+(-)^n N_{n-2,2}G_2 = 2e_n.$$

3.28. The symmetric function relation

$$na_n-s_1a_{n-1}+s_2a_{n-2}-\ldots+(-)^n s_n = 0,$$

when transformed to operators, yields

$$nD_n-d_1D_{n-1}+d_2D_{n-2}-\ldots+(-)^n d_n = 0.$$

From the symmetric function relation before us we can proceed to another, viz :—

$$n(2^n)-s_2(2^{n-1})+s_4(2^{n-2})-\ldots+(-)^n s_{2n} = 0,$$

and this, when transformed to operators, yields

$$(3.281)\quad n\frac{(d_2^n)}{n!}-(d_2)\frac{(d_2^{n-1})}{(n-1)!}+(d_4)\frac{(d_2^{n-2})}{(n-2)!}-\ldots+(-)^n d_{2n} = 0.$$

When the operand is W,

$$\frac{(d_2^n)}{n!}W = (-)^{n+1}N_{2n,2} \text{ or } (-)^n N_{2n,2},$$

according as n is or is not greater than 2.

It may be observed that in the definition (of representation) of $N_{2n,2}$ zeros and primes both occur in the compositions and representations. Zero, in fact, plays the part of a prime, and is regarded as such.

The result of this convention is that

$$N_{0,2} = 1,$$

the representation being $0 = 0+0$.

The relation obtained from (3.281), with the operand W, may therefore be written

$$nN_{2n,2}+e_2N_{2n-2,2}+e_4N_{2n-4,2}+\ldots+e_{2n-6}N_{6,2}$$
$$-e_{2n-4}N_{4,2}-e_{2n-2}N_{2,2}-e_{2n}N_{0,2} = 0,$$

which has validity when n is zero or any positive integer.

Tables of g_n and G_n are attached.

TABLE OF g_n.

n	g_n	n	g_n	n	g_n	n	g_n	n	g_n
0	1	6	0	12	−3	18	12	24	21
1	0	7	2	13	−2	19	−11	25	40
2	−1	8	0	14	8	20	−4	26	−31
3	1	9	−2	15	1	21	25	27	−16
4	1	10	2	16	−8	22	4	28	82
5	−1	11	4	17	8	23	−24	29	14

TABLE OF $\sum_{t=0}^{n} g_t g_{n-t} = G_n$.

n	G_n	n	G_n	n	G_n	n	G_n	n	G_n
0	1	6	−1	12	−18	18	82	24	151
1	0	7	8	13	−12	19	−88	25	334
2	−2	8	−1	14	42	20	−54	26	−338
3	2	9	−10	15	4	21	188	27	−260
4	3	10	9	16	−58	22	18	28	760
5	−4	11	16	17	38	23	−248	29	120

4. It may be noticed that the Euler-Goldbach conjecture may be stated in a form which involves the properties of all integers, unity alone excepted :

"Every integer, greater than unity, is the arithmetic mean of two primes."

Even the exception of unity disappears if we regard zero as a prime, and it appears to present itself as a prime in the theory which is being developed.

We may avoid consideration of the even prime 2 by the statement : "Every even integer, greater than 4, is the sum of two uneven primes."

The number of representations of $2n$ as the sum of two primes has been denoted above by $N_{2n,2}$. It is clear that we may say

$N_{2n,2} = 1/n$ (sum of primes of form $2n-p$, where p is a prime).

If we specify a sum of two uneven primes, $N_{2n,2}$ remains unaltered for all values of $n > 2$.

4.1. We will now restrict consideration to uneven primes, and to that end construct a symmetric function which is derived from W by striking out all functions whose partitions involve the part 2, viz :—

$$V = 1 + (3) + (5) + (7) + (3^2) + (11) + (13) + (5, 3) + (17) + \ldots.$$

There is a function corresponding to every uneven integer n, but the weights of the functions, being potencies of uneven integers, may be even or uneven.

4.11. Here we have $D_n V = V$ or zero,

according as n is or is not an uneven prime.

We apply the operator formulae to the operand V, and write

$$V^{-1} d_n V = f_n.$$

Since
$$d_n V = \Sigma(-)^{n+\Sigma\lambda} \frac{(\Sigma\lambda-1)!\, n}{\lambda_3!\, \lambda_5!\, \lambda_7! \ldots} D_3^{\lambda_3} D_5^{\lambda_5} D_7^{\lambda_7} \ldots V$$

$$= \Sigma(-)^{n+\Sigma\lambda} \frac{(\Sigma\lambda-1)!\, n}{\lambda_3!\, \lambda_5!\, \lambda_7! \ldots} V,$$

where
$$3\lambda_3+5\lambda_5+7\lambda_7+\ldots = n,$$

$$f_n = \Sigma(-)^{n+\Sigma\lambda} \frac{(\Sigma\lambda-1)!\, n}{\lambda_3!\, \lambda_5!\, \lambda_7! \ldots}.$$

This number f_n is an integer because it is equal to the algebraic sum of those coefficients in the symmetric function formula

$$s_n = \Sigma(-)^{n+\Sigma\lambda} \frac{(\Sigma\lambda-1)!\, n}{\lambda_1!\, \lambda_2!\, \lambda_3! \ldots} a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3} \ldots,$$

for which the attached literal products involve only suffixes which are uneven primes.

4.12. We are now led, by an operator formula, to the difference equations

$$f_n - f_{n-3} - f_{n-5} - \ldots - f_{n-p} - \ldots = n \text{ or zero}$$

(p being an uneven prime),

according as n is or is not an uneven prime.

The initial values are $f_1 = f_2 = 0$, and we calculate the table.

TABLE OF f_n.

n	f_n	n	f_n	n	f_n	n	f_n	n	f_n	n	f_n	n	f_n	n	f_n
1	0	7	7	13	39	19	152	25	655	31	3286	37	15910	43	75809
2	0	8	8	14	35	20	175	26	897	32	4296	38	20406	44	
3	3	9	3	15	38	21	241	27	1164	33	5503	39	26640	45	
4	0	10	15	16	72	22	308	28	1463	34	7259	40	34703	46	
5	5	11	22	17	85	23	414	29	2001	35	9357	41	44856	47	
6	3	12	15	18	111	24	551	30	2538	36	12147	42	58481	48	

It will be noticed that f_n is never negative. For this discussion this fact gives f_n an advantage over e_n.

In what follows n will be taken to be greater than 2, unless the contrary is expressly stated.

It is a necessary consequence of the expression of f_n in the form of the above given finite series that f_n is divisible by n when n is an uneven prime.

4.13. THEOREM IX.—*When the symmetric function V is expressed in terms of $s_2, s_3, s_4, \ldots$, the functions s_2, s_4 do not present themselves.*

We have also

$$V = \prod_{k=1}^{\infty} \exp(-)^{k+1} \frac{1}{k} f_k s_k = \exp \sum_{k=1}^{\infty} (-)^{k+1} \frac{1}{k} f_k s_k,$$

$$\exp \sum_{k=1}^{\infty} (-)^{k+1} \frac{1}{k} f_k \alpha_k = 1+\alpha^3+\alpha^5+\alpha^7+\alpha^{11}+\ldots+\alpha^p+\ldots,$$

where p is an uneven prime.

4.14. In V, if V_σ be the sum of the symmetric functions of weight σ

$$D_\sigma V_\sigma = \Sigma \frac{\partial_{s_1}^{\sigma_1} \partial_{s_2}^{\sigma_2} \partial_{s_3}^{\sigma_3} \ldots}{\sigma_1!\, \sigma_2!\, \sigma_3! \ldots} V_\sigma = V_\sigma \text{ or zero},$$

according as σ is or is not an uneven prime (*cf.* 1.7 *ante*).

THEOREM X.—*If we take the sum of the symmetric functions of weight σ which correspond to partitions into uneven prime parts, and express it in terms of $s_2, s_3, s_4, \ldots$, the sum of the numerical coefficients is unity or zero, according as σ is or is not an uneven prime.*

4.2. For the discussion of the representations of an integer as the sum of m uneven primes we have

$$d_n V^m = m f_n V^m,$$

and we derive from (D′)

$$D_n V^m = \Sigma(-)^{n+\Sigma\lambda} \frac{(mf_3)^{\lambda_3} (mf_5)^{\lambda_5} (mf_6)^{\lambda_6} \ldots}{3^{\lambda_3} 5^{\lambda_5} 6^{\lambda_6} \ldots \lambda_3!\, \lambda_5!\, \lambda_6! \ldots} V^m;$$

and thence the number of representations of the integer n is

$$N_{n,m} = \Sigma(-)^{n+\Sigma\lambda} \frac{(mf_3)^{\lambda_3} (mf_5)^{\lambda_5} (mf_6)^{\lambda_6} \ldots}{3^{\lambda_3} 5^{\lambda_5} 6^{\lambda_6} \ldots \lambda_3!\, \lambda_5!\, \lambda_6! \ldots},$$

where $$3\lambda_3+5\lambda_5+6\lambda_6+\ldots = n.$$

4.21. In particular

$$N_{n,1} = \Sigma(-)^{n+\Sigma\lambda} \frac{f_3^{\lambda_3} f_5^{\lambda_5} f_6^{\lambda_6} \dots}{3^{\lambda_3} 5^{\lambda_5} 6^{\lambda_6} \dots \lambda_3 ! \, \lambda_5 ! \, \lambda_6 ! \dots} = \text{unity or zero,}$$

according as n is or is not an uneven prime.

As an example,

$$N_{10,m} = -\tfrac{1}{10} m f_{10} + \tfrac{1}{21} m^2 f_3 f_7 + \tfrac{1}{50} m^2 f_5^2 = 3 \binom{m}{2},$$

the representations being

$$7+3+\text{zero } m-2 \text{ times},$$

$$5+5+\text{zero } m-2 \text{ times},$$

yielding

$$2\binom{m}{2} + \binom{m}{2} = 3\binom{m}{2}.$$

4.22. We find that

$$V^{-2} D_m V^2 = \Sigma(-)^{\Sigma\lambda} \frac{(2f_3)^{\lambda_3} (2f_5)^{\lambda_5} (2f_6)^{\lambda_6} \dots}{3^{\lambda_3} 5^{\lambda_5} 6^{\lambda_6} \dots \lambda_3 ! \, \lambda_5 ! \, \lambda_6 ! \dots},$$

where

$$3\lambda_3 + 5\lambda_5 + 6\lambda_6 + \dots = 2n,$$

so that

$$N_{2n,2} = \Sigma(-)^{\Sigma\lambda} \frac{(2f_3)^{\lambda_3} (2f_5)^{\lambda_5} (2f_6)^{\lambda_6} \dots}{3^{\lambda_3} 5^{\lambda_5} 6^{\lambda_6} \dots \lambda_3 ! \, \lambda_5 ! \, \lambda_6 ! \dots},$$

a fourth expression for $N_{2n,2}$.

4.221. The verification for $n = 5$, 6 is

$n = 5$,

$$N_{10,2} = -\frac{2f_{10}}{10} + \frac{4f_3 f_7}{21} + \frac{4f_5^2}{50} = -3+6 = 3,$$

$n = 6$,

$$N_{12,2} = -\frac{2f_{12}}{12} + \frac{4f_6^2}{72} + \frac{4f_3 f_9}{27} + \frac{4f_5 f_7}{35} - \frac{8f_3^2 f_6}{108} + \frac{16 f_3^4}{81.24} = -2+4 = 2.$$

4.23. A fifth expression, derived from the operator relation

$$\frac{1}{n}(d_2^n) = \sum_0^{2n} D_s D_{n-s},$$

is

$n > 2$,

$$N_{2n,2} = \Sigma(-)^{\Sigma\lambda+1} \frac{f_6^{\lambda_3} f_8^{\lambda_4} f_{10}^{\lambda_5} \dots}{3^{\lambda_3} 4^{\lambda_4} 5^{\lambda_5} \dots \lambda_3 ! \, \lambda_4 ! \, \lambda_5 ! \dots},$$

where

$$3\lambda_3 + 4\lambda_4 + 5\lambda_5 + \dots = n.$$

4.231. The verification for $n = 5$, 6 is

$n = 5$, $\qquad N_{10,2} = \frac{1}{5}f_{10} = 3,$

$n = 6$, $\qquad N_{12,2} = \frac{1}{6}f_{12} - \frac{1}{18}f_6^2 = 2.$

4.24. In analogy with what was found in the case of the operand W, we have a sixth expression, viz :—

$$2N_{2n,2} = \Sigma(-)^{\Sigma\lambda} \frac{(2f_3)^{\lambda_3}(2f_5)^{\lambda_5}(2f_6)^{\lambda_6}\dots}{3^{\lambda_3}5^{\lambda_6}6^{\lambda_6}\dots\lambda_3!\,\lambda_5!\,\lambda_6!\dots},$$

where $$3\lambda_3+5\lambda_5+6\lambda_6+\dots = 2n,$$

and we retain, in the summation, only those terms which do not involve f's with even suffixes exclusively.

E.g. $$N_{10,2} = \tfrac{1}{2}\left\{\frac{4f_3f_7}{21} - \frac{8f_3^2f_4}{72} + \frac{16f_2^2f_3^2}{144}\right\} = \tfrac{1}{2}(4-2+4) = 3.$$

4.3. When we utilize the operator δ_n with the operand W, we write

$$V^{-1}\delta_n V = i_n,$$

then $$i_n = \Sigma \frac{f_3^{\lambda_3}f_5^{\lambda_5}f_6^{\lambda_6}\dots}{3^{\lambda_3}5^{\lambda_5}6^{\lambda_6}\dots\lambda_3!\,\lambda_5!\,\lambda_6!\dots},$$

where $$3\lambda_3+5\lambda_5+6\lambda_6+\dots = n.$$

(4.31) $$f_n = \Sigma(-)^{\Sigma\mu+1}\frac{(\Sigma\mu-1)!\,n}{\mu_1!\,\mu_2!\dots} i_{m_1}^{\mu_1} i_{m_2}^{\mu_2}\dots,$$

where $$\mu_1m_1+\mu_2m_2+\dots = n.$$

(4.32) $$i_n = \Sigma(-)^{n+\Sigma\mu}\frac{(\Sigma\mu)!}{\mu_1!\,\mu_2!\dots},$$

where $\mu_1m_1+\mu_2m_2+\dots = n$ and m_1, m_2, ... are restricted to be uneven primes.

(4.33) $$\Sigma(-)^{n+\Sigma\mu}\frac{(\Sigma\mu)!}{\mu_1!\,\mu_2!\dots} i_{m_1}^{\mu_1} i_{m_2}^{\mu_2}\dots$$

(where $\mu_1m_1+\mu_2m_2+\dots = n$)

is unity or zero according as n is or is not an uneven prime.

(4.34) $$ni_n = f_n+f_{n-1}i_1+f_{n-2}i_2+\dots+f_1i_{n-1},$$

(4.35) $$i_n = i_{n-3}+i_{n-5}+i_{n-7}+\ldots+i_{n-p}+\ldots,$$

where p denotes an uneven prime and $i_0 = 1$.

(4.36) $$ni_n-(n-3)i_{n-3}-(n-5)i_{n-5}-\ldots-(n-p)i_{n-p}-\ldots = f_n.$$

4.4. With the operand V^2,

$$\delta_n V^2 = \Sigma \frac{(2f_3)^{\lambda_3}(2f_5)^{\lambda_5}(2f_6)^{\lambda_6}\ldots}{3^{\lambda_3}5^{\lambda_5}6^{\lambda_6}\ldots\lambda_3!\,\lambda_5!\,\lambda_6!\ldots} V^2,$$

where $$3\lambda_3+5\lambda_5+6\lambda_6+\ldots = n.$$

4.41. Moreover

$$\delta_n V^2 = \sum_{t=0}^{n} \delta_t V.\delta_{n-t}V = \sum_{t=0}^{n} i_t i_{n-t} = I_n \text{ suppose,}$$

$$2f_n = \Sigma(-)^{n+\Sigma\mu}\frac{(\Sigma\mu-1)!\,n}{\mu_1!\,\mu_2!\ldots} I_{m_1}^{\mu_1} I_{m_2}^{\mu_2}\ldots,$$

where $$\mu_1 m_1+\mu_2 m_2+\ldots = n.$$

(4.42) $$I_n = \Sigma(-)^{n+\Sigma\mu}\frac{(\Sigma\mu)!}{\mu_1!\,\mu_2!\ldots} N_{m_1,2}N_{m_2,2}\ldots,$$

where $$\mu_1 m_1+\mu_2 m_2+\ldots = n.$$

(4.43) $$N_{n,2} = \Sigma(-)^{n+\Sigma\mu}\frac{(\Sigma\mu)!}{\mu_1!\,\mu_2!\ldots} I_{m_1}^{\mu_1} I_{m_2}^{\mu_2}\ldots.$$

(4.44) $$nI_n = 2f_n+2f_{n-1}I_1+2f_{n-2}I_2+\ldots+2f_2 I_{n-2}.$$

(4.45)

$$I_n-N_{3,2}I_{n-3}+N_{4,2}I_{n-4}-\ldots+(-)^{n-1}N_{n-1,2}I_1+(-)^n N_{n,2}I_0 = 0,$$

where, in the last written formula, I_0 is to be taken equal to unity, and, in $N_{n,2}$, the representations involve zeros and uneven primes only.

The last formula but one establishes that n and I_n cannot both be uneven. Tables of i_n and I_n are attached.

TABLE OF i_n.

n	i_n	n	i_n	n	i_n	n	i_n	n	i_n
0	1	6	1	12	3	18	21	24	103
1	0	7	1	13	7	19	28	25	125
2	0	8	2	14	7	20	33	26	169
3	1	9	1	15	8	21	47	27	220
4	0	10	3	16	14	22	58	28	277
5	1	11	4	17	15	23	76	29	373

TABLE OF $I_n = \sum_{t=0}^{n} i_t i_{n-t}$.

n	I_n	n	I_n	n	I_n	n	I_n	n	I_n
0	1	6	3	12	11	18	99	24	607
1	0	7	2	13	26	19	140	25	764
2	0	8	6	14	29	20	169	26	1059
3	2	9	4	15	34	21	252	27	1424
4	0	10	9	16	62	22	322	28	1845
5	2	11	14	17	68	23	430	29	2546

TABLE OF $N_{n,2}$ (2 included in the primes).

n	N	n	N	n	N	n	N	n	N	n	N	n	N	n	N	n	N	n	N
0	1	10	3	20	4	30	6	40	6	50	8	60	12	70	10	80	8	90	16
1	0	11	2	21	2	31	4	41	2	51	0	61	4	71	2	81	2	91	2
2	2	12	2	22	5	32	4	42	8	52	6	62	5	72	12	82	7	92	8
3	2	13	4	23	2	33	2	43	4	53	2	63	2	73	4	83	2	93	0
4	1	14	3	24	6	34	7	44	6	54	8	64	10	74	9	84	16	94	9
5	4	15	2	25	2	35	0	45	2	55	2	65	0	75	2	85	2	95	0
6	1	16	4	26	5	36	8	46	7	56	6	66	12	76	10	86	9	96	14
7	4	17	2	27	0	37	2	47	2	57	0	67	2	77	0	87	0	97	2
8	2	18	4	28	4	38	3	48	10	58	7	68	4	78	14	88	8	98	6
9	2	19	4	29	2	39	2	49	2	59	2	69	2	79	2	89	2	99	2

5. What has been written above appears to include the material available, from the adopted standpoint, for the examination of the Euler-Goldbach conjecture.

It has not appeared, so far, that the results that have been reached are incompatible with the assumption that for a particular value of n

$$N_{2n,\,2} = 0.$$

In general, the results that have been found with the operand V are of most promise for dealing with the subject.

We have established the formula

$$\Sigma(-)^{\Sigma\lambda} \frac{f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} \dots}{1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots \lambda_1!\, \lambda_2!\, \lambda_3! \dots} = 0, \qquad \text{(X)}$$

where $$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots = 2n;$$

$$\Sigma(-)^{\Sigma\lambda} \frac{f_2^{\lambda_1} f_4^{\lambda_2} f_6^{\lambda_3} \dots}{1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots \lambda_1!\, \lambda_2!\, \lambda_3! \dots} = -N_{2n\ 2}, \qquad \text{(Y)}$$

where $$\lambda_1+2\lambda_2+3\lambda_3+\ldots = n;$$

and $$f_n-f_{n-3}-f_{n-5}-\ldots-f_{n-p}-\ldots = n \text{ or zero,} \qquad (Z)$$

according as n is or is not a prime (zero counting as a prime).

The formula (Y) is obtained by doubling the subscripts of $f_1, f_2, f_3, \ldots$ in the formula X, so far as the left-hand sides are concerned.

In view of (X) and (Z), is it possible to have the left-hand side of (Y) equal to zero for any particular value of n?

It is clear that, if we had the difference equation

$$f_{2n}-f_{2n-6}-f_{2n-10}-\ldots-f_{2n-2p}-\ldots = n \text{ or zero,}$$

according as n is or is not a prime, the left-hand side of (Y) would be zero for every value of n. This difference equation does not exist as a matter of course from what we know concerning the values of $N_{2n,2}$ and for other reasons.

It does not appear that any conclusion can be drawn, at present, with regard to the conjecture.

THE NULL PENCIL OF BINARY QUARTICS.

By R. VAIDYANATHASWAMY.

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THE particular variety of pencils of quartics in which any two members are apolar (which I call the "Null pencil of quartics") does not seem to have received any attention. I set down here some of its fundamental properties, and point out the specialities which it introduces into the usual geometric representations of quartic involutions. I also extend two of its properties to forms of higher order, the more striking of these extensions being :

The Jacobian ("functional determinant") of a Null system of binary 2n-ics is a perfect square; where by a Null system is meant a system of the form $\lambda_1\phi_1+\lambda_2\phi_2+\ldots+\lambda_n\phi_n$ (the ϕ's being linearly independent $2n$-ics), possessing the property that *any* two of its members are mutually apolar.

1. By a "Null pencil of quartics" we shall understand a pencil $\lambda a_x^4+\mu b_x^4$ in which *any two* members are apolar. The necessary and sufficient conditions for this are :

$$(a_x^4,\ a_x^4)^4 = (a_x^4,\ b_x^4)^4 = (b_x^4,\ b_x^4)^4 = 0. \tag{1}$$

In particular, it follows from this definition that every member of a Null pencil is self-apolar (*i.e.* apolar to itself); conversely a pencil containing *three* self-apolar quartics is composed wholly of such quartics, and is a Null pencil. For if

$$F(\lambda,\ \mu) = \left((\lambda a_x^4+\mu b_x^4),\quad (\lambda a_x^4+\mu b_x^4)\right)^4$$

vanishes for three values of $\lambda:\mu$, it must vanish identically, and equations (1) result.

The invariant j of a quartic being of the third degree in the coefficients, it follows that any pencil of quartics must contain three members with vanishing j. Since, however, i vanishes for all members of a Null pencil,

it follows that a Null pencil must contain three members for which $i = j = 0$, that is to say, three members possessing a cubed linear factor. Hence :

THEOREM (A).—*The Jacobian of a Null pencil is the square of a cubic f.*

Now it is known that there are *five* pencils of quartics with a given Jacobian.* We can *directly* specify the five pencils in the particular case in which the given Jacobian is the square of a cubic f. Supposing the linear factors of f to be X, Y, Z, these five pencils are (1) the Null pencil with which we started, (2) the singular pencil $XYZa_x$, a_x being an arbitrary linear form, (3) the three singular pencils of the type $\lambda XY^3+\mu XZ^3$. It would appear, therefore, that the Null pencil is the only non-singular pencil which has f^2 for its Jacobian [this is a particular case of the fact that only one non-singular pencil of binary n-ics is determined by a Jacobian $(2n-2)$-ic with only three distinct linear factors, the multiplicity of none of these being greater than $n-1$].†

This suggests the converse of Theorem (A), namely :

THEOREM (*a*).—*A non-singular pencil of quartics is necessarily a Null pencil if its Jacobian is a perfect square.*

To prove this we have only to remark that the pencil being non-singular will necessarily have to contain three members, which possess cubed linear factors and are therefore self-apolar. From this the truth of the theorem follows immediately.

The Null pencil, the Jacobian of which is f^2, is thus *uniquely* determined by the cubic $f(=XYZ)$. If X^3X', Y^3Y', Z^3Z' (where X', Y', Z' are linear forms) belong to the Null pencil, it would *therefore* follow that $X'Y'Z'$ is a covariant of f, and hence its cubicovariant. To verify this, let the linear relation between X, Y, Z be $X+Y+Z=0$, so that the linear factors of the cubicovariant are (except for numerical multipliers) $Y-Z$, $Z-X$, $X-Y$. Since

$$\Sigma X^3(Y-Z) \equiv -(Y-Z)(Z-X)(X-Y)(X+Y+Z) = 0,$$

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* Stephanos, *Mem. de l'Institut*, 27 (1883), 2 ; Meyer, *Apolarität*, § 29. From the geometrical representation of § 3 below, it will be seen that this follows from the fact that six planes in four dimensions have five common transversals.

† Cf. *Proc. Camb. Phil. Soc.*, 22 (1924), 51.

it follows that the three quartics $X^3(Y-Z)$, $Y^3(Z-X)$, $Z^3(X-Y)$ belong to a pencil which is precisely the Null pencil with the Jacobian

$$f^2 = X^2Y^2Z^2.$$

2. THEOREM (B).—*The Null pencil defined by the cubic* $f(=XYZ)$ *consists of all quartics which are apolar to each of the three quartics*

$$Y^2Z^2,\ Z^2X^2,\ X^2Y^2.$$

For since there is no linear relation between Y^2Z^2, Z^2X^2, X^2Y^2, the quartics apolar to them generate a pencil Γ. Now that member of Γ which contains the factor X must (since it is to be apolar to X^2Y^2 and X^2Z^2) contain the factor X^3. Thus Γ contains three members with the cubed linear factors X^3, Y^3, Z^3 and is therefore the Null pencil defined by the cubic $f = XYZ$.

THEOREM (C).—*The Null pencil of quartics, defined by the cubic* f, *consists of the Jacobians of* f *with all cubics* ϕ *apolar to* f.

For any cubic ϕ apolar to $f(=XYZ)$ with $X+Y+Z=0$ is a linear combination of X^3, Y^3, Z^3. Now the Jacobian of XYZ and X^3 differs only by a numerical factor from $X^3(Y-Z)$. Hence the Jacobian of f and ϕ is of the form

$$aX^3(Y-Z)+bY^3(Z-X)+cZ^3(X-Y),$$

and is therefore a member of the Null pencil defined by f.

As a corollary of this theorem we have:

Two Null pencils of quartics defined respectively by the cubics f *and* ϕ *do or do not have a common member, according as* f *and* ϕ *are or are not apolar.*

3. *Geometrical representation in four dimensions.*—The known geometrical representation of binary quartics by points in a four-fold space, with a fundamental norm-curve, furnishes simple proofs of the above theorems. Thus, let the point $(X_0X_1X_2X_3X_4)$ of S_4 be made to correspond with the binary quartic $(X_0X_1X_2X_3X_4)(x_1, x_2)^4$. The points which correspond to quartics which are perfect fourth powers will then trace the norm-curve

$$N_4: \quad \rho X_r = t^{4-r} \quad (r = 0, 1, \ldots, 4),$$

where t is a parameter. It can then be shown (either analytically or purely from binary considerations of apolarity) that there are four points $t_1 t_2 t_3 t_4$ on N_4, the osculating S_3's at which pass through a given point X, and that the $S_3 t_1 t_2 t_3 t_4$ is the locus of points which correspond to quartics *apolar* to the quartic "X." The correspondence between the point X and the $S_3 t_1 t_2 t_3 t_4$ is the "polarity" of the norm-curve.

Further, in this correspondence *self-apolar* quartics correspond to points lying on the quadric

$$N_2: \quad X_0 X_4 - 4X_1 X_3 + 3X_2^2 = 0.$$

It is well known (and may be immediately verified) that the polarity of this quadric is identical with the polarity of the norm-curve, that is to say, conjugate points with respect to N_2 also correspond to apolar quartics. On this account the quadric N_2 may be called "the *equivalent* quadric of the norm-curve N_4," and *vice versa*, the curve N_4 "an *equivalent* norm-curve of the quadric N_2."

Now a pencil of quartics Γ corresponds to a line γ in S_4; the Jacobian of Γ corresponds to the six points on N_4, the oscillating planes at which meet γ. If Γ is a Null pencil, then γ is a *generating line* of the equivalent *quadric*. The Theorems (A), (a) now take the geometrical forms :

THEOREM (A_1).—*Every generating line of the equivalent quadric meets three tangent lines of the norm-curve;* or, to put it differently, *any generating line of a quadric in S_4 meets three tangent lines of any equivalent norm-curve.*

THEOREM (a_1).—*Any line which meets three tangents of a norm-curve N_4 is a generating line of its equivalent quadric.*

Since (as may be easily seen) the tangents of N_4 are generating lines of its equivalent quadric, the truth of (a_1) is geometrically obvious. To prove (A_1), suppose that the generating line γ of the quadric meets the osculating plane P_α at the point α of the norm-curve. Now P_α is the polar region (with respect to the norm-curve and therefore also with respect to the equivalent quadric) of the tangent T_α at the point α of the curve. Hence P_α touches the equivalent quadric along T_α. Since γ meets P_α, it must meet that part of P_α which lies in the quadric, that is to say, γ meets T_α. Since meeting T_α is equivalent to meeting two consecutive P_α's it follows that γ meets *three* tangents of the norm-curve.

The geometrical proof of Theorem (B) turns upon the evident fact that the unique line-transversal of three tangents to a norm-curve is the intersection of the three S_3's determined by every two of them.

4. We shall now examine, with reference to the Null pencil, the three usual modes of representing pencils of quartics geometrically by means of the twisted cubic.

It is known (Meyer, *Apolarität*, § 19) that the pairs of lines intersecting the tangent-quadruples of an involution of the fourth order on the space-cubic generate one regulus of a quadric Q (which may be called the "Sturm quadric" of the involution), the other regulus of which belongs to the complex of the curve. If, now, the involution be a Null involution, all the tangent-quadruples will be equianharmonic, and therefore their transversal pairs will coalesce into lines belonging to the complex of the curve. Hence, for this case, *both* reguli of the quadric Q will have to belong to the complex of the curve. This is, however, not possible unless Q is two coincident planes. Hence:

The Sturm quadric of a Null involution on the twisted cubic is a pair of coincident planes. Conversely, the tangent-quadruples of the curve meeting the complex lines lying in any given plane P constitute a Null involution, the Jacobian of which is the square of the cubic cut out of the curve by P.

It will be noticed that the latter part of the theorem is the direct geometrical expression of Theorem (C).

It will be sufficient merely to state the specializations which occur in the two other usual modes of representing quartic pencils on the twisted cubic. A pencil Γ of quartics may be represented (i) as the set of tetrahedra inscribed in a twisted cubic C, and self-polar with respect to a quadric Σ "in-polar" to the curve (that is, apolar when considered as a class-quadric to all quadrics containing the curve), (ii) as the set of ∞^1 tetrahedra inscribed in C, and circumscribed to a second cubic curve C_1. If Γ is a Null pencil, the Jacobian of which corresponds to the points X, Y, Z of the curve (the points $X'Y'Z'$ on the curve being the cubi-covariant points of XYZ), then:

(1) The quadric Σ touches the curve at X, Y, Z; the poles with respect to Σ of the osculating planes at X, Y, Z are the points X', Y', Z'. Further, Σ cuts the plane at XYZ in the polar conic of its nul point with respect to the triangle XYZ.

(2) The four other quadrics corresponding to the four singular co-Jacobian pencils of Γ are, as loci, three cones touching the curve at two of the points XYZ and having their vertex at the third, and the squared plane XYZ; and as envelopes, the three squared points X, Y, Z and a conic which has XYZ for a self-polar triangle.

(3) The twisted cubic C_1 touches the lines $X'X$, $Y'Y$, $Z'Z$ at the points X', Y', Z' respectively, and osculates the osculating planes at X, Y, Z to C. The osculating planes at X', Y', Z' to C_1 contain the tangents at X, Y, Z to C.

This relation between the curves C and C_1 may be compared with the corresponding relation which obtains in the case (discussed in detail by Meyer *Apolarität*, § 177) in which Γ contains two perfect fourth powers.

5. It is well known that the conics of a plane which are apolar to a fundamental class-conic S, and which cut out the quadruples of an involution on S, belong to a pencil. The quadrangle which determines this pencil is *self-polar* with respect to S (that is to say, each of the four points is the centre of perspective of the triangle formed by the other three and its polar triangle). This quadrangle, together with its polar quadrilateral with respect to S, constitutes a Desargues configuration $[D]$ of *ten* points and *ten* lines, out of which there can be formed *five* self-polar quadrangles (and quadrilaterals) of S. These five quadrangles determine (according to Stephanos*) *five co-Jacobian pencils of quartics* on S.

Now one of the co-Jacobian pencils of the Null pencil defined by the cubic $f = XYZ$ is the singular pencil $XYZ\alpha_x$. The quadrangle corresponding to this pencil is obviously the quadrangle $XYZO$, where X, Y, Z are points on S corresponding to the linear forms XYZ, and O is the centre of perspective of XYZ and xyz (x, y, z being the respective tangents at X, Y, Z). The corresponding Desargues configuration $[D]$ consists therefore of the ten points :

$$X,\ Y,\ Z,\ O,\ (yz),\ (zx),\ (xy),\ (x, YZ),\ (y, ZX),\ (z, XY).$$

The five self-polar quadrangles formed out of these are :

(1) $[XYZO]$,

(2) $[X,\ (yz),\ (y, ZX),\ (z, XY)]$,

(3) $[Y,\ (zx),\ (z, XY),\ (x, YZ)]$,

(4) $[Z,\ (xy),\ (x, YZ),\ (y, ZX)]$,

(5) $[O,\ (yz),\ (zx),\ (xy)]$.

* "Les faisceaux de formes binaires", *Mémoires de l'Institut*, 27 (1883), 1, *loc. cit.*

The quadrangles (2), (3), (4) determine the three singular pencils of the type $\lambda XY^3+\mu XZ^3$; the quadrangle (5) determines the Null pencil.

The conclusion may be stated thus:

A Desargues configuration $[D]$, *self-reciprocal with respect to a conic* S, *determines a Null pencil (and its co-Jacobian pencils) on* S, *if and only if three of its points* X, Y, Z *(and therefore also three of its lines) belong to* S; *when this happens, the Null pencil is determined by that particular self-polar quadrangle of* $[D]$ *which has* XYZ *for its harmonic triangle.*

6. The fact was already mentioned that only one non-singular pencil of binary n-ics is determined by a Jacobian of the form $X^pY^qZ^r$ (where X, Y, Z are linear forms, $p+q+r=2n-2$ and p, q, $r<n$). We can specify this unique non-singular pencil in a manner completely analogous to Theorem (B):

THEOREM (B′).—*The unique non-singular pencil of binary n-ics defined by the Jacobian* $X^pY^qZ^r$ *is composed of all the n-ics which are apolar to each of the three forms*

$$Y^{n-q}Z^{n-r}, \quad Z^{n-r}X^{n-p}, \quad X^{n-p}Y^{n-q}.$$

To prove this we notice that the n-ics apolar to the three forms constitute a linear ∞^κ-system, where

$$\kappa = 6n-3-2(p+q+r)-2n = 1;$$

that is, the system of apolar n-ics is a pencil Γ. Now, from a known theorem in Apolarity,* the n-ic $X^{p+1}a_x^{n-p-1}$ is necessarily apolar to X^{n-p} and therefore also to each of the forms $X^{n-p}Y^{n-q}$, $X^{n-p}Z^{n-r}$ (whatever form of order $n-p-1$ is represented by a_x^{n-p-1}). Further, the condition that $X^{p+1}a_x^{n-p-1}$ is apolar to $Y^{n-q}Z^{n-r}$ gives $n-p-1$ equations which exactly suffice for determining a_x^{n-p-1} apart from a numerical multiplier. The form $X^{p+1}a_x^{n-p-1}$ thus determined belongs to Γ, so that it follows that Γ contains three members with the respective factors X^{p+1}, Y^{q+1}, Z^{r+1}. Thus Γ is identical with the (non-singular) pencil determined by the Jacobian $X^pY^qZ^r$—which proves the theorem.

This theorem may be interpreted geometrically in space of n dimensions in the same manner as was already noticed for Theorem (B) (§3).

7. The geometrical Theorem (A_1) can be extended without any difficulty to a space S_{2n} of $2n$ dimensions, and thereby be made to yield an

* Grace and Young, *Algebra of Invariants*, 225, 227.

algebraic extension of Theorem (A). A norm-curve N_{2n} in S_{2n} determines—as is well known—a polarity of the "quadric" variety, which corresponds to the apolar relation between binary $2n$-ics. The quadric which produces the same polarity—that is to say, the "equivalent quadric"—contains the norm-curve, and is uniquely determined by the further condition that it contains all the $(n-1)$-dimensional osculating regions of the curve. Now any region P_{n-1} of $n-1$ dimensions in S_{2n} is met by $n(n+1)$ osculating P_n's of the curve.* Suppose now that P_{n-1} is a generating region of the equivalent quadric, and that it meets O_n^α, the osculating region of n dimensions at the point α of the norm-curve. Now O_n^α is the polar region (with respect to the norm-curve and therefore also with respect to the equivalent quadric) of O_{n-1}^α; hence since O_{n-1}^α is a generating region, it follows that O_n^α *touches* the equivalent quadric along O_{n-1}^α. Since P_{n-1} lies wholly in the quadric and meets O_n^α, it must meet that part of O_n^α which lies in the quadric; that is, P_{n-1} meets O_{n-1}^α. Since meeting O_{n-1}^α is equivalent to meeting two consecutive O_n^α's we have finally the theorem:

THEOREM (A_1').—*Any generating region (of the maximum number of dimensions $n-1$) of a quadric in S_{2n} meets $n(n+1)/2$ osculating regions of $n-1$ dimensions of any equivalent norm-curve.*

Since the number $n(n+1)/2$ is just one greater than the number of points necessary to determine a quadric in a P_{n-1} which meets the $n(n+1)/2$ osculating regions, we may conclude that the converse of this theorem is also true †; namely:

THEOREM (a_1').—*Any P_{n-1} which meets $n(n+1)/2$ osculating regions of $n-1$ dimensions of a norm-curve N_{2n} is a generating region of its equivalent quadric.*

In the algebraical intepretation a region P_{n-1} in S_{2n} corresponds to a linear ∞^{n-1}-system of binary $2n$-ics; the linear factors of the Jacobian ‡

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* Veronese, *Math. Annalen*, 19 (1881), 161–234.

† This method of proving the converse has been suggested by a referee.

‡ The Jacobian of a ∞^κ-system $\lambda_0\phi_0 + \lambda_1\phi_1 + \ldots + \lambda_\kappa\phi_\kappa$ of binary m-ics is

$$\begin{vmatrix} \frac{d^\kappa \phi_0}{dx_1^\kappa} & \frac{d^\kappa \phi_0}{dx_1^{\kappa-1}dx_2} & \cdots & \frac{d^\kappa \phi_0}{dx_2^\kappa} \\ \vdots & & & \\ \frac{d^\kappa \phi_\kappa}{dx_1^\kappa} & \cdots & \cdots & \frac{d^\kappa \phi_\kappa}{dx_2^\kappa} \end{vmatrix}$$

If $\phi_p = a_{p_x}^m$ $(p = 0, 1, \ldots, \kappa)$, this determinant is a numerical multiple of

$$(\alpha_0\alpha_1)\ldots(\alpha_0\alpha_\kappa)(\alpha_1\alpha_2)\ldots(\alpha_{\kappa-1}\alpha_\kappa)\,\alpha_{0_x}^{m-\kappa}\ldots\alpha_{\kappa_x}^{m-\kappa}.$$

of the system correspond to the points a of the norm-curve, such that O_n^a meets P_{n-1}. A generating region P_{n-1} of the equivalent quadric corresponds to a Null system of binary $2n$-ics—that is, a system any two members of which are apolar, and the dimensions of which are the greatest possible consistent with this condition. Hence the algebraic extension of Theorem (A) derived from A_1' is :

THEOREM (A′).—*The Jacobian of a Null system of binary* $2n$*-ics is the square of a form* f *of order* $n(n+1)/2$.

The algebraic form of (a_1') would be :

THEOREM (a').—*A linear* ∞^{n-1}*-system of binary* $2n$*-ics, the Jacobian of which is a perfect square, is necessarily a Null system (in accordance with our definition), provided it is not singular.*

[If a be a root of the Jacobian of a linear ∞^{κ}-system of binary m-ics, then there exists a form $p_x^{m-\kappa-1}$, such that $(ax)^{\kappa+1}p_x^{m-\kappa-1}$ belongs to the system. If there exists more than one such form, a may be said to be a *singular* point of the system. A system is *singular* if it possesses one or more singular points.

A singular point is necessarily a repeated root of the Jacobian. The converse is not true, that is, a repeated root of the Jacobian is not necessarily a singular point of the system. It is, however, true that a repeated root of the Jacobian J is a singular point of *one at least* of the co-Jacobian systems determined by J.]

The actual number of Null systems with a given Jacobian is the number of P_{n-1}'s which meet $n(n+1)/2$ arbitrary P_{n-1}'s in S_{2n}. This number has not been evaluated for the general case. For $2n = 6$ its value is 16 (Segre, *Ency. Math. Wiss.*, III_2, 7, 817, note 157).

NOTE ON THE CONVERSE OF ABEL'S CONTINUITY THEOREM

By A. E. INGHAM.

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1. MR. LITTLEWOOD's well known "Converse of Abel's theorem"* asserts that *if* $\sum_1^\infty a_n e^{-\lambda_n s}$ *is a Dirichlet's series for which* $\lambda_n \sim \lambda_{n-1}$ *and* $a_n = O\{(\lambda_n - \lambda_{n-1})/\lambda_n\}$, *and* $\Sigma a_n e^{-\lambda_n s}$ *(then certainly convergent for* $s > 0$*)* $\to A$ *as* $s \to +0$, *then* Σa_n *is convergent to sum* A. Mr. Littlewood showed by an example† that in this theorem the restriction on the order of magnitude of $|a_n|$ is the minimum possible of its kind, and further, in the special case $\lambda_n = n$ (power series), *conjectured*‡ that if we relax this condition, however little, we cannot infer even *summability by Cesàro's means* from the modified hypotheses. More precisely: *if* $\chi(n)$ *is any function which tends to infinity with* n *(however slowly),*

$$a_n = O[\chi(n)/n] \quad \textit{and} \quad \Sigma a_n e^{-ns} \to A$$

do not necessarily involve summability (C, k) *of* Σa_n, *however large* k *may be.*

Mr. Littlewood verified his conjecture for the particular function $\chi(n) = (\log n)^a$ $(a > 0)$.‡ The object of the present note is to prove it in its general form,§ or rather to establish the natural extension of this result to Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$ and Riesz's methods of summation by typical means of type λ. When $\lambda_n = n$, these methods are equivalent to Cesàro's,‖ so that Mr. Littlewood's assertion will in fact follow as a

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* J. E. Littlewood, "The Converse of Abel's Theorem on Power Series", *Proc. London Math. Soc.* (2), 9 (1911), 434–448 (438, Theorem B).

† *Loc. cit.*, 444 (Theorem C).

‡ *Loc. cit.*, 447–8 (§ 10).

§ I am indebted to Mr. Littlewood for suggesting this problem.

‖ M. Riesz, "Une méthode de sommation équivalente à la méthode des moyennes arithmétiques", *Comptes Rendus* (12th June, 1911).

corollary from that particular case of our general theorem obtained by taking $\lambda_n = n$. Our ultimate aim, then, is to prove the following theorem, which gives in fact a little more information than is actually required.

THEOREM.—*If* $\lambda_n \sim \lambda_{n-1}$ *and* $\chi(n)$ *is any function which tends to infinity with* n, *there is a series* Σa_n *such that*

(i) $$a_n = O\{\chi(n)(\lambda_n - \lambda_{n-1})/\lambda_n\},$$

(ii) $\Sigma a_n e^{-\lambda_n s} \to 0$ *when* $s \to 0$ *in the angle** $$|\arg s| \leqslant \tfrac{1}{2}\pi - \delta;$$

but

(iii) Σa_n *is not summable* (R, λ, κ) *for any* $\kappa \geqslant 0$.

We can also find a series for which (i) *and* (ii) *are satisfied, and*

(iii)′ Σa_n *has an arbitrarily prescribed index of summability* $\kappa_0 \geqslant 0$.

The condition $\lambda_n \sim \lambda_{n-1}$, which appears in the above assertions, is, as Mr. Littlewood observes, not necessary in the "converse of Abel's theorem" itself, but some such restriction is quite essential for the truth of negative results of the kind with which we are here concerned.†

The example used for the proof of the above theorem is modelled on that constructed by Mr. Littlewood for the special case

$$\lambda_n = n, \ \chi(n) = (\log n)^\alpha.$$

The main difficulty seems to be the discussion of the existence of Abel's limit, which would appear to require an argument based on Cauchy's theorem, and so, when $\chi(n)$ is quite general, the use of analytic functions which (in particular) tend to infinity arbitrarily slowly when the variable tends to infinity along the real axis. It is plain, then, that no simple combination of the elementary functions of analysis will serve our

* In statements of this kind it is to be understood that δ is an arbitrarily small positive constant, and that $s \to 0$ *in any manner* in the specified angle, so that (ii) means: given ϵ, $\delta(\epsilon > 0, 0 < \delta < \frac{1}{2}\pi)$, we can find $r_0 = r_0(\epsilon, \delta) > 0$ such that

$$|\Sigma a_n e^{-\lambda_n s}| < \epsilon \quad \text{for} \quad 0 < |s| \leqslant r_0, \ |\arg s| \leqslant \tfrac{1}{2}\pi - \delta.$$

† Littlewood, *loc. cit.*, 437, 446 (§ 8).

purpose. It is, indeed, possible to *construct* functions (depending explicitly on χ) with suitable properties, but it has been found simpler for the present purpose to employ functions which are not analytic and to use, instead of Cauchy's theorem, the two-dimensional form of Green's theorem.

2. We consider a pair of real functions $f(x)$ and $\phi(x)$ satisfying some or all of the conditions A 1–4, B 1–2 below, and (later) a further condition C, which will be introduced in §6. We shall ultimately select functions fulfilling *all* of these requirements, but for some of our preliminary lemmas certain of these assumptions are irrelevant. We shall indicate precisely at each stage the hypotheses upon which our assertions rest.

A 1. $f'(x)$ *and* $\phi''(x)$ *are continuous for* $x \geqslant 0$.

2. $f(0) = 0$.

3. $\phi'(x) \geqslant \rho > 0 \quad (x \geqslant 0)$.

4. $f'/\phi'^2 \to 0, \quad f\phi''/\phi'^3 \to 0, \quad as \quad x \to \infty$.

B 1. $f(x) = O(e^{\epsilon e^{\epsilon x}})$ *for every* $\epsilon > 0$.

2. *There is a constant* c *such that*

$$0 < c < \tfrac{1}{2}\pi, \quad fe^{-c\phi'} \to 0 \ as \ x \to \infty .$$

We write

$$a(x) = \frac{d}{dx}\left(\frac{f(x)}{i\phi'(x)}\, e^{i\phi(x)}\right) = \left(f + \frac{f'}{i\phi'} - \frac{f\phi''}{i\phi'^2}\right) e^{i\phi} \quad (x \geqslant 0),$$

$$a(u) = a(\log u)/u \quad (u \geqslant 1),$$

and begin by discussing the behaviour of

$$\int_1^\infty a(u) e^{-su}\, du, \qquad \int_1^\omega \left(1 - \frac{u}{\omega}\right)^\kappa a(u)\, du \quad (\kappa \geqslant 0),$$

when $s \to 0$, $\omega \to \infty$ respectively.

3. LEMMA 1.—*If* $F(z)$ *is a regular function of* $z(= x+iy)$ *in a rectangle* R $(a \leqslant x \leqslant A,\ b \leqslant y \leqslant B)$, *and* $G(z) = G(x, y)$ *is a (complex) function of* (x, y) *which, together with its partial derivatives* G_x, G_y, *is*

continuous in R, then

$$\int_C FGdz = \iint_R F.(iG_x - G_y)\,dx\,dy,$$

the line integral being taken round the boundary of R in the positive sense.

We have only to apply Green's theorem [which obviously holds for complex functions of (x, y)] to the integral

$$\int_C (FGdx + iFGdy)$$

and observe that, $F(z)$ being regular, $iF_x = F_y = iF'(z)$ and so

$$(iFG)_x - (FG)_y = (iG_x - G_y)F.$$

4. LEMMA 2. (A, B).*—*We have*

$$\int_1^\infty a(u)e^{-su}\,du = o(1)$$

when $s \to 0$ *in the angle* $-\frac{1}{2}\pi+\delta \leqslant \arg s \leqslant \frac{1}{2}\pi-c-\delta$.

We suppose throughout

$$0 < \delta < \tfrac{1}{2}\pi - c, \quad s = re^{i\theta}, \quad r > 0, \quad -\tfrac{1}{2}\pi+\delta \leqslant \theta \leqslant \tfrac{1}{2}\pi-c-\delta,$$

so that

$$\Re(se^{iy}) = r\cos(\theta+y) \geqslant r\sin\delta = \gamma r \quad (0 \leqslant y \leqslant c), \tag{1}$$

where

$$\gamma = \sin\delta > 0.$$

Using the substitution $u = e^x$ and integrating by parts, we obtain, all the integrals being (absolutely) convergent and the integrated term vanishing by A 1–3, B 1,

$$\int_1^\infty a(u)e^{-su}\,du = \int_0^\infty se^x e^{-se^x} g(x) e^{i\phi(x)}\,dx = I, \tag{2}$$

say, where

$$g = f/i\phi'.$$

To estimate I we apply Lemma 1 to the functions

$$F_1(z) = se^z e^{-se^z}, \qquad G(x, y) = g(x)\,e^{i\phi(x) - y\phi'(x)}$$

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* The letters in parentheses at the beginning of a lemma indicate those of the hypotheses A–C which are used in its proof. In Lemma 2, c denotes, of course, any constant for which B 2 is satisfied. A stands for the set of hypotheses A1–4.

and the rectangle $R(0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant c)$, where X will ultimately be made to tend to infinity. If we denote the double integral by I' and the integrals along the four sides of R in order by I_1, I_2, I_3, I_4 [I_1 being the integral along the segment $(0, X)$ of the real axis], this gives

$$I_1+I_2+I_3+I_4 = I'. \tag{3}$$

In this and subsequent proofs we shall denote by $\epsilon_1(x)$, $\epsilon_2(x)$, ... functions of x only,* which are positive and decreasing for $x \geqslant 0$ and tend to zero as $x \to \infty$.

We may then write

$$|iG_x - G_y| = |(ig' - g\phi' - iyg\phi'' + g\phi')\, e^{i\phi - y\phi'}| = |i(g' - yg\phi'')\, e^{i\phi - y\phi'}|$$

$$\leqslant \epsilon_1(x)(1+y\phi')\,\phi'\, e^{-y\phi'} \quad (x \geqslant 0,\ \ y \geqslant 0), \tag{4}$$

since $g'/\phi' = f'/i\phi'^2 - f\phi''/i\phi'^3 \to 0$, $g\phi''/\phi'^2 = f\phi''/i\phi'^3 \to 0$ by A 4. This inequality combined with (1) gives

$$|I'| < \int_0^\infty re^x e^{-\gamma re^x} \epsilon_1(x)\, dx \int_0^\infty \{1+y\phi'(x)\}\, e^{-y\phi'(x)}\, \phi'(x)\, dy.$$

The inner integral here is equal to 2, and we have, writing $\gamma e^x = u$,

$$|I'| < \int_0^\infty re^{-ru} \epsilon_2(u)\, du. \tag{5}$$

In the same way, using A 3, B 2, we obtain

$$|I_3| \leqslant \int_0^X re^x e^{-\gamma re^x} |g(x)|\, e^{-c\phi'(x)}\, dx < \int_0^\infty re^{-ru} \epsilon_3(u)\, du.$$

Finally $I_4 = 0$ since $g(0) = f(0) = 0$. We now use these results in (3) and make $X \to \infty$, keeping s fixed; then will $I_2 \to 0$ by A 3, B 1, and (1), and $I_1 \to I$, and we shall have (putting $\epsilon_2 + \epsilon_3 = \epsilon_4$)

$$|I| \leqslant \int_0^\infty re^{-ru} \epsilon_4(u)\, du = \int_0^{r^{-\frac{1}{2}}} + \int_{r^{-\frac{1}{2}}}^\infty$$

$$\leqslant \epsilon_4(0)\, r^{\frac{1}{2}} + \epsilon_4(r^{-\frac{1}{2}}).$$

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* These functions may, of course, depend on *parameters* such as ρ, c, δ, κ. In (4) we may take, for example, $\epsilon_1(x)$ to be the upper bound of $|f/\phi'^2| + |f\phi''/\phi'^3|$ in the interval (x, ∞); and then in (5) we may take

$$\epsilon_2(u) = 2\gamma^{-1}\epsilon_1\{\log(u/\gamma)\} \quad (u \geqslant \gamma), \qquad \epsilon_2(u) = 2\gamma^{-1}\epsilon_1(0) \quad (0 \leqslant u < \gamma).$$

The right-hand side is independent of θ and tends to zero with r; the stated result therefore follows from (2).

5. LEMMA 3 (A).—*We have for fixed* $\kappa \geqslant 0$,

$$\int_1^\omega \left(1-\frac{u}{\omega}\right)^\kappa a(u)\,du = f(\xi)\,e^{i\phi(\xi)}\,\frac{\Gamma(\kappa+1)\,\Gamma[i\phi'(\xi)]}{\Gamma[\kappa+1+i\phi'(\xi)]}+o(1) \quad (\xi=\log\omega)$$

as $\omega\to\infty$.

The result is trivial when $\kappa=0$, for the left-hand side is then identical with the first term on the right. Suppose, therefore, $\kappa>0$. We then have, writing $u=e^x$ and integrating by parts,

$$(6) \qquad \int_1^\omega \left(1-\frac{u}{\omega}\right)^\kappa a(u)\,du = \int_0^\xi \kappa e^{x-\xi}(1-e^{x-\xi})^{\kappa-1}\,g(x)\,e^{i\phi(x)}\,dx = J,$$

say. We now apply Lemma 1 to the functions*

$$F_2=\kappa e^{z-\xi}(1-e^{z-\xi})^{\kappa-1}, \qquad G=g(x)\,e^{i\phi(x)-y\phi'(x)},$$

and the rectangle $0\leqslant x\leqslant\xi'$, $0\leqslant y\leqslant Y$, where $0<\xi'<\xi$, $Y>0$; we shall eventually make $Y\to\infty$ and then $\xi'\to\xi-0$. With conventions similar to those adopted in Lemma 2, we may write the result of Lemma 1 in the form

$$J_1+J_2+J_3+J_4=J'.$$

Considering first J', we have, writing

$$\mu=\mu_\kappa=+1 \quad (\kappa\geqslant 1),\ -1\ (0<\kappa<1),$$

$$|F_2(z)|\leqslant\kappa e^{x-\xi}(1+\mu e^{x-\xi})^{\kappa-1} \quad (x<\xi,\ y\gtreqless 0),$$

which, combined with our earlier inequality (4), gives

$$|J'|<\int_0^\xi \kappa e^{x-\xi}(1+\mu e^{x-\xi})^{\kappa-1}\,2\epsilon_1(x)\,dx<\omega^{-\kappa}\int_0^\omega \kappa(\omega+\mu u)^{\kappa-1}\epsilon_5(u)\,du$$

$$=\omega^{-\kappa}\left(\int_0^{\sqrt{\omega}}+\int_{\sqrt{\omega}}^{\omega}\right)\leqslant\omega^{-\kappa}\left[\kappa 2^{\kappa+1}\omega^{\kappa-1}\omega^{\frac{1}{2}}\epsilon_5(0)+2^\kappa\omega^\kappa\epsilon_5(\omega^{\frac{1}{2}})\right] \quad (\omega\geqslant 4)$$

$$\leqslant\epsilon_6(\omega).$$

* By $(1-e^{z-\xi})^{\kappa-1}$ we mean the branch which is regular for $z<\xi$ and positive for $z<\xi$.

Also $J_4 = 0$ since $g(0) = f(0) = 0$. Keeping ξ, ξ' fixed let $Y \to \infty$; in this process J_3 will tend to zero and J_2 will converge, since, by A 3, $F_2 G = O(e^{-\rho y})$ uniformly in $0 \leqslant x \leqslant \xi'$ as $y \to \infty$. Hence, writing J_2 at length and putting for brevity $\xi - \xi' = \zeta$, we obtain

$$(7) \qquad \left| J_1 + \kappa g(\xi') e^{i\phi(\xi')} \int_0^\infty e^{-\zeta+iy}(1-e^{-\zeta+iy})^{\kappa-1} e^{-y\phi'(\xi')} i dy \right| \leqslant \epsilon_6(\omega).$$

The integral on the left is

$$= -e^{i\zeta\phi'(\xi')} \int_\zeta^{\zeta-\infty i} e^{-z}(1-e^{-z})^{\kappa-1} e^{-i\phi'(\xi')z} dz = -e^{i\zeta\phi'(\xi')} \int_\zeta^\infty *$$

$$= -e^{i\zeta\phi'(\xi')} \int_0^{e^{-\zeta}} (1-x)^{\kappa-1} x^{i\phi'(\xi')} dx$$

$$\to -B[\kappa, i\phi'(\xi)+1]$$

as $\xi' \to \xi - 0$, where (with the usual notation) $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. For, writing the last integral as

$$\int_0^1 - \int_{e^{-\zeta}}^1 = j_1 - j_2,$$

we have $$|j_2| \leqslant \kappa^{-1}(1-e^{-\zeta})^\kappa \to 0$$

and $$j_1 = B[\kappa, i\phi'(\xi')+1] \to B[\kappa, i\phi'(\xi)+1]$$

by continuity ; also evidently $e^{i\zeta\phi'(\xi')} \to 1$.

By making $\xi' \to \xi - 0$ in (7), observing that $J_1 \to J$ and that ϕ and $g = f/i\phi'$ are continuous, we therefore obtain

$$|J - f(\xi)e^{i\phi(\xi)} B[\kappa+1, i\phi'(\xi)]| \leqslant \epsilon_6(\omega),$$

which, combined with (6), gives the desired result.

6. We pass now to the Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$ where the sequence (λ_n) satisfies the usual conditions $\lambda_1 < \lambda_2 < \ldots$, $\lambda_n \to \infty$, but is (for the present) otherwise unrestricted, and a_n is defined as follows. First completing the definition of $a(u)$ for all real u by putting $a(u) = 0$ $(u < 1)$,

* By Cauchy's Theorem ; the transformation is legitimate since $\phi'(\xi') > 0$.

and denoting by λ_0 any number $< \text{Min}\,(\lambda_1, 1)$, we define a_n as*

$$a_n = \int_{\lambda_{n-1}}^{\lambda_n} a(u)\,du \quad (n \geqslant 1).$$

We here introduce a new restriction on f and ϕ, namely

C. $$\bar{a}_n \equiv \int_{\lambda_{n-1}}^{\lambda_n} |a(u)|\,du = o(1) \quad as \quad n \to \infty.$$

LEMMA 4 (A, B, C).—*We have*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = o(1),$$

when $s \to 0$ *in the angle* $-\frac{1}{2}\pi + \delta \leqslant \arg s \leqslant \frac{1}{2}\pi - c - \delta$.

LEMMA 5 (A, C).—*For fixed* $\kappa \geqslant 0$ *we have*

$$\sum_{\lambda_n < \omega} a_n \left(1 - \frac{\lambda_n}{\omega}\right)^{\kappa} = f(\hat{\xi})\, e^{i\phi(\hat{\xi})}\, \frac{\Gamma(\kappa+1)\,\Gamma[i\phi'(\hat{\xi})]}{\Gamma[\kappa+1+i\phi'(\hat{\xi})]} + o(1) \quad (\hat{\xi} = \log \omega)$$

as $\omega \to \infty$.

To prove these results we first define a function $\eta(u)$ for $u > \lambda_0$ as follows:

$$\eta(u) = -\int_{\lambda_{n-1}}^{u} a(v)\,dv \quad (\lambda_{n-1} < u \leqslant \lambda_n).$$

C evidently implies $\eta(u) \to 0$ as $u \to \infty$.

Hence, assuming B 1, C, and observing that $a(u) = \eta(u) = 0$ for $\lambda_0 < u < 1$, we have

$$\sum_{1}^{\infty} a_n e^{-s\lambda_n} - \int_1^{\infty} a(u)\, e^{-su}\, du = \sum_{n=1}^{\infty} \int_{\lambda_{n-1}}^{\lambda_n} a(u)(e^{-s\lambda_n} - e^{-su})\,du \quad (\Re s > 0)$$

$$= \sum_{n=1}^{\infty} \int_{\lambda_{n-1}}^{\lambda_n} \eta(u)\, se^{-su}\,du = s\int_1^{\infty} \eta(u)\, e^{-su}\,du$$

$$= o(1)$$

when $s \to 0$ in $|\arg s| \leqslant \frac{1}{2}\pi - \delta$. Lemma 4 therefore follows from Lemma 2.

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* Since $a(u) = 0$ $(u < 1)$, a_1 does not depend on the choice of λ_0. For large n, we have, of course,

$$a_n = \Delta\left(\frac{f(\log \lambda_n)}{i\phi'(\log \lambda_n)}\, e^{i\phi(\log \lambda_n)}\right) \quad \{\Delta F(n) = F(n) - F(n-1)\}.$$

In the same way, if N is the integer defined by $\lambda_{N-1} < \omega \leqslant \lambda_N$, we deduce from C that

$$\sum_{\lambda_n < \omega} a_n(\omega-\lambda_n)^\kappa - \int_1^\omega a(u)(\omega-u)^\kappa\,du$$

$$= \sum_{n=1}^{N-1} \int_{\lambda_{n-1}}^{\lambda_n} a(u)\{(\omega-\lambda_n)^\kappa-(\omega-u)^\kappa\}\,du - \int_{\lambda_{N-1}}^{\omega} a(u)(\omega-u)^\kappa\,du$$

$$= \int_1^\omega \eta(u)\,\kappa(\omega-u)^{\kappa-1}\,du + \nu_\kappa\,\omega^\kappa\,\eta(\omega) \quad \{\nu_0 = 1, \quad \nu_\kappa = 0 \ \ (\kappa > 0)\}$$

$$= o(\omega^\kappa)$$

as $\omega \to \infty$, and Lemma 5 follows from Lemma 3.

7. LEMMA 6.—*Given a function $\psi(x)$ which tends to infinity with x, we can find a function $f(x)$ having a continuous derivative $f'(x)$ for $x \geqslant 0$ and such that, when $x \to \infty$,*

$$f = O(\psi), \quad f \to \infty, \quad f' \to 0.$$

We first choose, as is clearly possible, a function $\psi_1(x)$, continuous and increasing for $x \geqslant 0$, and such that

$$\psi_1 \to \infty, \quad \psi_1 = O(\psi), \quad \psi_1 = o(x) \quad (x \to \infty).$$

It is then obviously sufficient to take

$$f(x) = \int_{\frac{1}{2}x}^{x} \frac{\psi_1(v)}{v+1}\,dv.$$

8. *Proof of the main theorem.*—We shall show, with the help of the foregoing results, that, under the hypotheses of our theorem, we can choose f and ϕ in such a way that the series Σa_n, which we have defined in terms of these functions, has all the required properties.

Since $\lambda_n \sim \lambda_{n-1}$ and therefore $\lambda_n/(\lambda_n-\lambda_{n-1}) \to \infty$, we can choose a sequence (ϵ_n) such that

$$\epsilon_n \to 0, \quad \lambda_n \epsilon_n/(\lambda_n-\lambda_{n-1}) \to \infty.$$

We now define $\psi(x)$ for $x \geqslant 0$ thus:

$$\psi(x) = \text{Min}\left(\chi(n), \ \frac{\lambda_n \epsilon_n}{\lambda_n-\lambda_{n-1}}\right) \quad \text{when} \quad \lambda_{n-1} < e^x \leqslant \lambda_n.$$

Clearly $\psi \to \infty$ with x. We can therefore, by Lemma 6, choose $f(x)$ with continuous $f'(x)$ so that

$$f = O(\psi), \quad f \to \infty, \quad f' \to 0.$$

For the first part of the theorem we now choose ϕ so that

$$\phi' = (\log f)^2$$

for large x. A 1 and A 3 are obviously satisfied for sufficiently large x, and it is clear that, by suitably modifying f and ϕ over a finite range $0 \leqslant x < x_0$, we can secure the complete fulfilment of A 1–3. A 4 also will be satisfied since $\phi' \to \infty$ and

$$f'/\phi' = O(|f'|) = o(1). \tag{8}$$

$$f\phi''/\phi'^2 = O(|f\phi''/\phi'|) = O(|f'|/\log f) = o(1). \tag{9}$$

B 1 is an obvious consequence of $f' \to 0$, and B 2 holds—and indeed with an arbitrarily small c—since $fe^{-c(\log f)^2} \to 0$ when $f \to \infty$.

Further, recurring to the original definition of $a(u)$, and using (8) and (9), we have

$$\bar{a}_n = O\int_{\lambda_{n-1}}^{\lambda_n} \{f(\log u)+1\}\,\frac{du}{u} = O(\lambda_{n-1}^{-1})\int_{\lambda_{n-1}}^{\lambda_n} \psi(\log u)\,du \tag{10}$$

$$= O(\lambda_{n-1}^{-1}\lambda_n \epsilon_n) = O(\epsilon_n) = o(1),$$

that is C.

A, B, C are therefore completely satisfied, and we may apply Lemmas 4 and 5. The first shows, since c may be arbitrarily small, that

$$\sum_1^\infty a_n e^{-\lambda_n s} = o(1),$$

when $s \to 0$ in the angle $|\arg s| \leqslant \frac{1}{2}\pi - 2\delta$. The second gives, on the other hand, since $\phi' \to \infty$,

$$\sum_{\lambda_n < \omega} a_n \left(1-\frac{\lambda_n}{\omega}\right)^\kappa = \Gamma(\kappa+1)\,\frac{f(\xi)}{[\phi'(\xi)]^{\kappa+1}}\,e^{i[\phi(\xi)-\frac{1}{2}\pi(\kappa+1)]}\{1+o(1)\}+o(1). \tag{11}$$

When $\omega \to \infty$ the real and imaginary parts of the right-hand side oscillate infinitely, since $\phi \to \infty$ and $f/\phi'^{\kappa+1} \sim f/(\log f)^{2(\kappa+1)} \to \infty$ with f. Σa_n is therefore not summable (R, λ, κ) for any $\kappa \geqslant 0$.

Finally, using (10), we have

$$a_n = O(\tilde{a}_n) = O(\lambda_{n-1}^{-1}) \int_{\lambda_{n-1}}^{\lambda_n} \psi(\log u)\, du = O[\lambda_n^{-1}(\lambda_n - \lambda_{n-1})\chi(n)].$$

Our series therefore satisfies the stated conditions (i), (ii), (iii).

If we require a series satisfying (iii)′ instead of (iii), we take f as before and define ϕ so that

$$\phi' = f^{1/(\kappa_0+1)}$$

for large x (with suitable definitions over a finite range $0 \leqslant x < x_0$). We then have the hypotheses A–C and the conclusions (i) and (ii) as before, while (11) shows that the real and imaginary parts of the κ-th Rieszian mean of Σa_n tend to 0 if $\kappa > \kappa_0$, oscillate finitely if $\kappa = \kappa_0$, and oscillate infinitely if $0 \leqslant \kappa < \kappa_0$. The conclusion (iii)′ follows, and the proof of the theorem is completed. For the critical value $\kappa = \kappa_0$ we can secure, instead of finite oscillation as in the case just considered, summability or infinite oscillation; for the former we may take

$$\phi' = f^{1/(\kappa_0+1)} \log f,$$

and for the latter

$$\phi' = f^{1/(\kappa_0+1)}/(\log f).$$

ON A CASE OF INTEGRABILITY OF DARBOUX'S DIFFERENTIAL EQUATION OF DIMENSION TWO.

By ARTHUR BERRY.

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1. This paper is a contribution to the problem of the integration, and in particular of the algebraic integration, of an algebraic differential equation of the first order and degree

$$P\,dx+Q\,dy=0, \qquad (1)$$

where P and Q are polynomials in x, y.*

If we write x/z for x, y/z for y, the equation assumes the homogeneous form introduced by Darboux †

$$P\,dx+Q\,dy+R\,dz \equiv (Mz-Ny)dx+(Nx-Lz)dy+(Ly-Mx)dz=0, \qquad (2)$$

where L, M, N are homogeneous polynomials in x, y, z of the same order m, and P, Q, R are consequently polynomials of order $m+1$ connected by the identical equation

$$x\,P+y\,Q+z\,R=0. \qquad (3)$$

The equation (2) may conveniently be called Darboux's equation.

If P, Q, R have no common factor, an assumption which I make

* The literature of the subject is, of course, very extensive. Elementary results and some theory are given in standard books on Analysis and on Differential Equations. Fuller treatment of the modern analytical theory is to be found in Painlevé's *Stockholm Lectures* (1897), Forsyth's *Theory of Differential Equations*, 2 (1900), and Boutroux' *Leçons sur les fonctions définies par les équations différentielles du premier ordre* (1908). References to the literature are given in the articles in the *Encykl. d. Math. Wiss.* by Painlevé (2, A, 4, *a*, 1900) and Hilb (2, B, 6, 1916). Hilb, however, does not notice three important papers by Dulac [*Journal de l'Ecole Polyt.* (2), cah. 9 (1904), 5–125, *Ann. de Grenoble*, 17 (1905), 1–51, *Journal de Math.* (6), 2 (1906), 381–402], which deal with the behaviour of an integral in the neighbourhood of a singular point. I have found nothing of importance subsequent to the date of Hilb's article.

I have only added in the text a few references to papers directly connected with my particular problem.

† *Bull. des Sci. Math. et Astr.* (2), 2 (1878), pt. 1.

throughout, the integer m is called the *dimension* of the equation. Equations of the form (2) can in the first instance be classified according to their dimensions.

The case of $m = 0$ is trivial; in the case of $m = 1$ the equation is equivalent to an equation often called Jacobi's, which can be completely integrated by elementary methods.*

In the next higher case, $m = 2$, no general method of integration is known, nor any complete criterion for determining whether the equation has one or more algebraic particular integrals, or an algebraic general integral. Darboux has, however, given a large number of special cases in which the integration can be effected completely or reduced to quadratures; in all these cases the existence of one or more particular algebraic integrals is postulated, and in many the general integral is algebraic.

I have given a good deal of time to the case of $m = 2$ in the hope that in this apparently simple case it might be possible to obtain definite criteria for algebraic integrability, but have only obtained special results.

One of these results, which is precise and fairly general, is the subject of this paper.

The points common to $P = Q = R = 0$ (for an equation of any dimension) are called by Darboux *singular points*, by some other writers *critical points*, and play a fundamental part in the theory. Darboux has shown that, for an equation of dimension m, the number of them is, "in general," m^2+m+1 (consequently 7, if $m = 2$), but less if the intersections of $P = Q = R = 0$ are multiple.

Autonne† called attention to the importance of a particular kind of singular point, the *dicritic point* (*nœud dicritique*), which had occurred incidentally in Darboux's paper, and in a somewhat different connexion in earlier writers.

A dicritic point is defined as one at which

$$d(P\,dx+Q\,dy+R\,dz) \equiv \frac{\partial P}{\partial x}\,dx^2+\ldots+\ldots+\left(\frac{\partial Q}{\partial z}+\frac{\partial R}{\partial y}\right)dy\,dz+\ldots+\ldots = 0 \tag{4}$$

for all values of $dx : dy : dz$; provided that corresponding conditions of higher order are not satisfied. If such higher conditions are satisfied the point is *tricritique*, ... *polycritique*; but such points cannot occur if $m = 2$. A special kind of dicritic point, which Autonne calls

* See, *e.g.*, Goursat's *Cours d'Analyse*, 2, § 368.

† "Théorie des équations différentielles du premier ordre et du premier degré", *Journal de l'Ecole Polyt.*, cah. 61 (1891), 35–122. This is the first of a long series of papers on substantially the same subject.

hypercritique, occurs when all nine derivative $\partial P/\partial x, \ldots, \partial R/\partial z$ are zero, so that the curves $P = Q = R = 0$ have a common double point. As, however, Poincaré* has used the same word in connexion with the same problem in a different sense, I prefer to speak of a *general* or *special* dicritic point, according as it is not or is also *hypercritique*.

If a dicritic point is taken to be $x = y = 0$, P and Q are of the form

$$P \equiv ayz^m + \ldots, \tag{5}$$

$$Q \equiv -axz^m + \ldots,$$

when a is a constant, and the terms not written down are of higher order in x and y. In the *general* case $a \neq 0$, and can be taken to be 1; in the *special* case $a = 0$. Conversely, if P, Q are of the form (5), the point is dicritic. In the case of $m = 2$, if there is at least one general dicritic point, the differential equation (as was noticed by Darboux) reduces at once to a Riccati differential equation, and, if there is a special dicritic point, to a linear equation.

I have found that, if there is a dicritic point, the equation can be integrated completely in terms of what may fairly be called "known" functions, and that the conditions for algebraic integrability can be stated precisely. The functions involved are enumerated in § 4.

2. I consider first the case of a *general* dicritic point, which I take to be $x = y = 0$.

Since $m = 2$, P, Q, R are cubics, which, from the identity (3), can be written

$$P \equiv yz^2 - zu_2 + yw_2, \tag{6}$$

$$Q \equiv -xz^2 - zv_2 - xw_2,$$

$$R \equiv xu_2 + yv_2,$$

where u_2, v_2, w_2 are homogeneous and quadratic in x, y.

R is independent of z, and its linear factors equated to zero are obviously integrals of the differential equation. Three cases arise according as the factors of R are all distinct, two coincident, all coincident; the corresponding forms of R can, without loss of generality, be taken to be $xy(x-y)$, xy^2, y^3.

These three cases will now be considered in order.

* "Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré", *Rend. di Palermo*, 5 (1891), 161–191; 11 (1897), 193–239.

2.1. *Case* I.—$R \equiv xy(x-y)$.

There are six singular points, distinct from the dicritic point, lying in pairs on $x=0$, $y=0$, $x-y=0$. Choosing $z=0$ to go through one of the first pair and one of the third pair, we have

$$P = y\{z^2+(a-1)xz+(1-b)yz+cx(x-y)\}, \tag{7}$$

$$Q = -x\{z^2+axz-byz+cx(x-y)\}.$$

Taking $y=1$, and working with non-homogeneous coordinates x, z, we have the Riccati equation

$$x(x-1)\frac{dz}{dx}+z^2+(a-1)xz+(1-b)z+cx(x-1)=0. \tag{8}$$

The usual transformation

$$z = x(x-1)\frac{du}{dx}\Big/u$$

gives a linear differential equation of the second order; taking for convenience new constants defined by

$$\alpha+\beta=a, \quad \alpha\beta=c, \quad \gamma=b,$$

we have the hypergeometric equation

$$x(x-1)\frac{d^2u}{dx^2}+\{(\alpha+\beta+1)x-\gamma\}\frac{du}{dx}+\alpha\beta u=0. \tag{9}$$

Our equation can accordingly be completely integrated in terms of hypergeometric functions.

2.11. The cases in which the hypergeometric equation has one or more algebraic integrals are familiar;* any such integral gives a corresponding algebraic integral of the Riccati equation (8), and of the original Darboux equation. If the general integral of the hypergeometric equation is algebraic, so is that of the Darboux equation.

As, however, u need not be algebraic when $(du/dx)/u$ is algebraic, there may be non-algebraic integrals of the hypergeometric equation which give algebraic integrals of the Riccati equation.

I have examined this possibility elsewhere,† and have found that

* See, *e.g.*, the well known paper by H. A. Schwarz, *Journal für Math.*, 75 (1873), 292–335; *Ges. Math. Abh.*, 2, 211–259.

† *Quart. J. of Math.*, 49 (1923), 303 308.

the only cases are the obvious ones when the hypergeometric equation has one or more integrals of the well known forms

$$u = x^{\lambda}(x-1)^{\mu} P(x), \tag{10}$$

when λ and μ are constants and P is a polynomial.

Any such integral obviously gives an algebraic and rational integral of the Riccati equation, but is only an algebraic integral of the hypergeometric equation if λ and μ are both rational numbers. It is known that if there are two integrals u, v of the form (10), one at least must be algebraic, and if one is non-algebraic, though the Riccati equation has two independent and rational integrals, the general integral

$$z = x(x-1)\left(\frac{du}{dx}+c\frac{dv}{dx}\right)\Big/(u+cv) \tag{11}$$

is not algebraic.

Hence in Case I (the factors of R all distinct) the Darboux equation has a general algebraic integral if and only if the general integral of the corresponding hypergeometric equation is algebraic; it has one algebraic integral (involving z) if the hypergeometric equation has one integral of the form (10), whether the indices λ, μ are rational or not; it has two algebraic integrals (involving z) if the hypergeometric equation has two integrals of the form (10), one algebraic, the other not. In all these cases the Darboux equation has the particular integrals $x = 0$, $y = 0$, $x-y = 0$, which are not in the ordinary sense integrals of the Riccati equation.

2.2. *Case* II.—$R \equiv xy^2$.

The singular points other than the dicritic point lie on $x = 0$, $y = 0$. Choosing $z = 0$ to pass through one of each set, we have

$$P \equiv y(z^2+azx+byz+cxy), \tag{12}$$

$$Q \equiv -x(z^2+azx+byz+cxy)-xyz.$$

2.21. If a is not 0, we can reduce it to 1 by the substitution x for ax. The Riccati equation is

$$x\frac{dz}{dx}+z^2+xz+bz+cx = 0, \tag{13}$$

and the substitution

$$z = x\frac{du}{dx}\Big/u \tag{14}$$

gives the linear equation

$$x\frac{d^2u}{dx^2}+(x+b+1)\frac{du}{dx}+cu=0. \tag{15}$$

The further substitution

$$u=e^{-\frac{1}{2}x}x^{-(b+1)/2}v \tag{16}$$

reduces it to the normal form

$$\frac{d^2v}{dx^2}+[-\tfrac{1}{4}+\{c-\tfrac{1}{2}(b+1)\}/x+\tfrac{1}{4}(1-b^2)/x^2]v=0. \tag{17}$$

This is an equation derived by Whittaker* by a limiting process from the hypergeometric equation; two independent integrals are denoted by $W_{k,m}(x)$, $W_{-k,m}(-x)$, where $m=\frac{1}{2}b$, $k=c-\frac{1}{2}(b+1)$, and are sometimes called confluent hypergeometric functions.

Thus, in Case II, the Darboux equation can be completely integrated by means of functions $W_{k,m}$.

2.211. If the Riccati equation (13) has an algebraic integral, any branch of the integral must be expressible as a descending series of rational powers of x, convergent for $|x|$ large. Substituting such a series and equating coefficients, we easily find two possible integrals only,

$$z_1=-c-c(c-b)/x+\ldots, \tag{18}$$
$$z_2=-x+(c-b-1)+\ldots,$$

the corresponding integrals of (15) being

$$u_1=x^{-c}\{1+c(c-b)/x+\ldots\}, \tag{19}$$
$$u_2=e^{-x}x^{c-b-1}\{1+\ldots\}.$$

On the other hand, we find that, if (15) has integrals of this form, they are

$$u_1=x^{-c}\{1+c(c-b)/x+c(c+1)(c-b)(c-b+1)/(2!\,x^2)+\ldots\}, \tag{20}$$
$$u_2=e^{-x}x^{c-b-1}\{1-(c-1)(c-b-1)/x$$
$$+(c-1)(c-2)(c-b-1)(c-b-2)/(2!\,x^2)+\ldots\},$$

where enough terms have been written down to make the form of the general terms obvious.

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* *Bull. Amer. Math. Soc.*, 10 (1904), 125-134, and Whittaker and Watson's *Modern Analysis*, ch. 16.

Each of these series is evidently divergent "in general," and is, in fact, known to be an asymptotic integral of (15) for a certain region in the x plane;* and either series can only converge if it terminates, in which case the series in brackets is a polynomial in $1/x$.

The first series terminates if $c = -n$ or $c-b = -n$, the second if $c-1 = n$ or $c-b-1 = n$, where n is a positive integer or zero.

Moreover, these four cases are not essentially distinct, since the linear transformation z, x for $z+x, -x$, applied to the Darboux equation, which is equivalent to an interchange of the two singular points on $y = 0$, converts the Riccati equation into a like equation with $-c+b+1$ for c; and similarly the transformation z for $z+bx$ changes c into $c-b$. It is, therefore, enough to consider the case $c = -n$; u_1 is now a polynomial of order n, and the corresponding integral z_1 of the Riccati equation a rational function.

The second series u_2 terminates if $c-b-1 = n'$, a positive integer or zero, and if this is so, z_2 is also a rational function.

But, even if both of these conditions are satisfied, the general integral of the Riccati equation involves e^{-x} and is not algebraic, though it is an elementary function.

Hence, in the case considered, the general integral of the Darboux equation is never algebraic, but there may be one or two algebraic integrals, each giving z as a rational function of x. In addition there are always the two integrals, independent of z, $x = 0$ and $y = 0$.

2.22. If the constant a in equations (12) of §2.2 is 0, the Riccati equation is

$$x\frac{dz}{dx}+z^2+bz+cx = 0. \tag{21}$$

2.221. If $c \neq 0$, it is convenient to take $4cx$ instead of x as the independent variable, so that the Riccati equation becomes

$$x\frac{dz}{dx}+z^2+bz+\tfrac{1}{4}x = 0, \tag{22}$$

and the corresponding linear equation becomes

$$x\frac{d^2u}{dx^2}+(b+1)\frac{du}{dx}+\tfrac{1}{4}u = 0. \tag{23}$$

* Whittaker and Watson, *loc. cit.*

The further substitutions, $x = \xi^2$, $u = \xi^{-b} v$, transform this into

$$\xi^2 \frac{d^2v}{d\xi^2} + \xi \frac{dv}{d\xi} + (\xi^2 - b^2) = 0, \qquad (24)$$

which is Bessel's equation of order b.

Thus in this case the Riccati equation and the original Darboux equation can be integrated completely in terms of Bessel functions of $\sqrt{x}$.

2.2211. We can now prove, as in 2.211, that, if the Riccati equation (22) has an algebraic integral, the corresponding Bessel equation must have an integral of the form

$$v = e^{\pm i\xi} \xi^{-\frac{1}{2}} \{1 + a_1 \xi^{-1} + a_2 \xi^{-2} + \ldots\}, \qquad (25)$$

convergent if $|\xi|$ is large.

Bessel's equation is known to have formal solutions of this kind,* which are divergent for general values of b; they converge only if they terminate, as is the case for both series if and only if $b = n - \frac{1}{2}$, where n is an integer. If this occurs, we have two algebraic integrals of the Riccati equation (22), both rational functions of $\sqrt{x}$, but, as before, the general integral is not algebraic, though an elementary function, as it involves $e^{i\xi}$.

The corresponding Darboux equation

$$y\{z^2 + (n - \tfrac{1}{2})yz + \tfrac{1}{4}xy\}dx - x\{z^2 + (n + \tfrac{1}{2})yz + \tfrac{1}{4}xy\}dy + xy^2 dz = 0 \qquad (26)$$

accordingly has two or no algebraic integrals (involving z), as n is or is not an integer (positive, negative, or zero), and each such integral gives z as a rational function of $\sqrt{x}$.

As before, there are also the integrals $x = 0$, $y = 0$ independent of z.

2.222. If the constant c in equation (21) of §2.22 is zero, the Riccati equation assumes the very simple form

$$x \frac{dz}{dx} + z^2 + bz = 0, \qquad (27)$$

the integral of which is

$$z + b = Czx^o, \qquad (28)$$

* They are those linear combinations of the standard solutions which Nielsen calls cylinder functions of the third kind.

algebraic if b is any rational number other than 0, with the special form

$$1/z = \log x + c, \tag{29}$$

when $b = 0$.

Restoring the homogeneous variable, we have the Darboux equation

$$y(z^2+byz)dx - x\{z^2+(b+1)yz\}dy + xy^2dz = 0 \tag{30}$$

with the general integrals

$$(z+by)y^b = Czx^b \tag{31}$$

or

$$y/z = \log x/y + C,$$

according as $b \neq 0$ or $b = 0$.

The former integral is algebraic if b is any rational number other than 0, and includes always the particular integrals $x = 0$, $y = 0$, $z = 0$, $z+by = 0$.

The latter is obviously transcendental for general values of c, but includes the particular integrals $x = 0$, $y = 0$, $z = 0$.

The equation (30) is accordingly the only one belonging to Case II ($R \equiv xy^2$), the general integral of which is ever algebraic.

The cubics obviously have a double point at $y = z = 0$, so that there is a special dicritic point there, and the equation belongs to a class considered by Autonne with *two* dicritic points.

If $b \neq 0$, the equation belongs also to a type considered by Darboux, with four linear particular integrals.

2.3. *Case* III.—$R \equiv y^3$.

The Riccati equation is of the form

$$\frac{dz}{dx} + z^2 + A_1 z + A_2 = 0, \tag{32}$$

where A_1 is a linear function, A_2 a quadratic function of x.

Since the integral of a Riccati equation can have no variable branch point, and there is no singularity of the equation except at ∞, it follows that any integral is a one-valued function of x, and therefore if algebraic is rational. It is not difficult, by assuming a rational integral $z = P_1(x)/P_2(x)$, P_1, P_2 polynomials, to determine the conditions of existence of such an integral and, if these are satisfied, to find P_1, P_2. But to integrate the equation completely, whether there is or is not an algebraic integral, it is better to proceed as before by simplifying the equation, and then transforming into a linear equation.

Since $R = 0$ only when $y = 0$, all the singular points lie on $y = 0$; choosing $z = 0$ to go through one of them, so that $P = Q = 0$ at $y = z = 0$, we have

$$P = y(z^2+axz+byz+hxy+cy^2), \tag{33}$$

$$Q = -x(z^2+axz+byz+hxy+cy^2)-y^2z.$$

There are three singular points at $x = y = 0$; $y = z = 0$; $y = 0$, $z+ax = 0$, the two latter coinciding if $a = 0$, giving six-point contact of the three cubics at $y = z = 0$. I consider separately the cases $a \neq 0$, $a = 0$.

2.31. If $a \neq 0$, by taking $z = 0$ to be the common tangent to the cubics at $y = z = 0$, we make h vanish, and choosing $(ax+by)/\sqrt{a}$, $z/\sqrt{a}$ as new variables in place of x, z, we have the simplified forms

$$P \equiv y(z^2+xz+cy^2), \tag{34}$$

$$Q \equiv -x(z^2+xz+cy^2)-y^2z.$$

The Riccati equation is

$$\frac{dz}{dx}+z^2+xz+c = 0. \tag{35}$$

The transformation

$$z = \frac{du}{dz}\Big/u \tag{36}$$

gives the linear equation

$$\frac{d^2u}{dx^2}+x\frac{du}{dx}+cu = 0. \tag{37}$$

The further transformation

$$u = e^{-\frac{1}{4}x^2}w \tag{38}$$

reduces this equation to the normal form

$$\frac{d^2w}{dx^2}+(c-\tfrac{1}{2}-\tfrac{1}{4}x^2)w = 0. \tag{39}$$

This is the equation of the parabolic cylinder, otherwise called Weber's equation,* integrable in terms of the function $D_n(x)$ (with $n = c-1$), a special case of the function $W_{k,m}$ occurring in §2.21.

Thus, in the case considered in this paragraph, the Darboux equation can be integrated completely in terms of the function $D_n(x)$.

* Whittaker and Watson, *Modern Analysis*, ch. 16, §§ 16.5 *seq.*

2.311. If the Riccati equation (35) has an algebraic integral, we find, as before, that for $|x|$ large it is expansible in one of the two descending series

$$z_1 = -cx^{-1} - c(c+1)x^{-3} + \ldots, \tag{40}$$

$$z_2 = -x + (c-1)x^{-1} + \ldots.$$

The corresponding solutions of (37) must be of the form

$$u_1 = x^{-c}\{1 + \tfrac{1}{2}c(c+1)x^{-2} + \ldots\}, \tag{41}$$

$$u_2 = e^{-\frac{1}{2}x^2} x^{c-1}(1 + \ldots).$$

On the other hand, the usual process of solving in series gives the two formal solutions of (37),

$$u_1 = x^{-c}\{1 + c(c+1)/(2x^2) + c(c+1)(c+2)(c+3)/(2 \,.\, 4 \,.\, x^4) + \ldots\}, \tag{42}$$

$$u_2 = e^{-\frac{1}{2}x^2} x^{c-1}\{1 - (c-1)(c-2)/(2x^2) + (c-1)(c-2)(c-3)(c-4)/(2 \,.\, 4 \,.\, x^4) + \ldots\},$$

which for general values of c are divergent (giving asymptotic solutions), and can converge only if they terminate. The first terminates if c is a negative integer or zero, the second if c is a positive integer; in either case the series in brackets is a polynomial in $1/x^2$, and the corresponding solution of the Riccati equation is a rational function of x as required. The two conditions are obviously incompatible, so that if c is an integer there is one algebraic integral only (involving z) of the Darboux equation.

Moreover, as before, the two cases are not essentially distinct. The linear transformation z for $z+x$, which is equivalent to an interchange of the two singular points on $y = 0$ (other than $x = y = 0$), followed by the transformation ix, $-iz$ for x, z, converts c into $-c+1$.

We have accordingly the result that the Darboux equation

$$y(z^2 + xz + cy^2)dx - \{x(z^2 + xz + cy^2) + y^2 z\}dy + y^3 dz = 0 \tag{43}$$

has, in addition to the obvious integral $y = 0$, no algebraic integral unless c is an integer; if so, there is one further algebraic integral giving z/y as a rational function of x/y.

2.32. If the constant a occurring in equation (33) of §2.3 is zero, by taking $z + \frac{1}{2}by$ as a new variable instead of z, we reduce P, Q to the forms

$$P \equiv y(z^2 + hxy + cy^2), \tag{44}$$

$$Q \equiv -x(z^2 + hxy + cy^2) - y^2 z.$$

It is convenient to distinguish the cases $h \neq 0$, $h = 0$.

2.321. If $h \neq 0$, by taking $hx+cy$ as a new variable instead of x, and multiplying the variables by suitable constants, we reduce P, Q further to

$$P \equiv y(z^2+xy), \tag{45}$$

$$Q \equiv -x(z^2+xy)-y^2z.$$

The Riccati equation is now

$$\frac{dz}{dx}+z^2+x = 0, \tag{46}$$

and the corresponding linear equation is

$$\frac{d^2u}{dx^2}+xu = 0. \tag{47}$$

The further transformations

$$x = \lambda\xi^{\frac{2}{3}}, \quad u = \xi^{\frac{1}{2}}v, \quad \text{when} \quad 4\lambda^3 = 9,$$

convert this into

$$\xi^2\frac{d^2v}{d\xi^2}+\xi\frac{dv}{d\xi}+(\xi^2-\tfrac{1}{9})v = 0, \tag{48}$$

Bessel's equation of order 1/3.

Thus in this case the Darboux equation can be completely integrated in terms of Bessel functions of order 1/3.

2.3211. We can prove, as in preceding cases, that the Riccati equation has no algebraic integral. Or, more simply, if an integral z of the Riccati equation (46) is algebraic, assuming for the integral an expansion in descending powers of x, $z = \lambda x^{\kappa}+\ldots$, convergent for $|x|$ large, we find, on substitution, $\kappa = \frac{1}{2}$. This is impossible, since we have already seen that in all cases when $R \equiv y^3$, the algebraic integral, if any, must be rational.

Accordingly $y = 0$ is the only algebraic integral of the Darboux equation defined by (45).

2.322. If the constant h in equations (44) of §2.32 is zero, the Darboux equation is

$$(z^2+cy^2)(y\,dx-x\,dy)+y^2(y\,dz-z\,dy) = 0. \tag{49}$$

This has the particular integrals

$$z^2+cy^2 = 0, \; y = 0, \tag{50}$$

and the general integral

$$\sqrt{c}\,.\,x/y+\tan^{-1}(z/\sqrt{c}\,.\,y) = C, \tag{51}$$

which is transcendental, unless $c = 0$.

If $c = 0$, the equation is

$$z^2(y\,dx-x\,dy)+y^2(y\,dz-z\,dy) = 0, \tag{52}$$

the general integral of which is

$$xz-y^2 = C\,yz. \tag{53}$$

This is the only equation belonging to Case III ($R \equiv y^3$), the general integral of which is algebraic.

The equation of this section has a second dicritic point, which is *special*, at $y = z = 0$, the cubics having a double point there (*cf.* §2.22).

This completes the discussion of the case when there is a *general* dicritic point.

3. I consider next the case of a *special* dicritic point at $x = y = 0$, each of the cubics having a double point there. As before, there are three cases according as R is of the form $xy(x-y)$, xy^2, y^3.

3.1. *Case* I.—$R \equiv xy(x-y)$.

We can express P, Q in the forms

$$P \equiv -yz\{ax+(b-1)y\}+y(a'x^2+hxy+b'y^2), \tag{54}$$

$$Q \equiv xz\{(a-1)x+by\}-x(a'x^2+hxy+b'y^2).$$

The dicritic point absorbs four at least of the seven singular points; the others are given by

$$\begin{aligned} &x = 0, \quad (b-1)z = b'y; \\ &y = 0, \quad (a-1)z = a'x, \\ &x = y, \quad (a+b-1)z = (a'+h+b')y. \end{aligned} \tag{55}$$

We can clearly permute these points as we please, as the factors of R can be permuted by linear transformations of x and y.

3.11. I assume first that two of these points, which I take to be the first and second, are distinct from $x = y = 0$; then $b \neq 1$, $a \neq 1$, and, choosing $z = 0$ to pass through these two points, we have $a' = b' = 0$.

Putting $y = 1$ as before, the Darboux equation becomes the linear equation

$$x(x-1)\frac{dz}{dx} - (ax+b-1)z + hx = 0, \tag{56}$$

the integral of which is

$$z(1-x)^{-a-b+1}x^{b-1} - h\int (1-x)^{-a-b}x^{b-1}\,dx = C. \tag{57}$$

The integral occurring in this equation is of a familiar type, and is sometimes called the incomplete Beta-function.

Thus the general integral of the Darboux equation defined by (54), subject to $a \neq 1$, $b \neq 1$, is expressible in terms of the incomplete B-function.

3.111. The conditions that the incomplete B-function should be algebraic can be inferred from the theory of the hypergeometric function, but a direct proof given in the following lemma may be of interest.

LEMMA.—

$$B(m, n, x) \equiv \int x^{m-1}(1-x)^{n-1}\,dx$$

is algebraic if and only if (i) m and n are rational numbers, and (ii) one of the three numbers m, n, $p \equiv 1-m-n$ is a positive integer, the other two not being both integers $\leqslant 0$.

The first condition is obviously necessary, as otherwise there would be a transcendental singularity at $x = 0$ or $x = 1$.

The transformation $1-x$ for x converts B into an integral of the same form with m, n interchanged; the transformation $1/x$ for x similarly interchanges m and p, a root of unity occurring as a factor being in each case ignored. We can therefore permute m, n, p as we please.

By an "obvious" result, which appears to have been first proved in a more general form by Abel,* B if algebraic is of the form

$$x^m(1-x)^n R(x) + C,$$

where R is a rational function of x. Since the integrand has no singularity except at 0, 1, ∞, the denominator of R can have no factor other than x or $1-x$; taking out factors x, $1-x$, if any, which occur in the numerator or denominator of R, we have $B = x^{m+\mu}(1-x)^{n+\nu}P(x) + C$, where μ, ν are integers and P is a polynomial not having x or $1-x$ as a factor. Expanding B in ascending powers of x, differentiating and

* Cf. Hardy's tract, *Integration of a function of a single variable*, § 8.

comparing the index of the first term with that in the expansion of the integrand, we have "in general" $m+\mu = m$, $\mu = 0$, but (exceptionally) $m+\mu = 0$ (so that the first term in the expansion is a constant and vanishes on differentiation) and then, from the next term in the expansion, $m(= -\mu) > 0$.

Similarly either $\nu = 0$ or (exceptionally) $n = -\nu > 0$; and from the expansion at ∞, if ρ is the order of the polynomial P, either $\mu+\nu+\rho+1 = 0$ or (exceptionally) $p \equiv 1-m-n = 1+\mu+\nu+\rho > 0$.

The three "general" cases $\mu = 0$, $\nu = 0$, $\mu+\nu+\rho+1 = 0$ are incompatible, ρ being $\geqslant 0$, so that one of the three "exceptional" cases must occur, one of m, n, p must be positive, and, μ, ν, ρ being integers, must be a positive integer.

Let m be a positive integer; then x^{m-1} can be expressed as a polynomial of order $m-1$ in $1-x$, and B is obviously algebraic unless the integrand contains a term in $1/(1-x)$, giving a logarithm on integration. This logarithmic case occurs if and only if $n-1$ is one of the negative integers $-1, -2, \ldots, -m$; and then p is also a negative integer or zero; conversely if p and n are both integers $\leqslant 0$, $n-1 \equiv -m-p \geqslant -m$ and the logarithmic case occurs.

Permuting m, n, p we have the result as enunciated.

A slight modification of the last part of the proof shows that, whether m, n are rational numbers or not, $x^{-m}(1-x)^{-n}B(m, n, x)$ is, for a suitable choice of the constant of integration, a rational function of x, provided that condition (ii) of the lemma holds.

3.112. The conditions of algebraic integrability of the linear equation (58) are now obvious.

If $h \neq 0$ in equation (57) of §3.11, the general integral is by the lemma algebraic if and only if (i) a and b are rational numbers and (ii) one of a, b, $1-a-b$ is a positive integer, the other two not being both integers $\leqslant 0$; if a and b are not both rational numbers, there is a particular integral giving z as a rational function of x, provided that one of a, b, $1-a-b$ is a positive integer.

If $h = 0$, the general integral is algebraic if a and b are rational numbers; if either is irrational, only the particular integral $z = 0$ is algebraic.

In all these cases the Darboux equation has also the particular integrals $x = 0$, $y = 0$, $x-y = 0$.

3.12. If two of the singular points given by equations (55) of §3.1 coincide with $x = y = 0$, we can take them to be first two, so that

$a=1$, $b=1$; then $a' \neq 0$, $b' \neq 0$ (as otherwise P, Q, R would have a common factor), also $a+b-1=1$, so that the remaining singular point is distinct from $x=y=0$, and we choose $z=0$ to pass through it.

We now have

$$P \equiv -xyz+y(x-y)(a'x-b'y), \tag{58}$$

$$Q \equiv xyz-x(x-y)(a'x-b'y),$$

and the linear equation is

$$x(x-1)\frac{dz}{dx}-xz+(x-1)(a'x-b')=0. \tag{59}$$

The integral is

$$z/(x-1)+(a'-b')\log(x-1)+b'\log x = C, \tag{60}$$

which is always logarithmic since $b' \neq 0$, so that the Darboux equation defined by (58) has no algebraic integrals other than $x=0$, $y=0$, $x-y=0$.

3.2. *Case* II.— $R \equiv xy^2$.

We have

$$P \equiv yz(ax+by)+y(a'x^2+2h'xy+b'y^2), \tag{61}$$

$$Q \equiv -xz\{ax+(1+b)y\}-x(a'x^2+2h'xy+b'y^2).$$

The remaining two singular points are given by

$$x=0, \quad bz+b'y=0; \tag{62}$$

$$y=0, \quad az+a'y=0,$$

and we have to distinguish as before according as they coincide or not with $x=y=0$.

3.21. If a is not 0, we can conveniently take it to be -1; the second of the two singular points is distinct from $x=y=0$ so that we can take $z=0$ to pass through it, and further to be the common tangent of the three cubics at this point; we now have $a'=h'=0$, and the linear equation is

$$x\frac{dz}{dx}-z(x-b)+b'=0. \tag{63}$$

The integral is

$$e^{-x}x^b z+b'\int e^{-x}x^{b-1}dx = C, \tag{64}$$

giving z in terms of the "incomplete" Γ-function.

The general integral is never algebraic; if $b'=0$ there is a particular algebraic integral $z=0$; and if $b' \neq 0$ there is a particular integral giving z as a rational function of x, provided that b is a positive integer.

3.22. If $a = 0$, the second singular point coincides with $x = y = 0$, and the cubics have six-point contact there.

The linear equation is

$$x\frac{dz}{dx}+bz+a'x^2+2h'x+b' = 0, \tag{65}$$

and its integral is

$$zx^b+\int x^{b-1}(a'x^2+2h'x+b')dx = C, \tag{66}$$

which is transcendental or algebraic, for general values of C, according as b is irrational or rational, with the exceptional cases $b = 0$, $b = -1$ (and $h' \neq 0$), $b = -2$, for each of which it is logarithmic.

If b is irrational there is a particular algebraic integral.

In all cases of §3.2 ($R \equiv xy^2$) the Darboux equation has also the particular algebraic integrals $x = 0$ and $y = 0$.

3.3. *Case* III.— $R \equiv y^3$.

We have
$$P \equiv yz(ax+by)+y(a'x^2+2h'xy+b'y^2), \tag{67}$$
$$Q \equiv -z(ax^2+bxy+y^2)-x(a'x^2+2h'xy+b'y^2),$$

and the remaining singular point is at

$$y = 0, \quad az+a' = 0. \tag{68}$$

3.31. If a is not 0, we can conveniently take it to be -1; the last singular point is distinct from $x = y = 0$, and, as in §3.21, we can choose $z = 0$ to be the common tangent to the cubics at this singular point, so that $a' = h' = 0$; we can further simplify by taking $x-by$ as a new variable in place of x. We now have

$$P \equiv -xyz+b'y^3, \tag{69}$$
$$Q \equiv z(x^2-y^2)-b'xy^2;$$

and the integral of the Darboux equation is

$$e^{-\frac{1}{2}x^2/y^2}z+b'y\int e^{-\frac{1}{2}x^2/y^2}d(x/y) = Cy, \tag{70}$$

giving z in terms of the error-function, which is a particular case of the incomplete Γ-function.

If $b' = 0$, $z = 0$ is a particular integral, otherwise the only algebraic integral is $y = 0$.

3.32. If the constant a in equations (67) vanishes, the remaining singular point coincides with $x = y = 0$, so that the cubics have seven-point contact there; as before $a' \neq 0$.

We now have

$$P \equiv by^2z + y(a'x^2 + 2h'xy + b'y^2), \tag{71}$$

$$Q \equiv -yz(bx+y) - x(a'x^2 + 2h'xy + b'y^2);$$

and the integral, in non-homogeneous coordinates, is

$$e^{bx}z + \int e^{bx}(a'x^2 + 2h'x + b')dx = C. \tag{72}$$

If $b \neq 0$, this can be written

$$e^{bx}(z + ax^2 + \beta x + \gamma) = C, \tag{73}$$

a, β, γ being new constants replacing a', h', b'.

This is "in general" transcendental, but for $C = 0$ gives the algebraic integral

$$z + ax^2 + \beta x + \gamma = 0. \tag{74}$$

The Darboux equation as before has also the algebraic integral $y = 0$. If $b = 0$, the integral is

$$3z + a'x^3 + 3h'x^2 + 3b'x = C,$$

or, in homogeneous coordinates,

$$3y^2z + a'x^3 + 3h'x^2y + 3b'xy^2 = Cy^3, \tag{75}$$

a cuspidal cubic.

4. We have now examined all cases of Darboux's equation of dimension 2, with one dicritic point, equations derivable from one another by linear transformation of the homogeneous variables not being regarded as distinct.

The integration can always be effected completely in terms of elementary functions, hypergeometric functions (§2.1), Whittaker's confluent hypergeometric functions $W_{k,m}(x)$ (§2.21), Bessel functions (§2.221 and §2.321), Weber's functions of the parabolic cylinder $D_n(x)$ (§2.31), "incomplete" B-functions (§3.11), "incomplete" Γ-functions (§3.21), and the error-function (§3.31).

[I have to express my gratitude to a referee for having pointed out a number of small errors, and for suggesting some improvements in expression.—December 1923.]

A SUGGESTION FOR THE DEVELOPMENT OF THE ORDINARY DIFFERENTIAL OPERATOR

By J. BRILL.

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THERE appear to be four fundamental laws in the development of the infinitesimal calculus as applied to modern physics. In the first place we have the relations

$$d(u+v) = du+dv, \tag{1}$$

$$d(uv) = vdu+udv, \tag{2}$$

where u and v are functions of the independent variables expressing the continuum, and d is the symbol of an infinitesimal variation due to a displacement of the representative point. Secondly, adopting the notation of the calculus of tensors, we have

$$du = \frac{\partial u}{\partial x_r}(dx)^r. \tag{3}$$

Finally, taking δ as the symbol of another infinitesimal variation, we have

$$\delta dx - d\delta x = 0, \tag{4}$$

where x may be any one of the independent variables.

These may be taken as the axiomatic laws governing the operating symbols, d, δ, $\partial/\partial x_r$, and a calculus may be developed from them. It is to be noted that (3) merely defines the law of the first application of the operator $\partial/\partial x_r$. In the development of the differential calculus we consider the operators $\partial/\partial x_r$ and $\partial/\partial x_s$ as commutative. This arises as a direct consequence of (4).

It is proposed to retain (1), (2), and (3); and, replacing (4) by the relation

$$\delta(dx)^r - d(\delta x)^r = a^r_{jk}(\delta x)^j(dx)^k, \tag{5}$$

to give a preliminary discussion of the effect on the resulting calculus, it being understood that the symbols bear an extended interpretation, which only coincides with the ordinary one when the a's are severally zero.

The supposition (5) denies, in general, the possibility of what is known as a "parallel displacement." In order that (5) may be consistent, when we make δ identical with d, we must have

$$\left.\begin{aligned} a^r_{jk} &= 0, \quad j = k \\ a^r_{jk} + a^r_{kj} &= 0, \quad j \neq k \end{aligned}\right\}. \tag{6}$$

Consider the possibility of the existence of a stationary value for the integral

$$\int X_r (dx)^r,$$

taken along a line, within the continuum, the end points of which are fixed, it being assumed that the X's are functions, if necessary in a generalized sense, of the x's. We assume the symbol "$\int$" to be distributive, and to be commutative with δ; and it is in consequence of the retention of (1), (2), (3) that the following operations are valid.

We have

$$\delta \int X_r (dx)^r = \int \left\{ \frac{\partial X_s}{\partial x_r} (\delta x)^r (dx)^s + X_r \delta (dx)^r \right\}$$

$$= \int \left\{ \frac{\partial X_s}{\partial x_r} (\delta x)^r (dx)^s + X_r d(\delta x)^r + X_r a^r_{jk} (\delta x)^j (dx)^k \right\}$$

$$= \left[X_r (\delta x)^r \right] + \int \left\{ \left(\frac{\partial X_s}{\partial x_r} - \frac{\partial X_r}{\partial x_s} \right) (\delta x)^r (dx)^s + X_r a^r_{jk} (\delta x)^j (dx)^k \right\}.$$

The first part of this expression vanishes on account of the fixity of the limits of integration. As regards the portion under the integral sign, the coefficient of the variation of each independent variable must be made to vanish separately. Thus, reverting to the ordinary notation so far as the elements of the displacement of a representative point are concerned, if we write

$$[rs] = \frac{\partial X_s}{\partial x_r} - \frac{\partial X_r}{\partial x_s} \quad \text{and} \quad (jk) = X_r a^r_{jk},$$

we have, for the fourfold continuum, the conditions

$$\begin{aligned}
&(11)\,dx_1+\{[12]+(12)\}\,dx_2+\{[13]+(13)\}\,dx_3+\{[14]+(14)\}\,dx_4=0,\\
&\{[21]+(21)\}\,dx_1+(22)\,dx_2+\{[23]+(23)\}\,dx_3+\{[24]+(24)\}\,dx_4=0,\\
&\{[31]+(31)\}\,dx_1+\{[32]+(32)\}\,dx_2+(33)\,dx_3+\{[34]+(34)\}\,dx_4=0,\\
&\{[41]+(41)\}\,dx_1+\{[42]+(42)\}\,dx_2+\{[43]+(43)\}\,dx_3+(44)\,dx_4=0. \qquad (7)
\end{aligned}$$

These equations require the vanishing of the determinant

$$\begin{vmatrix}
(11), & [12]+(12), & [13]+(13), & [14]+(14)\\
[21]+(21), & (22), & [23]+(23), & [24]+(24)\\
[31]+(31), & [32]+(32), & (33), & [34]+(34)\\
[41]+(41), & [42]+(42), & [43]+(43), & (44)
\end{vmatrix} \qquad (8)$$

Thus, in general, loci, for which our integral has a stationary value, would only be possible within a certain threefold defined by the equation expressing the evanescence of the determinant (8). To secure that such loci should exist throughout the continuum, and not merely be confined to a special threefold within it, it would be necessary that the determinant (8) should vanish identically. Such loci, however, would be quite definite, and the direction of such a locus at a given point would be limited to a definite number of directions. To secure that, in general, two arbitrary points might be joined by a locus of the required type, it would be necessary that at least the first minors of (8) should be evanescent. This would reduce our system of equations (7) to two independent relations. The evanescence of the second minors would reduce the same set to a single relation. It is thus seen that to secure that

$$\delta\int(X_1dx_1+X_2dx_2+X_3dx_3+X_4dx_4)=0,$$

independently of the path, the separate evanescence of each element of (8) would be necessary.

Thus we should have

$$(rr)=0, \quad [rs]+(rs)=0.$$

Since $[rs]+[sr]=0$, it would follow that

$$(rs)+(sr)=0.$$

As the conditions

$$(rr) = 0, \quad (rs)+(sr) = 0,$$

must be satisfied independently of the X's, we arrive at the conditions (6) given above; and thus our calculus appears self-consistent.

If we put

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = d\phi,$$

the condition $[jk]+(jk) = 0$ may be written in the form

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} - \frac{\partial^2 \phi}{\partial x_k \partial x_j} + a^r_{jk} \frac{\partial \phi}{\partial x_r} = 0. \tag{9}$$

Thus (6) and (9) give the conditions of a generalized calculus. It is to be noted that what we have so far developed is an operational calculus of a logically consistent type. Before further development could be effected, it would be necessary to consider more fully the question of the definition of the operand, *i.e.* the generalized type of function with which the calculus is concerned. This might possibly necessitate more restricted definitions of the a's than we have at present laid down.

For simplicity, the argument has been confined to the case of a four-fold continuum, but can be made to apply to the higher continua, and the results expressed by (6) and (9) are general.

The operational calculus obtained bears a certain relationship to the theory of linear associative algebras. It would seem possible to obtain an operational calculus, as suggested by any known symbolical algebra, by the introduction of suitable forms for the axiomatic laws governing the fundamental operators, and of a suitable definition of the operand, or generalized function.

ON CUBIC FIELDS WITH A GIVEN DISCRIMINANT

By W. E. H. BERWICK.

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THE field of algebraic numbers defined by an irreducible equation

$$a_0\theta^n+a_1\theta^{n-1}+\ldots+a_n=0$$

possesses one arithmetical invariant, the discriminant D. To an assigned discriminant there belong only a finite number of these number-fields. For quadratic fields the defining equation can be taken to be $\theta^2-a=0$, where a is an integer divisible by no square factor, and then

$$D=a,\ 4a,\ \text{or}\ 4a,$$

according as $a\equiv 1,\ 2,$ or $3 \pmod 4$.

The present paper contains a method of obtaining all the cubic fields belonging to an assigned discriminant. It is based on the representation

$$(a+b\sqrt{m})(\gamma+\sqrt{m})^3+(a-b\sqrt{m})(\gamma-\sqrt{m})^3=0$$

of the equation defining the field and is intimately bound up with the classification of ideal numbers in the field $[\sqrt{m}]$. An unsettled question in the latter theory, whether there exists a field $[\sqrt{m}]$ the ideals in whose principal genus compound according to the laws of an assigned Abelian group, has its analogue in the present investigation. In this paper the question that arises is whether there exists a quadratic field whose principal genus contains an assigned number of exponents of irregularity, each divisible by 3. Probably neither question can be completely settled without more experimental evidence than that furnished by Gauss's tables of quadratic forms. Peculiarities of the prime number 3 as a factor of the discriminant account independently for other difficulties in the investigation. The present theory is complete, except in deciding whether certain types of discriminant actually exist.

1. As in elementary algebra the cubic equation

$$a(\xi) \equiv a_0\xi^3+a_1\xi^2+a_2\xi+a_3 = 0, \tag{1}$$

supposed to have rational coefficients, is identifiable with

$$A(\xi+\alpha)^3+B(\xi+\beta)^3 = 0,$$

provided

$$A+B = a_0, \quad 3(A\alpha+B\beta) = a_1, \quad 3(A\alpha^2+B\beta^2) = a_2, \quad A\alpha^3+B\beta^3 = a_3.$$

By eliminating A, B we see that the numbers α, β satisfy

$$3a_0\alpha\beta-a_1(\alpha+\beta)+a_2 = 0,$$

$$a_1\alpha\beta-a_2(\alpha+\beta)+3a_3 = 0,$$

and hence are the roots of the equation

$$(a_1^2-3a_0a_2)\,x^2-(a_1a_2-9a_0a_3)\,x+(a_2^2-3a_1a_3) = 0$$

involving the radical

$$\sqrt{[3(27a_0^2a_3^2+4a_0a_2^3+4a_1^3a_3-a_1^2a_2^2-18a_0a_1a_2a_3)]}$$

which is equal to $\sqrt{[-3D(a_0\xi)]}/a_0$. Denoting by m the unique (positive or negative) integer which contains no square factor and bears a rational square ratio to $-3D(a_0\xi)$, we have

$$\alpha = p+q\sqrt{m}, \qquad \beta = p-q\sqrt{m},$$

also

$$A = P+Q\sqrt{m}, \quad B = P-Q\sqrt{m},$$

where p, q, P, Q are rational numbers. (The case when $m = 1$ is excluded here and mentioned again in § 18 below). By this reduction the cubic (1) becomes

$$(P+Q\sqrt{m})(\xi+p+q\sqrt{m})^3+(P-Q\sqrt{m})(\xi+p-q\sqrt{m})^3 = 0.$$

Put $(\xi+p)/q = \gamma$, also $P\pm Q\sqrt{m} = c(a\pm b\sqrt{m})$, where a, b are rational integers with no common factor. The equation $a(\xi) = 0$ is then equivalent to

$$\left.\begin{aligned}(a+b\sqrt{m})(\gamma+\sqrt{m})^3+(a-b\sqrt{m})(\gamma-\sqrt{m})^3 = 0\\ \text{or} \qquad a\gamma^3+3mb\gamma^2+3ma\gamma+m^2b = 0\end{aligned}\right\}, \tag{2}$$

which in the sequel will be taken as the standard equation defining a cubic field. A few transformed equations are collected here for reference; putting

$$a\gamma = \theta, \quad \lambda = \theta + mb, \quad \mu = \theta - mb, \quad \kappa = mab/\theta, \quad \nu = \kappa + a,$$

then
$$\theta^3 + 3mb\theta^2 + 3ma^2\theta + m^2a^2b = 0, \tag{3}$$

$$\lambda^3 + 3m(a^2 - mb^2)\lambda - 2m^2b(a^2 - mb^2) = 0, \tag{4}$$

$$\mu^3 + 6mb\mu^2 + 3m(a^2 + 3mb^2)\mu + 4m^2b(a^2 + mb^2) = 0, \tag{5}$$

$$\kappa^3 + 3a\kappa^2 + 3mb^2\kappa + mab^2 = 0, \tag{6}$$

and
$$\nu^3 - 3(a^2 - mb^2)\nu + 2a(a^2 - mb^2) = 0. \tag{7}$$

The numbers $\theta, \lambda, \mu, \kappa, \nu$ are all integers of $[\gamma]$ and their discriminants are given by

$$D(\theta) = D(\lambda) = D(\mu) = -2^2 . 3^3 . m^3a^2(a^2 - mb^2)^2,$$

$$D(\kappa) = D(\nu) = -2^2 . 3^3 . mb^2(a^2 - mb^2)^2.$$

2. The quadratic integer $a + b\sqrt{m}$ will be called the *generator* of the field $[\gamma]$. Since, when

$$\gamma = -\gamma' \quad \text{and} \quad (a + b\sqrt{m})(\gamma + \sqrt{m})^3 + (a - b\sqrt{m})(\gamma - \sqrt{m})^3 = 0,$$

then
$$(a - b\sqrt{m})(\gamma' + \sqrt{m})^3 + (a + b\sqrt{m})(\gamma' - \sqrt{m})^3 = 0,$$

$a + b\sqrt{m}$ and $a - b\sqrt{m}$ generate the same field.

If $\delta = A\gamma^2 + B\gamma + C$, with A, B, C rational, the fields $[\gamma]$, $[\delta]$ are identical. Two fields $[\gamma]$, $[\delta]$ will be said to be *distinct* when $[\delta]$ is identical neither with $[\gamma]$ nor with one of its conjugates.

When two cubic fields $[\gamma]$, $[\delta]$ are identical, γ and δ are connected by a linear relation

$$\gamma = (p\delta + q)/(r\delta + s)$$

with p, q, r, s rational. Applying this transformation to (2),

$$(a + b\sqrt{m})\{p\delta + q + (r\delta + s)\sqrt{m}\}^3 + (a - b\sqrt{m})\{p\delta + q - (r\delta + s)\sqrt{m}\}^3 = 0,$$

or
$$(a' + b'\sqrt{m})(\delta' + \sqrt{m})^3 + (a' - b'\sqrt{m})(\delta' - \sqrt{m})^3 = 0,$$

with
$$a' + b'\sqrt{m} = (a + b\sqrt{m})(p + r\sqrt{m})^3$$

and
$$\delta' = [(p - mr^2)\delta + pq - mrs]/(ps - qr).$$

It follows that $[\gamma]$, $[\delta]$ are identical when, and only when, their generators $a+b\sqrt{m}$, $a'+b'\sqrt{m}$ are connected by the relation

$$\frac{a+b\sqrt{m}}{a-b\sqrt{m}}\left(\frac{p+r\sqrt{m}}{p-r\sqrt{m}}\right)^3=\frac{a'+b'\sqrt{m}}{a'-b'\sqrt{m}} \quad \text{or} \quad \frac{a'-b'\sqrt{m}}{a'+b'\sqrt{m}}.$$

3. The next step is to determine D, the discriminant of the field generated by $a+b\sqrt{m}$. D is a factor of $D(\theta)$ and $D(\theta)/D$ an integral square. Further, D is well known to be a common factor of the discriminants of all the integers of the field.

When $a+b\sqrt{m}$ contains the cube of an ideal factor in $[\sqrt{m}]$, put

$$(a+b\sqrt{m})=\mathfrak{a}\mathfrak{b}^3, \quad (a-b\sqrt{m})=\mathfrak{a}'\mathfrak{b}'^3,$$

where $\mathfrak{a}$ and $\mathfrak{a}'$, $\mathfrak{b}$ and $\mathfrak{b}'$ are conjugate and $\mathfrak{a}$, $\mathfrak{a}'$ contain no cubed factors. Put also

$$(\alpha)=\mathfrak{d}'\mathfrak{c}, \quad (\beta)=\mathfrak{b}'\mathfrak{c}, \quad (\alpha')=\mathfrak{d}\mathfrak{c}', \quad (\beta')=\mathfrak{b}\mathfrak{c}'.$$

Then

$$\tau\equiv\left[\frac{\alpha}{\beta}(\gamma+\sqrt{m})+\frac{\alpha'}{\beta'}(\gamma-\sqrt{m})\right]a$$

is an integer of the field and, if τ_1, τ_2, τ_3 correspond to γ_1, γ_2, γ_3 [the conjugate roots of (2)],

$$\tau_1-\tau_2=a\left(\frac{\alpha}{\beta}+\frac{\alpha'}{\beta'}\right)(\gamma_1-\gamma_2),$$

$$D(\tau)=-2^2.3^3.m^3a^2(a^2-mb^2)^2(a\beta'+a'\beta)^6/(\beta\beta')^6.$$

Hence $D(\tau)$ does not contain the factor $(\mathfrak{b}\mathfrak{b}')^6$ which appears in $(a^2-mb^2)^2$. It is therefore sufficient to assume that $a+b\sqrt{m}$ contains no cubed ideal factor.

The aggregate of integers in the cubic field is put down by the method of Woronoj,* a variety of cases requiring separate discussion. It is sufficient to give the details of three cases and then to tabulate them all.

I. When p, $\neq 2$ or 3, divides a^2-mb^2 and m (or a), and therefore a (or m), we use equation (7), noting that $D(\nu)\equiv 0 \pmod{p^3}$ exactly. The number ν^2/p is integral, so that D contains the prime factor p once only.

II. When $p=3$, $m\equiv 2 \pmod 3$ and $ab\equiv\pm 1 \pmod 3$, then $a^2-mb^2\equiv -1 \pmod 3$ and we use equation (5). $D(\mu)\equiv\pm 3^3 \pmod{3^4}$, and $a^2+mb^2\equiv 0 \pmod 3$. If $a^2+mb^2\equiv\pm 3 \pmod 9$ the coefficients of

* Sommer, *Vorlesungen über Zahlentheorie* (1907), § 42. Bachmann, *Allgemeine Arithmetik der Zahlkörper* (1905), 7.

the μ-equation are, in order, exactly divisible by 1, 3, 3, 3. Hence in this case there is no complex integer $(\mu^2+B\mu+C)/3$ and D must $\equiv \pm 3^3 \pmod{3^4}$. But when $a^2+mb^2 \equiv 0 \pmod 9$, $\mu^2/3$ is integral and $D \equiv \pm 3 \pmod{3^2}$.

III. When $p=2$, $m \equiv 1 \pmod 8$ and $ab \equiv 1 \pmod 2$, then

$$a^2-mb^2 \equiv 2^s \pmod{2^{s+1}},\ s \geqslant 3, \quad \text{and} \quad D(\lambda) \equiv 2^{2s+2} \pmod{2^{2s+3}}.$$

The coefficients of the λ-equation being, in order, exactly divisible by 1, 0, 2^s, 2^{s+1}, it immediately follows that

(i) if $s=3t$, $\mathfrak{o} \equiv \left(1, \frac{\lambda}{2^t}, \frac{\lambda^2}{2^{2t}}\right)$ (bas 2)* and $D \equiv 2^2 \pmod{2^3}$,

(ii) if $s=3t+1$, $\mathfrak{o} \equiv \left(1, \frac{\lambda}{2^t}, \frac{\lambda^2}{2^{t+1}}\right)$ (bas 2) and $D \equiv 2^2 \pmod{2^3}$,

(iii) if $s=3t-1$, $\mathfrak{o} \equiv \left(1, \frac{\lambda}{2^t}, \frac{\lambda^2}{2^{2t}}\right)$ (bas 2) and $D \equiv 1 \pmod{2^2}$.

4. In the last column of the table which follows is inserted the highest power of each possible prime factor of D which actually divides it, and in the preceding one an appropriate equation for obtaining the result.

I. $p \neq 2$ or 3.

Multiples of p.	*Non-multiples of p.*	*Equation.*	
a^2-mb^2	$a,\ mb,\ (a^2-mb^2)/p$	(4)	p^2
a^2-mb^2	$a,\ mb,\ (a^2-mb^2)/p^2$	(4)	p^2
	a^2-mb^2	(4)	1
$a,\ m;\ a^2-mb^2$		(7)	p
m	$a,\ b(a^2-mb^2)$	(4)	p
$m,\ b$	a	(4)	p
a	a^2-mb^2	(6)	1
b	$m(a^2-mb^2)$	(4)	1

* I write $\mathfrak{o} \equiv (1, \omega_2, \omega_3)$ (bas p) when ω_2, ω_3 are integers, and when there is no integer $(l_1+l_2\omega_2+l_3\omega_3)/p$ unless $l_1 \equiv l_2 \equiv l_3 \equiv 0 \pmod p$.

	Multiples of p.	*Non-multiples of p.*	*Equation.*	
	II. $p = 3$.			
$m \equiv 0 \pmod 3$		$ab \equiv \pm 1 \pmod 3$	(3)	3^4
	$a \equiv \pm 3 \pmod 9$	$b \equiv \pm 1 \pmod 3$	(6)	3^4
	$a \equiv 0 \pmod 9$	$b \equiv \pm 1 \pmod 3$	(6)	1
	$b \equiv 0 \pmod 3$	$a \equiv \pm 1 \pmod 3$	(3)	1
$m \equiv 2 \pmod 3$	$a^2+mb^2 \equiv \pm 3 \pmod 9$	$ab \equiv \pm 1 \pmod 3$	(5)	3^3
	$a^2+mb^2 \equiv 0 \pmod 9$	$ab \equiv \pm 1 \pmod 3$	(5)	3
$m \equiv 1 \pmod 3$	$a^2-mb^2 \equiv 3 \pmod 9$	$ab \equiv \pm 1 \pmod 3$	(4)	3^5
	$a^2-mb^2 \equiv 9 \pmod{27}$	$ab \equiv \pm 1 \pmod 3$	(4)	3^5
$m \equiv 1, 2 \pmod 3$	$a \equiv 0 \pmod 9$	$b \equiv \pm 1 \pmod 3$	(3)	3
	$b \equiv 0 \pmod 9$	$a \equiv \pm 1 \pmod 3$	(6)	3
	$a \equiv \pm 3 \pmod 9$	$b \equiv \pm 1 \pmod 3$	(3)	3^3
	$b \equiv \pm 3 \pmod 9$	$a \equiv \pm 1 \pmod 3$	(6)	3^3
	III. $p = 2$.			
$m \equiv 1 \pmod 8$	$a^2-mb^2 \equiv 8 \pmod{16}$	$ab \equiv 1 \pmod 2$	(4)	2^2
	$a^2-mb^2 \equiv 16 \pmod{32}$	$ab \equiv 1 \pmod 2$	(4)	2^2
	$a^2-mb^2 \equiv 32 \pmod{64}$	$ab \equiv 1 \pmod 2$	(4)	1
	$a \equiv 0 \pmod 2$	$b \equiv 1 \pmod 2$	(6)	1
	$b \equiv 0 \pmod 2$	$a \equiv 1 \pmod 2$	(3)	1
$m \equiv 5 \pmod 8$	All cases			1
$m \equiv 3 \pmod 4$	,,			2^2
$m \equiv 2 \pmod 4$	,,			2^3

Thus for the fields generated by

	$1+3\sqrt{6}$, $9+\sqrt{15}$,		$D \equiv \pm 1 \pmod 3$,
by	$7+2i$, $27+4\sqrt{11}$,	$19+\sqrt{10}$,	$D \equiv \pm 3 \pmod{3^2}$,
by	$5+2i$, $3+\sqrt{11}$,	$1+2\sqrt{7}$,	$D \equiv \pm 3^3 \pmod{3^4}$,
by	$1+\sqrt{6}$, $3+\sqrt{15}$,	$7+5i\sqrt{3}$,	$D \equiv \pm 3^4 \pmod{3^5}$,
and by	$1+\sqrt{7}$, $5+\sqrt{10}$,	$10+\sqrt{19}$,	$D \equiv \pm 3^5 \pmod{3^6}$.

Again taking either of the generators

$$1+\sqrt{13}, \quad 1+\sqrt{33}, \qquad D \equiv 1 \pmod 4,$$

taking $1+\sqrt{3}$, $2+\sqrt{7}$, $7+\sqrt{33}$, $D \equiv 4 \pmod 8$,

and taking $1+\sqrt{6}$, $2+\sqrt{10}$, $D \equiv 8 \pmod{16}$.

The results, when $p \neq 2$ or 3, can be conveniently summarised as follows:—

(i) Each prime factor of m occurs just once in D.

(ii) A prime which divides a^2-mb^2 just $3s+1$ or $3s+2$ times, and does not divide m, occurs just twice in D.

(iii) Primes contained in a^2-mb^2 just $3s$ times, and prime factors of a, b, apart from those which fall under (i) and (ii), are prime to the discriminant of the field.

5. When $D \equiv \pm 3^5 \pmod{3^6}$, m necessarily $\equiv 1 \pmod 3$, the generator $a+b\sqrt{m}$ contains an ideal factor of 3 just $3s+1$ or $3s+2$ times, and the corresponding cubic field will be said to be of the *third category*. N_3 fields of the third category belong to such a discriminant. If, when $m \equiv 1 \pmod 3$, a prime ideal factor of 3 divides the generator just $3s$ times, then $D \equiv \pm 3 \pmod{3^2}$ or $\pm 3^3 \pmod{3^4}$ and, after § 3, the corresponding cubic field can be defined by a generator containing no ideal factor of 3.

Generators free from ideal factors of 3 will be said to be *primary*, and the corresponding fields to be of the *first category*, when

$$D \equiv \pm 1 \pmod 3 \quad \text{or} \quad \pm 3 \pmod{3^2}.$$

If $D \equiv \pm 3^4 \pmod{3^5}$ or $\pm 3^3 \pmod{3^4}$ the field will be said to be of the *second category* and the generator to be *non-primary*. The conditions for a number $a+b\sqrt{m}$ to be primary are

$$\left.\begin{array}{lll}
m \equiv 0 \pmod 3 & \left\{\begin{array}{lll} & a \equiv \pm 1 \pmod 3, & b \equiv 0 \pmod 3, \\ \text{or} & a \equiv 0 \pmod 9, & b \equiv \pm 1 \pmod 3; \end{array}\right. \\
m \equiv 2 \pmod 3 & \left\{\begin{array}{lll} & ab \equiv \pm 1 \pmod 3, & a^2+mb^2 \equiv 0 \pmod 9, \\ \text{or} & a \equiv 0 \pmod 9, & b \equiv \pm 1 \pmod 3, \\ \text{or} & a \equiv \pm 1 \pmod 3, & b \equiv 0 \pmod 9; \end{array}\right. \\
m \equiv 1 \pmod 3 & \left\{\begin{array}{lll} & a \equiv 0 \pmod 9, & b \equiv \pm 1 \pmod 3, \\ \text{or} & a \equiv \pm 1 \pmod 3, & b \equiv 0 \pmod 9. \end{array}\right.
\end{array}\right\} \quad (8)$$

It is readily seen that the product of two primary numbers is primary and that when one factor of a primary product is primary the other factor is primary too. Further, if $a+b\sqrt{m}$, $a'+b'\sqrt{m}$ are non-primary, one and one only of the products

$$(a+b\sqrt{m})(a'+b'\sqrt{m}), \quad (a+b\sqrt{m})(a'-b'\sqrt{m})$$

is primary. It will be convenient to assign the primary character 1 or ρ or ρ^2 to each integer $a+b\sqrt{m}$ divisible by no prime ideal factor of 3. When $a_1+b_1\sqrt{m}$ is primary its character is 1 and, taking that of a non-primary number $a_2+b_2\sqrt{m}$ to be ρ, those of $a_3+b_3\sqrt{m}$, $a_4+b_4\sqrt{m}$ are ρ^2, ρ respectively when

$$(a_2+b_2\sqrt{m})(a_3+b_3\sqrt{m}), \quad (a_3+b_3\sqrt{m})(a_4+b_4\sqrt{m})$$

are primary.

The definition of primary numbers clearly leads to a supplementary law of reciprocity. It is not necessary to enter into this development on the present occasion.

Fields belonging to the *associated* discriminants D_0, 3^4D_0 or $3D_0$, 3^3D_0, with $dv(3, D_0) = 1$, will be found together and N_1 is the number of distinct fields of the first category, N_2 that of the second category.

6. We are now in a position to search for the cubic fields which belong to an assigned discriminant D. Put

$$D = \xi.2^u.3^v.p_1p_2\dots p_e q_1^2q_2^2\dots q_f^2, \tag{9}$$

where $\xi = \pm 1$ and the p's and q's are different positive primes. 2 is to be included among the p's when $D \equiv 8 \pmod{16}$, whence $u = 0$ or 2, but 3 is not so included. The field $[\sqrt{m}]$ is thus defined by

$$m = -\xi p_1p_2\dots p_e \quad \text{or} \quad m = -\xi.3p_1p_2\dots p_e, \tag{10}$$

according as v is odd or even. 2 is only to be included among the q's when $m \equiv 1 \pmod 8$ and $D \equiv 4 \pmod 8$, and 3 only when

$$D \equiv \pm 3^5 \pmod{3^6}.$$

Then, if $a+b\sqrt{m}$ is the generator of a cubic field whose discriminant is D, different prime ideal factors of every q divide $a+b\sqrt{m}$, $a-b\sqrt{m}$. (When $q = 2$, m must $\equiv 1 \pmod 8$ and ab must $\equiv 1 \pmod 2$). The

two prime ideal factors of 2 divide $\frac{1}{2}(a+b\sqrt{m})$, $\frac{1}{2}(a-b\sqrt{m})$ respectively. It will not be necessary to mention this obvious modification again.) Hence

$$a^2-mb^2 \equiv 0 \pmod{q_i} \quad \text{or} \quad z_i^2 \equiv m \pmod{q_i}, \tag{11}$$

and so it is necessary for m to be a quadratic residue of every q in order that D (or its associate) may be a discriminant. When this essential condition is satisfied the principal ideal

$$(q_i) = (q_i, z_i+\sqrt{m})(q_i, z_i-\sqrt{m}).$$

To simplify the notation it will be supposed, in the first instance, that the principal genus of ideals in $[\sqrt{m}]$ contains only one cycle, the class symbols being

$$C, \quad C^2, \quad \ldots, \quad C^{h-1}, \quad C^h \sim 1.$$

Then
$$(q_i, z_i+\sqrt{m}) \sim C^{r_i}P_i,$$

$$(q_i, z_i-\sqrt{m}) \sim C^{h-r_i}P_i^{-1} \sim C^{h-r_i}P_i,$$

where $P_i^2 \sim 1$. P_i is a product of some of the prime ideals

$$(p_1, \sqrt{m}), \quad (p_2, \sqrt{m}), \quad \ldots, \quad (p_e, \sqrt{m})$$

together with (or including) prime ideal factors of 2 and 3 when these ideals do not belong to the principal genus and when their squares are (2) and (3) respectively. Forming a product of f ideals, the i-th factor being one of the pair

$$(q_i, z_i+\sqrt{m}), \quad (q_i, z_i-\sqrt{m}), \quad i = 1, 2, \ldots, f,$$

we have
$$(q_1q_2\ldots q_f, c_0+\sqrt{m}) \sim C^{\Sigma\pm r_i}\prod_{1,f}P_i \sim C^{\Sigma x_i r_i}\prod_{1,f}P_i \tag{12}$$

where $x_i = +1$ or -1 according as $(q_i, z_i+\sqrt{m})$ or $(q_i, z_i-\sqrt{m})$ is a factor of the ideal on the left. Under the possible permutations of sign there are 2^f products of the type

$$(q_1q_2\ldots q_f, c_0+\sqrt{m}),$$

no two having the same c_0. Next it is necessary to find an ideal $\mathfrak{j}$ such that the ideal

$$\mathfrak{j}^3(q_1q_2\ldots q_f, c_0+\sqrt{m})$$

is principal. Taking $\mathfrak{j} \sim C^w\prod_{1,f}P_i$ it is only necessary to make

$$3w+x_1r_1+x_2r_2+\ldots+x_fr_f \equiv 0 \pmod{h}. \tag{13}$$

7. When h is prime to 3 the congruence (13) gives 2^f values (not always distinct) of $w \pmod h$, there being one for each permutation of signs

$$x_1 = \pm 1, \quad x_2 = \pm 1, \quad \ldots, \quad x_f = \pm 1.$$

Taking $(q, c+\sqrt{m})$ to be an ideal in the class $C^w \Pi P_i$ the ideal

$$(q, c+\sqrt{m})^3 (q_1 q_2 \ldots q_f, c_0+\sqrt{m})$$

is principal and expressible by a finite number of operations in the form

$$l(a+b\sqrt{m}) \quad \text{with} \quad dv(a, b) = 1.$$

After § 3 the discriminant of the field generated by $a+b\sqrt{m}$ is either D, as required, or else the associate of D, the exact power of 3 contained in the discriminant depending on the primary character of $a+b\sqrt{m}$ as recorded in § 4.

The generator $a+b\sqrt{m}$, just obtained, depends on the choice of the ideal $(q, c+\sqrt{m})$. Retaining the same set of values for $x_1, x_2, \ldots, x_f$, let $(q', c'+\sqrt{m})$ be another ideal such that

$$(q', c'+\sqrt{m})^3 \sim (q, c+\sqrt{m})^3.$$

Then

$$(q, c+\sqrt{m})^3 (q_1 q_2 \ldots q_f, c_0+\sqrt{m}) = l(a+b\sqrt{m}),$$

$$(q', c'+\sqrt{m})^3 (q_1 q_2 \ldots q_f, c_0+\sqrt{m}) = l'(a'+b'\sqrt{m})\eta,$$

with $dv(a', b') = 1$ and η a unit of the quadratic field. Also

$$(q, c-\sqrt{m})^3 (q_1 q_2 \ldots q_f, c_0-\sqrt{m}) = l(a-b\sqrt{m}),$$

$$(q', c'-\sqrt{m})^3 (q_1 q_2 \ldots q_f, c_0-\sqrt{m}) = l'(a'-b'\sqrt{m})\eta',$$

η' being conjugate to η. (The introduction of further units into the last two equations makes no effective difference in the sequel.) Since h is prime to 3 the equivalence of $(q, c+\sqrt{m})^3$, $(q', c'+\sqrt{m})^3$ involves that of $(q, c+\sqrt{m})$, $(q', c'+\sqrt{m})$ and there exist conjugate pairs (α, α'), (β, β'), such that

$$\alpha(q, c+\sqrt{m}) = \beta(q', c'+\sqrt{m}),$$

$$\alpha'(q, c-\sqrt{m}) = \beta'(q', c'-\sqrt{m}).$$

It follows that $$\frac{a+b\sqrt{m}}{a-b\sqrt{m}} = \left(\frac{\beta\alpha'}{\alpha\beta'}\right)^3 \frac{(a'+b'\sqrt{m})\eta}{(a'-b'\sqrt{m})\eta'},$$

and hence that the fields generated by $a+b\sqrt{m}$, $(a'+b'\sqrt{m})\eta$ are identical.

Further, when η, η_0 are different units, the fields generated by

$$(a'+b'\sqrt{m})\eta, \quad (a'+b'\sqrt{m})\eta_0$$

can only be the same when η/η_0 is a cube in $[\sqrt{m}]$. Since all the units of a quadratic field are comprised in the expression $\pm\epsilon^n$, $a+b\sqrt{m}$ is one of a set of three associated generators

$$a+b\sqrt{m}, \quad (a+b\sqrt{m})\epsilon, \quad (a+b\sqrt{m})\epsilon^2,$$

defining different fields, when $m>0$ or $=-3$, whereas this generator stands alone when $m=-1$, -2, or <-3.

No two different permutations of sign among the x's in (13) can give rise to the same cubic field. To prove this it is sufficient to consider the two cubic equations

$$(a_1+b_1\sqrt{m})(a_2+b_2\sqrt{m})(\gamma+\sqrt{m})^3+(a_1-b_1\sqrt{m})(a_2-b_2\sqrt{m})(\gamma-\sqrt{m})^3 = 0,$$

$$(a_1+b_1\sqrt{m})(a_2-b_2\sqrt{m})(\delta+\sqrt{m})^3+(a_1-b_1\sqrt{m})(a_2+b_2\sqrt{m})(\delta-\sqrt{m})^3 = 0,$$

where $a_1^2-mb_1^2$, $a_2^2-mb_2^2$ contain the respective prime factors q_1, q_2 once only. Were these two equations to define the same field

$$\frac{a_2+b_2\sqrt{m}}{(a_2-b_2\sqrt{m})\eta} \quad \text{would be a cube in } [\sqrt{m}], \quad \frac{\tau^3}{\sigma^3} \text{ say,}$$

whence $$(a_2+b_2\sqrt{m})\sigma^3 = (a_2-b_2\sqrt{m})\eta\tau^3.$$

The prime ideal $(q_2, a_2+b_2\sqrt{m})$ divides the left just $3M+1$ times and the right just $3M'$ times, leading to a contradiction, so the fields $[\gamma]$, $[\delta]$ are certainly distinct as stated. (With an obvious modification in the proof this argument, and the result, also apply when h is divisible by 3.)

8. Returning to the congruence (13) when h is divisible by 3, some of the r's, $r_1, r_2, \ldots, r_g$, say, will be prime to 3 and the others, $r_{g+1}, \ldots, r_f$, divisible by 3. Every permutation of sign among the latter set is permissible, but not among the former set, for

$$x_1r_1+x_2r_2+\ldots+x_gr_g \equiv 0 \pmod 3, \qquad (14)$$

and, when this congruence is satisfied, (13) gives a unique value of $w \pmod{\frac{1}{3}h}$.

Corresponding to each solution of (13) there are three generators, apart from combinations with units, in the quadratic field. If, as in § 7,

$$(q, c+\sqrt{m})^3(q_1q_2\dots q_f, c_0+\sqrt{m}) = l(a+b\sqrt{m}),$$

$$(q', c'+\sqrt{m})^3(q_1q_2\dots q_f, c_0+\sqrt{m}) = l'(a'+b'\sqrt{m}),$$

the equivalence of $(q, c+\sqrt{m})^3$, $(q', c'+\sqrt{m})^3$ does not involve that of $(q, c+\sqrt{m})$, $(q', c'+\sqrt{m})$. It can only be inferred that $(q', c'+\sqrt{m})$ is equivalent to one of the three ideals

$$(q, c+\sqrt{m}), \quad (q, c+\sqrt{m})(q'', c''+\sqrt{m})^{h/3}, \quad (q, c+\sqrt{m})(q'', c''+\sqrt{m})^{2h/3},$$

where $(q'', c''+\sqrt{m})$ is an ideal in the class C. Putting

$$(q, c+\sqrt{m})^3(q'', c''+\sqrt{m})^h(q_1q_2\dots q_f, c_0+\sqrt{m}) = l_1(a_1+b_1\sqrt{m}),$$

$$(q, c+\sqrt{m})^3(q'', c''+\sqrt{m})^{2h}(q_1q_2\dots q_f, c_0+\sqrt{m}) = l_2(a_2+b_2\sqrt{m}),$$

the fields generated by $a+b\sqrt{m}$, $a_1+b_1\sqrt{m}$, $a_2+b_2\sqrt{m}$ are distinct from each other, for otherwise, η_1, η_2 being units,

$$\eta_1(a_1+b_1\sqrt{m})/(a+b\sqrt{m}), \quad \eta_2(a_2+b_2\sqrt{m})/(a+b\sqrt{m})$$

would be cubes of numbers in $[\sqrt{m}]$. This is not the case since $(q'', c''+\sqrt{m})^{h/3}$ is not a principal ideal, so due to each solution of (13) to the modulus $\frac{1}{3}h$ there are three distinct generators

$$a+b\sqrt{m}, \quad a_1+b_1\sqrt{m}, \quad a_2+b_2\sqrt{m}.$$

With each of the generators units can be incorporated as in § 7, so the fields actually appear in sets of nine when $m > 0$, the generators of a set being

$$(a+b\sqrt{m})\epsilon^s, \quad (a_1+b_1\sqrt{m})\epsilon^s, \quad (a_2+b_2\sqrt{m})\epsilon^s, \quad s = 0, 1, 2.$$

At this stage it is necessary further to distinguish between positive and negative values of m.

9. When m is positive and h prime to 3,

$$N_3 \quad \text{or} \quad N_1+N_2 = 3 \cdot 2^{f-1}.$$

The generators of the distinct fields can be arranged in 2^{f-1} sets of three,

a typical set being

$$a+b\sqrt{m}, \quad (a+b\sqrt{m})\epsilon, \quad (a+b\sqrt{m})\epsilon^2$$

with ϵ, as primitive unit of $[\sqrt{m}]$, > 1.

It usually happens (as when $m < 29$) that ϵ is non-primary; in these circumstances one and only one generator of the triad is primary.

There exist certain values of m for which ϵ is primary. In the first hundred, twenty-five are of this type, viz.: 29, 43, 58, 62, 67, 69, 74, 77, 78, 82, 83, 85, 87, 93, 103, 106, 109, 113, 114, 122, 131, 137, 139, 151, 159.

The principal ideal $(q_i, z_i+\sqrt{m})^h = (a_i+b_i\sqrt{m})\epsilon^s$ and the generators of the $3 \cdot 2^{f-1}$ fields can be taken to be

$$(a_1+b_1\sqrt{m})(a_2 \pm b_2\sqrt{m})\ldots(a_f \pm b_f\sqrt{m})\epsilon^s, \quad s = 0, 1, 2. \tag{15}$$

10. Next, when $m > 0$ and $h \equiv 0 \pmod 3$, the generators of the distinct fields fall into sets of nine, a typical set being

$$(a+b\sqrt{m})\epsilon^s \alpha^t, \quad s = 0, 1, 2, \quad t = 0, 1, 2, \tag{16}$$

where (α) is the cube of an ideal in the class $C^{h/3}$. Whether ϵ is primary or not it usually happens that three of the nine integers

$$1, \quad \epsilon, \quad \epsilon^2, \quad \alpha, \quad \epsilon\alpha, \quad \epsilon^2\alpha, \quad \alpha^2, \quad \epsilon\alpha^2, \quad \epsilon^2\alpha^2,$$

have each of the primary characters $1, \rho, \rho^2$. (A change of α does not alter this property.) In such cases three of the generators (16) are primary and six non-primary. Within the limits of Gauss's tables,* the following values of m fall under this head: 79, 223, 226, 235, 254, 257, 321, 326, 730, 761, 839, 842, 874, 895, 898, 899, 906, 934, 985, 993.

Cases occur, however, when ϵ and α are both primary. Thus, if $m = 14189 = 7.2027$, $h = 3$,

$$\epsilon = \tfrac{1}{2}(2025+17\sqrt{14189}),$$

$$\alpha = \tfrac{1}{2}(117+\sqrt{14189}), \quad \alpha\alpha' = -5^3, \quad (5, \alpha) \backsim (1).$$

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* *Werke*, 2, 475–6.

For such values of m the nine generators (16) have all the same primary character as $a+b\sqrt{m}$.

11. When h is prime to 3 there is one cubic field, to be designated *ground-field*, for each value of m, not included among those obtained in § 7. Its generator is ϵ and discriminant as below:

Residue of m (mod 12)	1, 5,	2, 7, 10, 11,	3, 6,	9,
ϵ primary	$-3m$,	$-12m$,	$-\frac{4}{3}m$,	$-\frac{1}{3}m$,
ϵ non-primary	$-27m$,	$-108m$,	$-108m$,	$-27m$.

There are four ground-fields, generated by ϵ, α, $\epsilon\alpha$, $\epsilon^2\alpha$, when $h \equiv 0 \pmod 3$. In the most frequent cases (*i.e.* when $m = 79$, etc.) one of these generators is primary and one ground-field of the first category, three of the second. When exceptionally (as for $m = 14189$) the four numbers ϵ, α, $\epsilon\alpha$, $\epsilon^2\alpha$ are all primary, all four ground-fields are of the first category.

It is of interest to state the conditions that an assigned negative *prime* number $-p$ may be a discriminant. Clearly $p \equiv 3 \pmod 4$, $m = 3p$ and, when h is prime to 3, ϵ must be primary. When $h \equiv 0 \pmod 3$ one of the numbers ϵ, α, $\epsilon\alpha$, $\epsilon^2\alpha$ is primary and $-p$ is always a discriminant. These conditions are both necessary and sufficient,

e.g. $$m = 69, \quad \epsilon = \tfrac{1}{2}(25+3\sqrt{69}), \quad D = -23.$$

Up to 1000 the following prime values of $-D$ were recorded by the writer on a previous occasion*: 23, 31, 59, 83, 107, 139, 199, 211, 239, 283, 307, 331, 367, 379, 419, 431, 439, 491, 499, 503, 547, 563, 643, 743, 751, 823, 839, 863, 883, 907, 971, 983.

Belonging to each of these discriminants there is only one set of conjugate fields.

12. Passing to negative values of m the quadratic field $[\sqrt{m}]$ contains only the real units ± 1 unless $m = -1$ or -3. No special treatment of $[\sqrt{-1}]$ is necessary for, since $(\sqrt{-1})^3 = -\sqrt{-1}$, the generators

$$a+b\sqrt{-1}, \quad (a+b\sqrt{-1})\sqrt{-1}$$

* *Proc. London Math. Soc.*, 10 (1912), 137–138.

define the same field. The other exceptional value, $m = -3$, is reserved for consideration in § 17 below.

First, when h is prime to 3, put

$$(q_i) = \mathfrak{q}_i\mathfrak{q}_i', \quad \mathfrak{q}_i^h = (a_i + b_i\sqrt{m}), \quad i = 1, 2, \ldots, f.$$

The 2^{f-1} fields belonging to the associated discriminants are then defined by the generators

$$(a_1 + b_1\sqrt{m})(a_2 \pm b_2\sqrt{m}) \ldots (a_f \pm b_f\sqrt{m}). \tag{17}$$

Next when the class-symbols of the principal genus form a single cycle whose order is divisible by 3, the cube of an ideal $\mathfrak{a}$ in the class $C^{h/3}$ is the principal ideal (a). Also if

$$\mathfrak{a} \sim \mathfrak{b}', \quad \mathfrak{b}\mathfrak{b}' = (b_0), \quad \mathfrak{b}^3 = (\beta),$$

$\mathfrak{a}\mathfrak{b}$ is the principal ideal (γ) and $a\beta = \gamma^3$ which is always primary. Hence the primary character of a is independent of the particular ideal $\mathfrak{a}$ in the class $C^{h/3}$. This number a plays the same part as the primitive unit ϵ in the real theory, and 3.2^{f-1} fields belong to the associated discriminants. Usually a is non-primary as when

$$m = -23, \ -26, \ -29, \ -31, \ -38, \ -59, \ -106, \ -109, \ -155, \ -157.$$

In such cases the generators of the distinct fields fall into sets of three and one member of each triad is primary.

There exist certain values of m for which a is primary. A few of this type are

$$-107, \ -237, \ -262, \ -331, \ -397, \ -643, \ -687, \ -705,$$

$$-771, \ -1419, \ -2091, \ -3387, \ -4035, \ -4119, \ -6063.$$

13. When m is negative and h prime to 3, there is no ground-field corresponding to that generated by the primitive unit for a positive m. But when $h \equiv 0 \pmod 3$ there exists a ground-field generated by a of § 12. Its discriminant is as noted in § 11 above.

The conditions that a positive prime p may be a discriminant

necessarily of a ground-field, are:

(i) $p \equiv 1 \pmod 4$, (§ 4, III);

(ii) h, the class-number of $[\sqrt{-3p}]$, $\equiv 0 \pmod 3$;

(iii) $a \equiv 0 \pmod 9$, $b \equiv \pm 1 \pmod 3$ or $a \equiv \pm 1 \pmod 3$, $b \equiv 0 \pmod 3$, ensuring a field of the first category;

(iv) $a^2+3pb^2 = 3c^3$ or c^3 when ab is even,

$a^2+3pb^2 = 12c^3$ or $4c^3$ when ab is odd,

in the respective cases of (iii), a, b being co-prime integers;

(v) $3(a+b\sqrt{-3p})$ or $(a+b\sqrt{-3p})$, when ab is even,

$\frac{3}{2}(a+b\sqrt{-3p})$ or $\frac{1}{2}(a+b\sqrt{-3p})$, when ab is odd,

not to be the cube $(a'+b'\sqrt{-3p})^3$ of a number in the field—thus ensuring an irreducible equation.

It is only for certain exceptional values of p that these conditions can be satisfied: in fact positive prime discriminants appear to be rare. The four smallest known to the writer (who has made no systematic search), along with the corresponding generators, are:

$$229, \quad 9+\sqrt{-687}, \qquad (9^2+3\,.\,229 = 3\,.\,4\,.\,4^3),$$

$$257, \quad 27+\sqrt{-771}, \qquad (27^2+3\,.\,257 = 3\,.\,4\,.\,5^3),$$

$$1129, \quad 27+\sqrt{-3387}, \qquad (27^2+3\,.\,1129 = 3\,.\,4\,.\,7^3),$$

$$1373, \quad 45+\sqrt{-4119}, \qquad (45^2+3\,.\,1373 = 3\,.\,4\,.\,8^3).$$

14. The available data concerning quadratic forms show only one more feature which can be illustrated by concrete examples. In the simplest case the class-symbols of the principal genus are

$$C_1^{y_1} C_2^{y_2}, \quad 0 \leqslant y_1 < h_1, \quad 0 \leqslant y_2 < h_2,$$

where h_1, h_2 are both divisible by 3. In taking possible values of m from Gauss's tables* it is necessary to allow for the fact that the ideal theory agrees with Kronecker's quadratic forms $ax^2+bxy+cy^2$, which causes the

* *Werke*, 2, 450–474.

exclusion of many of his irregular determinants $8n+5$. Within the limit of the tables the thirteen exceptional values of m are:

$$-974, \quad -2437, \quad -5069, \quad -5857, \quad -6085, \quad -6221, \quad -6226,$$

$$-8242, \quad -9385, \quad -9497, \quad -9574, \quad -9934, \quad -10798.$$

For each of these there are eight non-equivalent ideals whose cubes are principal, *e.g.*, giving one of each conjugate pair,

$$\mathfrak{a} = (22, 7+\sqrt{-2437}), \qquad \mathfrak{a}^3 = (30-2\sqrt{-2437}),$$

$$\mathfrak{b} = (41, 8+\sqrt{-2437}), \qquad \mathfrak{b}^3 = (173-4\sqrt{-2437}),$$

$$\mathfrak{ab} \sim \mathfrak{c} = (29, 17+\sqrt{-2437}), \qquad \mathfrak{c}^3 = (121+2\sqrt{-2437}),$$

$$\mathfrak{a}^2\mathfrak{b} \sim \mathfrak{d} = (46, 1+\sqrt{-2437}), \qquad \mathfrak{d}^3 = (98+6\sqrt{-2437}).$$

The primary character of $\mathfrak{j}^3 \equiv (a+b\sqrt{m})$ on the right only depends on the class of $\mathfrak{j}$. For each of the thirteen values of m noted just one of the four generators is primary ($173-4\sqrt{-2437}$ in the above), so one ground-field is of the first category, and three are of the second.

15. In the most general case the class-symbols of the ideals in the principal genus of $[\sqrt{m}]$ are

$$C_1^{y_{1i}} C_2^{y_{2i}} \dots C_k^{y_{ki}} \dots C_l^{y_{li}}, \quad 0 \leqslant y_{ki} < h_k.$$

Assuming that $h_1, h_2, \dots h_k$ are divisible by 3 and the others not, there exist 3^k-1 non-equivalent ideals whose cubes are principal. Since μ, μ^2 generate the same field, there are $\frac{1}{2}(3^k-1)$ ground-fields when m is negative and $\frac{1}{2}(3^{k+1}-1)$ when m is positive, for units can be incorporated in the latter case. Putting $j=k$ when $m<0$, $j=k+1$ when $m>0$, the generators of the ground-fields are

$$\alpha_1^{z_1} \alpha_2^{z_2} \dots \alpha_j^{z_j}, \quad z_i = 0, 1, 2,$$

the first non-vanishing z being 1. If α_1, α_2 are both non-primary either $\alpha_1\alpha_2$ or $\alpha_1^2\alpha_2$ is primary and α_2 can be replaced by the primary one of the pair. Repeating, the generators of the ground-fields can be taken to be

$$\beta_1^{z_1} \beta_2^{z_2} \dots \beta_j^{z_j}, \quad z = 0, 1, 2,$$

where $\beta_2, \dots, \beta_j$ are all primary.

16. In the table which follows, G_1, G_2 denote the respective numbers of ground-fields of the first and second categories. The values of N_1, N_2 are obtained in the simpler cases by noting that the congruence (14) has $2\phi(g) \equiv \frac{2}{3}\{2^{g-1}-(-1)^{g-1}\}$ solutions, and $f-d$ is the number of primary factors $a_i+b_i\sqrt{m}$ in (15) or (17). When the fields are all of the third category, $N_3 = N_1+N_2$.

j	D	β_1	G_1	G_2	N_1	N_2	m
0	+	...	0	0	$2^{f-d}\phi(d)$	$2^{f-d}\phi(d+1)$	$-1, -2, -5$
1	−	ρ, ρ^2	0	1	2^{f-1}	2^f	2, 3, 5
1	−	1	1	0	$3.2^{f-d}\phi(d)$	$3.2^{f-d}\phi(d+1)$	29, 43
1	+	ρ, ρ^2	0	1	$2^{f-g}\phi(g)$	$2^{f-g+1}\phi(g)$	$-23, -31$
1	+	1	1	0			-107
2	−	ρ, ρ^2	1	3	$3.2^{f-g}\phi(g)$	$3.2^{f-g+1}\phi(g)$	79
2	−	1	4	0			14189
2	+	ρ, ρ^2	1	3			-2437
j	−	ρ, ρ^2	$\frac{1}{2}(3^{j-1}-1)$	3^{j-1}			
j	−	1	$\frac{1}{2}(3^j-1)$	0			
j	+	ρ, ρ^2	$\frac{1}{2}(3^{j-1}-1)$	3^{j-1}			
j	+	1	$\frac{1}{2}(3^j-1)$	0			

Most small values of m fall into one of the first four lines and additional entries for the next four are not to be found without labour.

Whether more than the eight types noted in the preceding pages actually exist is uncertain in the present state of knowledge, as supplied by Gauss's tables. The writer knows no reason why all four types should not exist for every positive j. Since there are four fields when

$$D = -42567 = -3.14189 = -3.7.2027$$

there is no possibility of proving that only one set of conjugate fields can belong to a discriminant with no square factor.

17. The value -3 for m, excluded from § 12, is exceptional in that the field $[\sqrt{-3}]$ contains the cyclotomic units

$$\omega \equiv \tfrac{1}{2}(-1+\sqrt{-3}) \equiv \exp(2\pi i/3), \qquad \omega^2 \equiv \tfrac{1}{2}(-1-\sqrt{-3}) \equiv \exp(4\pi i/3).$$

Every cubic field whose generator lies in $[\sqrt{-3}]$ has a square discriminant and, being cyclical, is identical with its conjugates. The fields

appear in sets of three, a typical set being defined by

$$(a+b\sqrt{-3})\omega^s(\gamma_s+\sqrt{-3})^3+(a-b\sqrt{-3})\omega^{2s}(\gamma_s-\sqrt{-3})^3=0,$$

$$s=0,\ 1,\ 2. \quad (18)$$

A prime number $q_i, \equiv 1 \pmod 6$, is known to be expressible as a product of *primary* factors in $[\sqrt{-3}]$, so 2^{f-1} fields of the first category have $D=q_1^2q_2^2\dots q_f^2$, and 2^f of the second have $D=81q_1^2q_2^2\dots q_f^2$. There is also a ground field, of discriminant 81, generated by ω.

By a theorem due to Kronecker the roots of a cyclical cubic equation, with $D=q_1^2q_2^2\dots q_f^2$, are expressible in terms of $q_1q_2\dots q_f$-th roots of unity. Also from (18)

$$\frac{\beta^s(\gamma_s+\sqrt{-3})}{\gamma_0+\sqrt{-3}}=\frac{\beta^{2s}\omega^t(\gamma_s-\sqrt{-3})}{\gamma_0-\sqrt{-3}}$$

where β is a ninth root of unity. Since the roots of each equation are real, γ_1 and γ_2 are expressible rationally in terms of γ_0 and $\cos 2\pi/9$. When $q=7$, the three cubic fields can be defined by

$$\theta^3+\theta^2-2\theta-1=0, \quad \chi^3-21\chi+35=0, \quad \psi^3-21\psi-28=0$$

with the respective discriminants 7^2, 63^2, 63^2.

If $\alpha=\exp(2\pi i/7)$ and $\beta=\exp(2\pi i/9)$, their roots are

$$\theta=\alpha+\alpha^6, \quad \theta'=\alpha^2+\alpha^5, \quad \theta''=\alpha^4+\alpha^3;$$

$$\chi=(\alpha+\alpha^6)(\beta+\beta^8)+(\alpha^2+\alpha^5)(\beta^4+\beta^5)+(\alpha^4+\alpha^3)(\beta^2+\beta^7),$$

$$\chi'=(\alpha+\alpha^6)(\beta^4+\beta^5)+(\alpha^2+\alpha^5)(\beta^2+\beta^7)+(\alpha^4+\alpha^3)(\beta+\beta^8),$$

$$\chi''=(\alpha+\alpha^6)(\beta^2+\beta^7)+(\alpha^2+\alpha^5)(\beta+\beta^8)+(\alpha^4+\alpha^3)(\beta^4+\beta^5);$$

$$\psi=(\alpha+\alpha^6)(\beta+\beta^8)+(\alpha^2+\alpha^5)(\beta^2+\beta^7)+(\alpha^4+\alpha^3)(\beta^4+\beta^5),$$

$$\psi'=(\alpha+\alpha^6)(\beta^4+\beta^5)+(\alpha^2+\alpha^5)(\beta+\beta^8)+(\alpha^4+\alpha^3)(\beta^2+\beta^7),$$

$$\psi''=(\alpha+\alpha^6)(\beta^2+\beta^7)+(\alpha^2+\alpha^5)(\beta^4+\beta^5)+(\alpha^4+\alpha^3)(\beta+\beta^8).$$

18. To complete the investigation it is necessary to mention the value $m=1$ excluded at the beginning. The corresponding cubic equation is a rational transformation of

$$\theta^3-n=0,$$

and it can be assumed that $n = p_1 p_2 \dots p_e q_1^2 q_2^2 \dots q_f^2$, where the p's and q's are different primes. Its integral basis is*

$$\left(1, \theta, \frac{\theta^2}{q_1 q_2 \dots q_f}\right) \quad \text{or} \quad \left(1, \theta, \frac{\theta^2 + n\theta + n^2}{3 q_1 q_2 \dots q_f}\right)$$

according as $n \not\equiv \pm 1$ or $\equiv \pm 1 \pmod 9$. These fields are the ones discussed by Dedekind in his classical memoir on cubic corpora.†

[*Addendum* (August 14th, 1924.)—A referee has drawn my attention to Fräulein Wanda Braun's Inaugural Dissertation, "Bestimmung der Körpersdiskriminante in einem kubischen Zahlkörper" (Strassburg, 1909). This tract discusses what prime factors divide the discriminant of a cubic field, defined as by (1) above. It obtains results that appear to be similar to most of those now tabulated in § 4 when translated into the present notation. Some of the conclusions, however, are incorrect, *e.g.* the author arrives at the result that a discriminant contains the prime factor 2 only once in certain cases.]

* Sommer, *Vorlesungen über Zahlentheorie*, 261.

† *Journal für Math.*, 121 (1900), 40–123.

ON THE OCCURRENCE OF A LINEAR DETERMINANTAL SYSTEM OF POINTS

By W. L. Marr.

[Received November 22nd, 1923.—Read December 13th, 1923.]

1.—*Introduction.*

In the *Proc. London Math. Soc.* (2), 18 (1920), 274–279, appears a paper by Dr. W. P. Milne entitled "Determinantal Systems of Co-apolar Triads on a Cubic Curve." These determinantal configurations of points present themselves thus :

Let there be given an "apolar triangle" ABC situated on a cubic curve. Then a singly infinite system of determinantal configurations of nine points

$$\begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

can be found on the cubic curve, such that the six triangles corresponding to the six terms in the expansion of the above determinantal form constitute each a triangle co-apolar with ABC. Following on this result, Dr. Milne suggested that I should make a special study of these determinantal configurations of nine points in the case of rational cubic curves. This led at once to the study of determinantal systems of points situated (i) on a straight line, (ii) on a conic, (iii) on a twisted cubic. The results of these investigations were published by me in two papers in the *Proc. Edinburgh Math. Soc.*, 37 (Session 1918–1919).

A determinantal system of points

$$\begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

on a straight line l can be defined as follows :—

(1) The six triads of points $P_\lambda Q_\mu R_\nu (\lambda \neq \mu \neq \nu)$ defined by the terms of the expansion of the above determinantal form are each apolar to the same triad ABC situated on l.

(2) The Hessian points of the above six triads $P_\lambda Q_\mu R_\nu$ are each apolar to the same pair of points HK on l.

(3) The Hessian points of ABC are apolar to HK.

It can easily be shown that (3) is a direct result of (1) and (2).

2.—*The fundamental property.*

The main result to be established in this paper presented itself to me unexpectedly when investigating the complete configuration of cubic curves passing through six points on a conic, and having the tangents at these six points concurrent. I took the particular case where the conic consists of two straight lines, and obtained the following result :—

Let $U_1 U_2 U_3$, $V_1 V_2 V_3$ be two triads of points lying respectively on the lines m, n. Let m and n intersect in the point O, and let l be the line joining the polar points of O with respect to the triads $U_1 U_2 U_3$ and $V_1 V_2 V_3$. The nine lines UV intersect l in a determinantal system of points.

To establish this result we have to establish (1) and (2) above.

It is plain that all cubic curves Ψ through the six points $U_1 U_2 U_3$, $V_1 V_2 V_3$ intersect any given line in triads of points apolar to a given triad. For, if Ψ_1 be any cubic through the six points, the general cubic Ψ through them is given by the equation

$$\Psi \equiv \Psi_1 + (\lambda x + \mu y + \nu z)mn = 0$$

where m and n are taken to represent respectively the two lines on which the points lie. Putting $z = 0$ in this equation, we see at once that the required triad is uniquely defined as being apolar to the three triads in which z is cut by the cubics Ψ_1, xmn, ymn. Thus all cubic curves through the six points $U_1 U_2 U_3$, $V_1 V_2 V_3$ cut l in triads of points apolar to the same triad ABC. Take as particular cases the nine

degenerate cubic curves, consisting each of three lines $U_r V_\rho$, $U_s V_\sigma$, $U_t V_\tau$, and we see that (1) is established.

Consider next the polar conics of O with respect to all the cubics Ψ. Plainly they form a pencil, since they all pass through the intersections of the polar conic of O with respect to Ψ_1 and the lines m, n, *i.e.* the polar points of O with respect to the triads $U_1 U_2 U_3$ and $V_1 V_2 V_3$. Thus the polar conics of O will cut l in a system of pairs of points forming an involution.

Consider now the six particular cases in which Ψ degenerates into a triangle $U_r V_\rho$, $U_s V_\sigma$, $U_t V_\tau$. Since the polar line and the polar conic of a triangle with respect to a given point are known to cut each other in the Hessian points of the triad of points in which the line is cut by the sides of the triangle, it follows that the Hessian points of the six triads, in which the six triangles $U_r V_\rho$, $U_s V_\sigma$, $U_t V_\tau$ cut l, belong to the above involution, and hence possess a pair of points on l to which they are all apolar. Thus the criteria (1) and (2) for a determinantal system of points are established.

3.—*Further remarks and properties.*

It easily follows from a consideration of the cubic Ψ that the Hessian points of the fundamental triad ABC are the points lm, ln, since ABC has already been shown to be apolar to the points of intersection of l with nm^2 and mn^2. In fact, the triad ABC consists of the three points where the apolar locus of $U_1 U_2 U_3$ and $V_1 V_2 V_3$ cuts l. Further, ABC is the Jacobian triad of the three points in which l is intersected by the cubic through O and the nine points of intersection of the pencils

$$(ml)[V_1 V_2 V_3] \quad \text{and} \quad (nl)[U_1 U_2 U_3].$$

4.—*Cubics through* $U_1 U_2 U_3$, $V_1 V_2 V_3$ *having* mn *as a polar conic.*

To find the pair of points H, K on l to which the Hessian points of all the triads $P_\lambda Q_\mu R_\nu$ are apolar, we have merely to find the double points of the involution defined on l by its points of intersection with the polar conics of O with respect to the system of cubics Ψ. Plainly the harmonic triangle of this pencil of polar conics has as its three vertices the point O itself and two points lying on l, since l is the polar line of O with respect to all the cubics Ψ, and therefore to the polar conics of

O with respect to the cubics Ψ. Thus H, K are the vertices of the two line pairs (other than m, n) which constitute the degenerate members of the above pencil of polar conics.

H and K may be otherwise viewed as the poles of the degenerate conic mn with respect to those two cubic curves Ψ each of which possesses the property that all their tangents at $U_1U_2U_3$, $V_1V_2V_3$ are concurrent. This follows from the fact that, if L be the point of concurrency, then, since mn is the polar conic of L, the polar conic of O will consist of two straight lines meeting in L. Hence L will coincide with either H or K, which have been shown above to be the vertices of the two degenerate polar conics of O. We note also that the Hessian points of ABC, viz. the points lm, ln, are apolar to H, K. This verifies (3) above.

I wish to express my indebtedness to Prof. W. P. Milne for the interest he has taken in this paper, and for revising the manuscript in its final form.

ON THE TRINOMIAL QUINTIC

By H. B. C. Darling.

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It is well known that the trinomial quintic $x^5-x-a=0$, to which form, as shown by Bring and Jerrard, the general algebraical equation of the fifth degree can be reduced without loss of generality, was solved by Hermite* in terms of elliptic functions. His work is, however, almost entirely in the domain of elliptic modular functions, and has little or no relation to algebraical considerations. It appeared to the present writer that an investigation commencing on purely algebraical lines, and coordinating subsequently with elliptic functions, might be feasible. He has succeeded in obtaining a solution equivalent to Hermite's, but based on simple algebra, and in which the elliptic function work involved is not of an advanced character.

Let
$$\left.\begin{aligned}\mu_1 &= \nu^{-\frac{2}{5}}\{\sqrt{(\nu^2+1)}+\nu\}^{\frac{1}{5}}\{\sqrt{(\nu^2+1)}+1\}^{\frac{2}{5}}\\ \mu_2 &= \nu^{-\frac{1}{5}}\{\sqrt{(\nu^2+1)}+\nu\}^{\frac{2}{5}}\{\sqrt{(\nu^2+1)}-1\}^{\frac{1}{5}}\end{aligned}\right\}, \tag{1}$$

where ν is a real positive quantity; and let

$$(\mu_1\mu_2)^{\frac{1}{2}}+(\mu_1\mu_2)^{-\frac{1}{2}}=m, \quad (\mu_2/\mu_1)^{\frac{1}{2}}-(\mu_2/\mu_1)^{-\frac{1}{2}}=n. \tag{2}$$

Then, by aid of the identities

$$\{\sqrt{(\nu^2+1)}\pm\nu+1\}^2=2\{\sqrt{(\nu^2+1)}\pm\nu\}\{\sqrt{(\nu^2+1)}+1\},$$

$$\{\sqrt{(\nu^2+1)}\pm\nu-1\}^2=2\{\sqrt{(\nu^2+1)}\pm\nu\}\{\sqrt{(\nu^2+1)}-1\},$$

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* *Comptes Rendus*, 46 (1858) (1), 508.

we find that

$$(\mu_1\mu_2)^{\frac{5}{2}}+(\mu_1\mu_2)^{-\frac{5}{2}} = (2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}},$$

$$(\mu_2/\mu_1)^{\frac{5}{2}}-(\mu_2/\mu_1)^{-\frac{5}{2}} = \frac{\nu-2}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}};$$

and therefore we have, from the identity

$$(x\pm x^{-1})^5\mp 5(x\pm x^{-1})^3+5(x\pm x^{-1}) = x^5\pm x^{-5},$$

$$m^5-5m^3+5m-(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}} = 0, \tag{3}$$

$$n^5+5n^3+5n-\frac{\nu-2}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}} = 0. \tag{4}$$

Equations (3) and (4) are well known forms of resolvable quintics.

Now let M be a quantity defined by the relation

$$(\nu^2-\nu+2)M = \nu(\nu-1)n^3+\sqrt{\{2\nu(\nu^2+1)\}}n^2+3\nu(\nu-1)n+2\sqrt{\{2\nu(\nu^2+1)\}}; \tag{5}$$

then, taking the square, cube, &c., of this equation, employing equation (4) to keep the powers of n below the fifth, and dividing both sides by $(\nu^2-\nu+2)$ at each stage; we find successively,

$$(\nu^2-\nu+2)M^2 = \nu(\nu+1)n^4+4\nu(\nu+1)n^2+(\nu-1)\sqrt{\{2\nu(\nu^2+1)\}}n+4(\nu^2+1),$$

$$(\nu^2-\nu+2)M^3 = \nu\sqrt{\{2\nu(\nu^2+1)\}}.n^4+3\nu(\nu-1)n^3+(4\nu+3)\sqrt{\{2\nu(\nu^2+1)\}}.n^2$$
$$+(8\nu^2-10\nu-2)n+(2\nu+6)\sqrt{\{2\nu(\nu^2+1)\}},$$

$$(\nu^2-\nu+2)M^4 = 4\nu(\nu+1)n^4-(\nu+1)\sqrt{\{2\nu(\nu^2+1)\}}.n^3+(2\nu^3+15\nu^2+17\nu)n^2$$
$$+(\nu-7)\sqrt{\{2\nu(\nu^2+1)\}}.n+4(\nu+3)(\nu^2+1),$$

$$(\nu^2-\nu+2)M^5$$
$$= 5\nu\sqrt{\{2\nu(\nu^2+1)\}}.n^4+10\nu(\nu-1)n^3+10(2\nu+1)\sqrt{\{2\nu(\nu^2+1)\}}.n^2$$
$$+5(5\nu^2-7\nu-2)n+(2\nu^3+9\nu^2+23\nu+2)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}.$$

From these equations we see that

$$M^5-5M^3+5M-(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}} = 0;$$

so that, by (3), M satisfies the same resolvable quintic as m.

If we put $(\mu_1\mu_2)^{\frac{1}{2}} = r$, the roots of equation (3) are known to be

$$r+r^{-1}, \quad \omega r+(\omega r)^{-1}, \quad \omega^2 r+(\omega^2 r)^{-1}, \quad \omega^3 r+(\omega^3 r)^{-1}, \quad \text{and} \quad \omega^4 r+(\omega^4 r)^{-1};$$

where $\omega(=e^{2\pi i/5})$ is a complex fifth root of unity; and therefore the equation has only one real root. But M and m are both real quantities; hence $M = m$, and so, by (5),

$$(\nu^2-\nu+2)m = \nu(\nu-1)n^3+\sqrt{\{2\nu(\nu^2+1)\}}\,.\,n^2+3\nu(\nu-1)n+2\sqrt{\{2\nu(\nu^2+1)\}}; \quad (6)$$

and therefore

$$(\nu^2-\nu+2)(m+n) = \nu(\nu-1)n^3+\sqrt{\{2\nu(\nu^2+1)\}}\,.\,n^2+(4\nu^2-4\nu+2)n+2\sqrt{\{2\nu(\nu^2+1)\}}; \quad (7)$$

whence, proceeding as before, we find successively,

$$(\nu^2-\nu+2)(m+n)^2 = \nu(3\nu-1)n^4+2\sqrt{\{2\nu(\nu^2+1)\}}\,.\,n^3 +(11\nu^2-3\nu+2)n^2+(\nu+3)\sqrt{\{2\nu(\nu^2+1)\}}n+4(\nu^2+1),$$

$$(\nu^2-\nu+2)(m+n)^4 = (3\nu-1)(\nu-2)n^4+(\nu-7)\sqrt{\{2\nu(\nu^2+1)\}}\,.\,n^3+(2\nu^3+21\nu^2-33\nu+16)n^2 -(\nu^2+21\nu+4)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}\,.\,n+\frac{(\nu^2+1)(12\nu^2+4\nu-16)}{\nu},$$

and, finally,

$$(\nu^2-\nu+2)(m+n)^5 = 5(\nu-1)(\nu-2)(2\nu+1)n^3+5(\nu-2)(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}\,.\,n^2 +\frac{5(\nu-2)(2\nu+1)(4\nu^2-4\nu+2)}{\nu}n +\frac{(2\nu^4+40\nu^3-50\nu^2+26\nu-4)}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}.$$

Multiplying this equation by $(-5)^{-\frac{5}{4}}u^{\frac{5}{4}}$, where

$$u = \nu(\nu-2)^{-1}(2\nu+1)^{-1}; \quad (8)$$

multiplying (7) by $(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}$, and adding the two products, we see that if

$$(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(m+n) = y, \quad (9)$$

then

$$(\nu^2-\nu+2)(y^5+y) = (-5)^{-\frac{5}{4}}u^{\frac{5}{4}}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}\,.\,\frac{(2\nu^4+20\nu^3-20\nu^2+46\nu-4)}{\nu};$$

or, dividing by $(\nu^2-\nu+2)$,

$$y^5+y = (-5)^{-\frac{1}{2}} u^{\frac{5}{4}} \sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}} \cdot \frac{(2\nu^2+22\nu-2)}{\nu};$$

so that if

$$\beta = -2^{\frac{3}{2}}(-5)^{-\frac{5}{4}} \frac{(\nu^2+11\nu-1)\sqrt{(\nu^2+1)}}{\nu^{\frac{1}{4}}(\nu-2)^{\frac{3}{4}}(2\nu+1)^{\frac{5}{4}}}, \tag{10}$$

then

$$y^5+y+\beta = 0. \tag{11}$$

Now

$$(\nu^2+11\nu-1)^2(\nu^2+1) = \nu(\nu-2)^5+(2\nu+1)^5$$

$$= \nu^{\frac{1}{2}}(\nu-2)^{\frac{3}{2}}(2\nu+1)^{\frac{5}{2}} \left\{\frac{\nu^{\frac{1}{2}}(\nu-2)^{\frac{5}{2}}}{(2\nu+1)^{\frac{5}{2}}} + \frac{(2\nu+1)^{\frac{5}{2}}}{\nu^{\frac{1}{2}}(\nu-2)^{\frac{1}{2}}}\right\}. \tag{12}$$

Again, if we put

$$\nu = \frac{1}{2q} \cdot \frac{\left\{(1-q^5)(1-q^{15})(1-q^{25})\dots\right\}^5}{(1-q)(1-q^3)(1-q^5)\dots}, \tag{13}$$

where $q = e^{-\pi K'/K}$ for the elliptic modulus k; then, since*

$$(1-q)(1-q^3)(1-q^5)\dots = 2^{\frac{1}{6}} q^{\frac{1}{24}} k^{-\frac{1}{12}} k'^{\frac{1}{6}}; \tag{14}$$

we have

$$\nu = 2^{-\frac{1}{3}} k^{\frac{1}{12}} k'^{-\frac{1}{6}} \lambda^{-\frac{5}{12}} \lambda'^{\frac{5}{6}}, \tag{15}$$

where λ, λ' are the modulus and co-modulus corresponding to q^5. Also, comparing the value of ν in (13) with that employed by the author in his paper† on certain identities enunciated by S. Ramanujan, we see that, by equation (45) of the same paper,

$$\frac{\nu^5(\nu-2)}{2\nu+1} = \frac{\lambda'^4}{4\lambda^2}; \tag{16}$$

so that

$$\frac{\nu(\nu-2)^5}{(2\nu+1)^5} = \nu^{-24} \frac{\lambda'^{20}}{4^5\lambda^{10}}$$

$$= \frac{k'^4}{4k^2}, \tag{17}$$

by (15) above; and therefore

$$\frac{\nu^{\frac{1}{2}}(\nu-2)^{\frac{5}{2}}}{(2\nu+1)^{\frac{5}{2}}} = \frac{k'^2}{2k}.$$

Thus, by (10) and (12),

$$\beta = -2(-5)^{-\frac{5}{4}} \frac{1+k^2}{k^{\frac{1}{2}}k'}. \tag{18}$$

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* Cayley, *Elliptic Functions* (1895), 290.

† *Proc. London Math. Soc.* (2), 19 (1921), 356, equation (33).

The trinomial quintic considered by Hermite* is $x^5-x-\alpha=0$, and he adopts for his parameter α the value

$$\alpha = 2(5)^{-\frac{1}{2}}\frac{1+k^2}{k^{\frac{1}{2}}k'}.$$

Noticing that our equation is $y^5+y+\beta=0$, we see that, *mutatis mutandis*, the parameters α and β are in agreement; since the latter quintic is reduced to Hermite's form by putting

$$y=(-1)^{-\frac{1}{4}}x; \quad \beta=-(-1)^{-\frac{5}{4}}\alpha.$$

Further, by (9), we see that if equation (7) be multiplied by $(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}$, and divided by $(\nu^2-\nu+2)$, it may be written

$$y=\frac{(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}}{(\nu^2-\nu+2)}$$

$$\times\left[\nu(\nu-1)n^3+\sqrt{\{2\nu(\nu^2+1)\}}.n^2+2(2\nu^2-2\nu+1)n+2\sqrt{\{2\nu(\nu^2+1)\}}\right].$$

Hence, since, by (4),

$$n^5+5n^3+5n-\frac{\nu-2}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}=0,$$

it follows that the trinomial quintic $y^5+y+\beta=0$, where β has the value given by (10) or (18), may be transformed by the Tschirnhausen substitution

$$y=az^3+bz^2+cz+d$$

into the resolvable quintic

$$z^5+5z^3+5z+p=0; \tag{19}$$

where

$$\left.\begin{aligned} p &= -\frac{\nu-2}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}, \\ a &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{\nu(\nu-1)}{\nu^2-\nu+2}, & b &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{\sqrt{\{2\nu(\nu^2+1)\}}}{\nu^2-\nu+2}, \\ c &= 2(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{(2\nu^2-2\nu+1)}{\nu^2-\nu+2}, & d &= 2b, \\ c-3a &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}, & \frac{a}{b} &= \frac{\nu(\nu-1)}{\sqrt{\{2\nu(\nu^2+1)\}}}. \end{aligned}\right\} \tag{20}$$

* *Œuvres*, 2 (1908), 11.

Again, if N be a quantity defined by the relation

$$(2\nu^2+\nu+1)N = -(\nu+1)m^3+\sqrt{\{2\nu(\nu^2+1)\}}\,m^2+3(\nu+1)m-2\sqrt{\{2\nu(\nu^2+1)\}}, \tag{21}$$

then, taking the several powers of this equation, employing equation (3) to keep the powers of m below the fifth, and dividing at each stage by $(2\nu^2+\nu+1)$, we find that N satisfies the same resolvable quintic (4) as n; and therefore, since this quintic has only one real root, we have $N = n$; so that, by (21),

$$(2\nu^2+\nu+1)(m+n) = -(\nu+1)m^3+\sqrt{\{2\nu(\nu^2+1)\}}\,.\,m^2+(2\nu^2+4\nu+4)m-2\sqrt{\{2\nu(\nu^2+1)\}}.$$

Thus, since, by (9),

$$y = (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(m+n),$$

and, by (3), $\quad m^5-5m^3+5m-(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}} = 0,$

we see that the trinomial quintic $y^5+y+\beta = 0$ may be transformed by the Tschirnhausen substitution

$$y = a'z'^3+b'z'^2+c'z'+d'$$

into the resolvable quintic

$$z'^5-5z'^3+5z'+p' = 0;$$

where

$$\left.\begin{aligned} p' &= -(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}, \\ a' &= -(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{\nu+1}{2\nu^2+\nu+1}, \qquad b' = (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{\sqrt{\{2\nu(\nu^2+1)\}}}{2\nu^2+\nu+1}, \\ c' &= 2(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\frac{(\nu^2+2\nu+2)}{2\nu^2+\nu+1}, \qquad d' = -2b', \\ c'+3a' &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}, \qquad \frac{a'}{b'} = -\frac{\nu+1}{\sqrt{\{2\nu(\nu^2+1)\}}}. \end{aligned}\right\} \tag{22}$$

It is seen from the above that the trinomial quintic $y^5+y+\beta = 0$ can be transformed by Tschirnhausen substitutions into two resolvable quintics,

$$z'^5-5z'^3+5z'+p' = 0,$$

$$z^5+5z^3+5z+p = 0,$$

such that the root $y = (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(m+n)$ of the trinomial quintic is made up of the product of the quantity $(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}$ by the sum of the roots, m and n respectively, of the two resolvable quintics. And it follows that each root of the trinomial quintic is made up of the product of $(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}$ by the sum of two quantities, one of which is a root of the first resolvable quintic, and the other a root of the second resolvable quintic, these roots being taken in appropriate order. The difference between the parameters of the two resolvable quintics is, by (20) and (22),

$$p-p' = \left\{\frac{2}{\nu}(\nu^2+1)\right\}^{\frac{5}{2}},$$

and their product is

$$pp' = 2\,\frac{(\nu-2)(2\nu+1)}{\nu}\left(\frac{\nu^2+1}{\nu}\right) = \frac{1}{u}\left\{\frac{2}{\nu}(\nu^2+1)\right\};$$

so that $$u = (p-p')^{\frac{2}{3}}/pp'.$$

By (9) and (2) we may write

$$y = (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}\{(\mu_1\mu_2)^{\frac{1}{2}}+(\mu_2/\mu_1)^{\frac{1}{2}}-(\mu_2/\mu_1)^{-\frac{1}{2}}+(\mu_1\mu_2)^{-\frac{1}{2}}\}. \qquad (23)$$

Thus y is the algebraical sum of four terms, say e, f, g, h respectively, whose product is $\frac{1}{5}u$. The values of e, f, g, h are, by (23),

$$\left.\begin{aligned} e &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(\mu_1\mu_2)^{\frac{1}{2}}, & f &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(\mu_2/\mu_1)^{\frac{1}{2}},\\ g &= -(-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(\mu_2/\mu_1)^{-\frac{1}{2}}, & h &= (-5)^{-\frac{1}{4}}u^{\frac{1}{4}}(\mu_1\mu_2)^{-\frac{1}{2}}; \end{aligned}\right\} \qquad (24)$$

where μ_1 and μ_2 are given by (1), and u by (8); and the five roots of the trinomial quintic $y^5+y+\beta = 0$ are now as follows :—

$$\left.\begin{aligned} y_0 &= e+f+g+h,\\ y_1 &= \omega e+\omega^2 f+\omega^3 g+\omega^4 h,\\ y_2 &= \omega^2 e+\omega^4 f+\omega^6 g+\omega^8 h,\\ y_3 &= \omega^3 e+\omega^6 f+\omega^9 g+\omega^{12} h,\\ y_4 &= \omega^4 e+\omega^8 f+\omega^{12} g+\omega^{16} h; \end{aligned}\right\} \qquad (25)$$

where $\omega(=e^{2\pi i/5})$ is a complex fifth root of unity.

If we adopt for ν the value given by (13), so as to make the trinomial quintic $y^5+y+\beta = 0$ in agreement, *mutatis mutandis*, with Hermite's

equation $x^5-x-\alpha=0$; then we have*

$$\nu^{-\frac{1}{5}}\{\sqrt{(\nu^2+1)}+\nu\}^{\frac{2}{5}}\{\sqrt{(\nu^2+1)}-1\}^{\frac{1}{5}}=q^{-\frac{2}{5}}\frac{(1-q^4)(1-q^6)(1-q^{14})(1-q^{16})\dots}{(1-q^2)(1-q^8)(1-q^{12})(1-q^{18})\dots};$$

so that, by (1) above

$$\mu_2=q^{-\frac{2}{5}}\frac{(1-q^4)(1-q^6)(1-q^{14})(1-q^{16})\dots}{(1-q^2)(1-q^8)(1-q^{12})(1-q^{18})\dots}. \tag{26}$$

Also, by (1),

$$\frac{\mu_1^2}{\mu_2}=\frac{\sqrt{(\nu^2+1)}+1}{\nu};$$

from which

$$\frac{\mu_1^2-\mu_2}{\mu_1^2+\mu_2}=\frac{\{\sqrt{(\nu^2+1)}-\nu\}^{\frac{1}{2}}}{\{\sqrt{(\nu^2+1)}+\nu\}^{\frac{1}{2}}}=\frac{1}{\sqrt{(\nu^2+1)}+\nu}.$$

But, by (1), $\mu_1\mu_2^2=\sqrt{(\nu^2+1)}+\nu$;

hence† $$\mu_1\mu_2^2\frac{\mu_1^2-\mu_2}{\mu_1^2+\mu_2}=1. \tag{26a}$$

From this it follows‡ that

$$\mu_1=q^{-\frac{1}{5}}\frac{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\dots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}. \tag{27}$$

In these forms we recognize μ_1 and μ_2 as functions whose properties have been investigated by Prof. L. J. Rogers§ and the late Mr. S. Ramanujan.§

Prof. Rogers proved in 1894 that

$$\mu_1=q^{-\frac{1}{5}}\left(1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\dots\right);$$

so that

$$\mu_2=q^{-\frac{2}{5}}\left(1+\frac{q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^6}{1+}\dots\right).$$

Prof. L. J. Mordell‖ has also considered these functions.

From (1) we readily obtain

$$\mu_1^5-\mu_1^{-5}-11=\frac{(\nu-2)^2(2\nu+1)}{\nu^2},\qquad \mu_2^5-\mu_2^{-5}-11=\frac{(\nu-2)(2\nu+1)^2}{\nu};$$

* *Proc. London Math. Soc.*, *loc. cit.*, 353 and 356, equations (18), (33), and (34).

† Cf. *loc. cit.*, 354, equation (25) between C_1 and C_2.

‡ *Loc. cit.*, 353, equation (19).

§ *Proc. Camb. Phil. Soc.*, 19 (1919), 211. For other references see *Proc. London Math. Soc.*, *loc. cit.*, 355, 359, and 387.

‖ *Proc. London Math. Soc.* (2), 20 (1922), 408.

so that, by (8),

$$(\mu_1^5-\mu_1^{-5}-11)(\mu_2^5-\mu_2^{-5}-11) = u^{-3}; \tag{28}$$

and thus the value of y given by (23) may be expressed entirely in terms of the functions μ_1, μ_2 as defined by (27) and (26), viz :—

$$y_0 = (-5)^{-\frac{1}{4}} \frac{(\mu_1\mu_2)^{\frac{1}{2}}+(\mu_2/\mu_1)^{\frac{1}{2}}-(\mu_2/\mu_1)^{-\frac{1}{2}}+(\mu_1\mu_2)^{-\frac{1}{2}}}{(\mu_1^5-\mu_1^{-5}-11)^{\frac{1}{10}}(\mu_2^5-\mu_2^{-5}-11)^{\frac{1}{10}}}. \tag{29}$$

The other four roots of the trinomial quintic may be at once written down from (24) and (25) in terms of μ_1, μ_2.

As a check on the accuracy of these results we will take a numerical example. Let

$$y = (-1)^{-\frac{1}{4}}x, \quad \beta = -(-1)^{-\frac{5}{4}}\alpha;$$

then our equation $y^5+y+\beta = 0$ becomes $x^5-x-\alpha = 0$. By (18) we have $\alpha = 2(5)^{-\frac{5}{4}}(1+k^2)/k^{\frac{1}{2}}k'$; from which k can be obtained in terms of α by solving two quadratics. For simplicity, take $\alpha = 1$; then

$$k = 0{\cdot}07269666.$$

The corresponding value of q is found from a table to be

$$q = 0{\cdot}000331176.$$

Writing $(-1)^{\frac{1}{4}}e = e'$, $(-1)^{\frac{1}{4}}f = f'$, etc., we determine e', f', g', h' by substituting this value of q in (24). If the values of the roots are required to six places of decimals only, we may neglect powers of q above the second. Thus, from equations (26), (27), and (28), we have $u = q(1+2q)$, $\mu_1\mu_2 = q^{-\frac{2}{5}}(1+q+q^2)$, $\mu_2/\mu_1 = q^{-\frac{1}{5}}(1-q+2q^2)$. Hence, $e' = 0{\cdot}9986156$, $f' = 0{\cdot}2010325$, $g' = -0{\cdot}0404969$, $h' = 0{\cdot}0081525$; the calculations being facilitated by observing that $e'h' = -f'g'$. When once we have obtained these quantities, the values of all five roots of the equation $x^5-x-1 = 0$ are readily calculated from (25), and we find

$$\begin{aligned}
x_0 &= 1{\cdot}167304 & x_0^5 &= 2{\cdot}167304\\
x_1 &= 0{\cdot}181232+1{\cdot}083954i & x_1^5 &= 1{\cdot}181230+1{\cdot}083956i\\
x_2 &= -0{\cdot}764884+0{\cdot}352472i & x_2^5 &= 0{\cdot}235117+0{\cdot}352470i\\
x_3 &= -0{\cdot}764884-0{\cdot}352472i & x_3^5 &= 0{\cdot}235117-0{\cdot}352470i\\
x_4 &= 0{\cdot}181232-1{\cdot}083954i & x_4^5 &= 1{\cdot}181230-1{\cdot}083956i;
\end{aligned}$$

where the fifth powers have been calculated by actual involution of the values of the roots set against them respectively. If tables for q are not available, and k is fairly small, as in the above example, we may use the relation obtained by inversion of the value of k in terms of q; thus

$$q = \tfrac{1}{16}k^2+\tfrac{1}{32}k^4+\tfrac{21}{1024}k^6+\ldots.$$

The value of y_0 given by (9) can be put in another form, as follows :— We have $(m+n)(m-n)^2 = m^3-m^2n-mn^2+n^3$; and if we express the several powers of m in terms of n, by aid of (5) and the following equations (in which $M = m$), using equation (4) to keep the powers of n below the fifth, we find that

$$(m^3-m^2n-mn^2+n^3)(\nu^2-\nu+2)$$

$$= (\nu-1)\sqrt{\{2\nu(\nu^2+1)\}}.n^4+(7\nu^2-5\nu+2)n^3+(3\nu+2)\sqrt{\{2\nu(\nu^2+1)\}}.n^2$$

$$+(14\nu^2-10\nu-6)n+10\nu\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}},$$

$= Q$, suppose. Also

$$mn(m^5+n^5) = m^6n+mn^6;$$

which, by (3),

$$= 5m^4n-5m^2n+(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}.mn+mn^6;$$

and, proceeding as above, we obtain

$$\left[5m^4n-5m^2n+(2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}.mn+mn^6\right](\nu^2-\nu+2)$$

$$= \frac{(\nu-2)(2\nu+1)}{\nu}Q = \frac{Q}{u},$$

by (8). Thus

$$u = \frac{(m+n)(m-n)^2}{mn(m^5+n^5)};$$

so that, by (9),

$$y_0^4 = -\tfrac{1}{5}\frac{(m+n)^5(m-n)^2}{mn(m^5+n^5)};$$

by which, as we see from (2), y_0 is again expressed entirely in terms of the functions μ_1, μ_2; but in a different form from that given by (29).

It will be seen from equations (25) that each root of the trinomial quintic $y^5+y+\beta=0$ is the sum of four quantities, each of which may be regarded as a fifth root of a root of a particular biquadratic

$$x^4+c_1x^3+c_2x^2+c_3x+c_4=0;$$

the fifth roots of unity being taken in a definite order. The order of the quantities e, f, g, h is also determinable; for, by (24) and (26a),

$$eh+fg=0, \tag{30}$$

$$e^2g+ef^2+fh^2+g^2h=0, \tag{31}$$

and g is identified as being the only one of the four quantities which is negative when multiplied by $(-5)^{+\frac{1}{4}}$. Equation (30) then signalizes f, and e is distinguished from h by (31). The trinomial quintic may be considered, in a certain sense, as generated by this biquadratic, whose coefficients are seen by (24) to be

$$c_1=-2(-5)^{-\frac{5}{4}}u^{\frac{5}{4}}\frac{(\nu^2+\nu-1)}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}}, \qquad c_2=(-5)^{-\frac{3}{2}}u^{\frac{3}{2}}.\frac{2}{\nu}(\nu^2+1),$$

$$c_3=(-5)^{-\frac{15}{4}}u^{\frac{15}{4}}\left\{\frac{2}{\nu}(\nu^2+1)\right\}^{\frac{3}{2}}, \qquad c_4=-(-5)^{-5}u^5=\left(\frac{u}{5}\right)^5.$$

Thus, whatever be the parameter β of the trinomial quintic, the following relation exists between the coefficients of the generating biquadratic, viz :—

$$(5c_2)^{15}c_4^3=c_3^{10}.$$

Also $c_1-20c_3^{\frac{1}{2}}=\beta$, and $5c_1+4c_2c_3^{-\frac{1}{2}}=\beta$; leading to another relation independent of β, viz.: $c_1c_3^{\frac{1}{2}}+c_2+5c_3^{\frac{3}{2}}=0$, and also to

$$(c_1-\beta)(\beta-5c_1)=80c_2, \quad \text{and} \quad (c_1-\beta)/(\beta-5c_1)=25c_4^{\frac{1}{5}}.$$

We see, further, that

$$3125c_2^5+c_2^2c_3^3+6c_2c_3^4+25c_3^2=0;$$

which, regarded as a quintic in c_2, is a particular case of Klein's* *Hauptgleichung*. When ν has the value given in (13), these coefficients may readily be expressed in terms of formal elliptic functions, *e.g.*

$$c_1=-2(-5)^{-\frac{1}{4}}\Lambda(1+k\lambda)/Kk^{\frac{1}{2}}k';$$

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* *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (1884), 182; trans. Morrice (1888), 201.

where λ is the modulus for q^5, and Λ the corresponding first quarter-period.

Finally, we will consider the quintic

$$x^5+10Bx^3+5(9B^2-AC)x-5^{-\frac{5}{4}}\Pi^{\frac{1}{4}}=0, \tag{32}$$

which forms the connecting link between Hermite's trinomial quintic and the quintic of Kronecker and Brioschi. The notation is Klein's,* Π being the discriminant of the Jacobian multiplier-equation

$$(z-A)^6-4A(z-A)^5+10B(z-A)^3-C(z-A)+5B^2-AC=0, \tag{33}$$

for the quintic transformation. Hermite's equation is arrived at by putting $B=0$ in (32), and Kronecker's and Brioschi's by putting $A=0$.

If we put $AC=5B^2$, and $x=y\sqrt{(2B)}$ in (32), it becomes

$$y^5+5y^3+5y-2^{-\frac{5}{2}}5^{-\frac{5}{4}}B^{-\frac{5}{2}}\Pi^{\frac{1}{4}}=0\,; \tag{34}$$

while (33) reduces to a quintic, on dividing by $(z-A)$. The discriminant of the latter quintic becomes, when multiplied by C^2, that of the sextic from which this quintic was degraded by division; thus,

$$\Pi=5^5(C^3-128B^5)^2.$$

The condition for $AC=5B^2$ is seen, by Klein's equation (19), to be $4A_0^2-A_1A_2=0$; whence, in his notation,

$$A=\tfrac{5}{4}A_1A_2, \qquad B=-\tfrac{1}{2}(A_1^{\frac{5}{2}}-A_2^{\frac{5}{2}})^2(A_1^{\frac{1}{2}}A_2^{\frac{1}{2}}),$$

$$C=(A_1^{\frac{5}{2}}-A_2^{\frac{5}{2}})^4, \qquad \Pi=5^5(A_1^{\frac{5}{2}}-A_2^{\frac{5}{2}})^{20}(A_1^{\frac{5}{2}}+A_2^{\frac{5}{2}})^4.$$

Thus, if we assign to A_1 the value μ_2/μ_1 of the present paper, and determine the magnitude still available by putting $A_1^{\frac{1}{2}}A_2^{\frac{1}{2}}=-1$, we have $A_0=-\frac{1}{2}$, $A=\frac{5}{4}$, $B=\frac{1}{2}\{(\mu_2/\mu_1)^5+(\mu_2/\mu_1)^{-5}\}^2$; and (34) becomes

$$y^5+5y^3+5y=(\mu_2/\mu_1)^5-(\mu_2/\mu_1)^{-5}=\frac{(\nu-2)}{\nu}\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}^5};$$

which is the resolvable quintic (4) satisfied by n; *i.e.*, by $(\mu_2/\mu_1)^5-(\mu_2/\mu_1)^{-5}$.

* *L.c.*, Ikosaeder, 151; trans. 165.

Also, putting $z-A = S/\xi$, where $S = (\mu_2/\mu_1)^{\frac{5}{2}}+(\mu_2/\mu_1)^{-\frac{5}{2}}$, the Jacobian equation (33) reduces to

$$\xi^5-5\xi^3+5\xi = S = \frac{(\nu^2-\nu+2)}{\nu}\sqrt{\frac{2}{\nu}};$$

which is the resolvable quintic satisfied by $(\mu_2/\mu_1)^{\frac{1}{2}}+(\mu_2/\mu_1)^{-\frac{1}{2}}$.

Similarly, if we put $AC = 5B^2$, $x = y\sqrt{(-2B)}$, $A_0 = -\frac{1}{2}$, $A = \frac{5}{4}$, $A_1 = -\mu_1\mu_2$, $A_1^{\frac{1}{2}}A_2^{\frac{1}{2}} = -1$, $B = -\frac{1}{2}\{(\mu_1\mu_2)^{\frac{5}{2}}-(\mu_1\mu_2)^{-\frac{5}{2}}\}^2$, equation (32) becomes

$$y^5-5y^3+5y = (\mu_1\mu_2)^{\frac{5}{2}}+(\mu_1\mu_2)^{-\frac{5}{2}} = (2\nu+1)\sqrt{\left\{\frac{2}{\nu}(\nu^2+1)\right\}};$$

which is the resolvable quintic (3) satisfied by m; *i.e.*, by $(\mu_1\mu_2)^{\frac{1}{2}}+(\mu_1\mu_2)^{-\frac{1}{2}}$; while if we write $z-A = R/\xi'$, where $R = (\mu_1\mu_2)^{\frac{5}{2}}-(\mu_1\mu_2)^{-\frac{5}{2}}$, the Jacobian equation (33) reduces, in this case to

$$\xi'^5+5\xi'^3+5\xi' = R = (2\nu^2+\nu+1)\sqrt{\frac{2}{\nu}};$$

which is the resolvable quintic satisfied by $(\mu_1\mu_2)^{\frac{1}{2}}-(\mu_1\mu_2)^{-\frac{1}{2}}$.

THE CONFORMAL REPRESENTATION OF THE AREA OF A PLANE BOUNDED BY TWO STRAIGHT OR CIRCULAR SLITS

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Introduction.

The problem of a two-dimensional electrical condenser has lately occupied the attention of Prof. A. E. H. Love.* The solution of the various questions connected therewith depends on that of the conformal representation of the area formed by a plane slit along two parallel straight lines. The representation of the area bounded by two perpendicular slits has been obtained in certain special cases by Thomae,† who refers for one result to a dissertation by Bähr, whose paper we have not been able to consult. In the following we consider the most general cases of planes slit along two straight lines or circular arcs, but otherwise unbounded.

For our purposes the slits must not intersect, nor both extend to infinity (which counts as an intersection). If they intersect once, the area of the plane is simply-connected; if they intersect twice, the plane is divided into two parts, and ceases to be a connected area at all. Subject to these restrictions the plane bounded by two slits is a doubly-connected area, as is seen most readily by projecting it stereographically upon the surface of a Neumann sphere.

We have next to choose standard doubly-connected areas upon which to map the slit plane, which we take as the plane of the complex variable w. We have found our problem amenable to two methods of treatment. One is based on the Schwarzian principle of symmetric continuation. In it (except for the cases we describe as being completely symmetrical) we adopt as the standard area the plane of a variable t slit along two

* Love, *Proc. London Math. Soc.* (2), 22 (1924), 337–369.

† Thomae, *Leipziger Berichte*, 58 (1906), 172–191.

portions of the axis of real numbers. In the other, we make a direct application of Dirichlet's principle and, following the work of Villat,* obtain the representations desired upon the area between two concentric circles in the plane of a variable z.

Analysis of the Problem.

We find that the problem divides itself into three.

Problem I.—The circles (lines) on which the slits lie intersect in two distinct points. Inversion with respect to one of these points transforms the slits into straight slits (not parallel).

Problem II.—The circles do not intersect. Inversion with respect to a limiting point of the system of circles coaxal with them transforms the slits into concentric circular arcs.

Problem III.—The circles touch one another. Inversion with respect to the common point transforms the slits into two parallel straight lines.

We suppose the appropriate inversion already performed in each case.

Two points in the figure are of special importance in our methods, viz, the intersection of the lines in Problem I, or the common centre of the circular arcs in Problem II, which we take as the origin of coordinates and always denote by O, and the point at infinity which we always denote by Ω. In Problem III we have one special point Ω.

[It will save considerable repetition if we note here that O, Ω are in no way special points from the point of view of the conformal representation. Since, in the neighbourhood of one of these points, a point in the transformed region corresponds uniquely to a point in the w-plane, and, conversely, a point in the w-plane corresponds uniquely to a point in the transformed region, it follows that O is a simple zero and Ω a simple pole of w expressed as a function of t, z, or any subsidiary variable we employ (with the obvious exception of $\log w$ which occurs in one particular discussion). The importance of O, Ω lies in the fact that we determine w as a function of the variables we employ by means of its zero and poles.]

Problems I and III require further subdivision into several cases, according to the positions of O and Ω with respect to the slits. Either of these points may lie on a slit, at its extremity, or in the interior of

* Villat, *Rend. di Palermo*, 33 (1912), 134-174.

the area whose representation is sought. We may treat as limiting cases the various cases which arise with either of these points as an extremity of a slit, *i.e.* when one slit is terminated at the origin or extends to infinity in one direction only. In Problem I we are then left with four cases :—

Case I.—Both slits are finite, neither passing through O.

Case II.—Both slits are finite, and one passes through O. We include in this case the figure obtained by inversion with respect to O when one slit is finite, not passing through O, and the other slit is doubly infinite (or, as we may say, passes through Ω) with a gap in which O lies.

Case III.—One slit is finite; the other passes through O and Ω.

Case IV.—One slit passes through O, the other passes through Ω.

Similarly, in Problem III, we have two cases :—

Case I.—Both slits are finite.

Case II.—One slit passes through Ω.

We further require to distinguish certain types of symmetry. If, in Problem I, the slits are self-inverse with respect to a circle with centre at O and of a suitably chosen radius, we say that the figure is partially symmetrical. Symmetry about a diameter of the circle we describe as complete symmetry. In Problem II symmetry about a diameter is the condition of partial symmetry; complete symmetry requires the existence of a circle with respect to which the slits are mutually inverse. In Problem III partial symmetry requires an axis of symmetry perpendicular to the slits, complete symmetry an axis parallel to them.

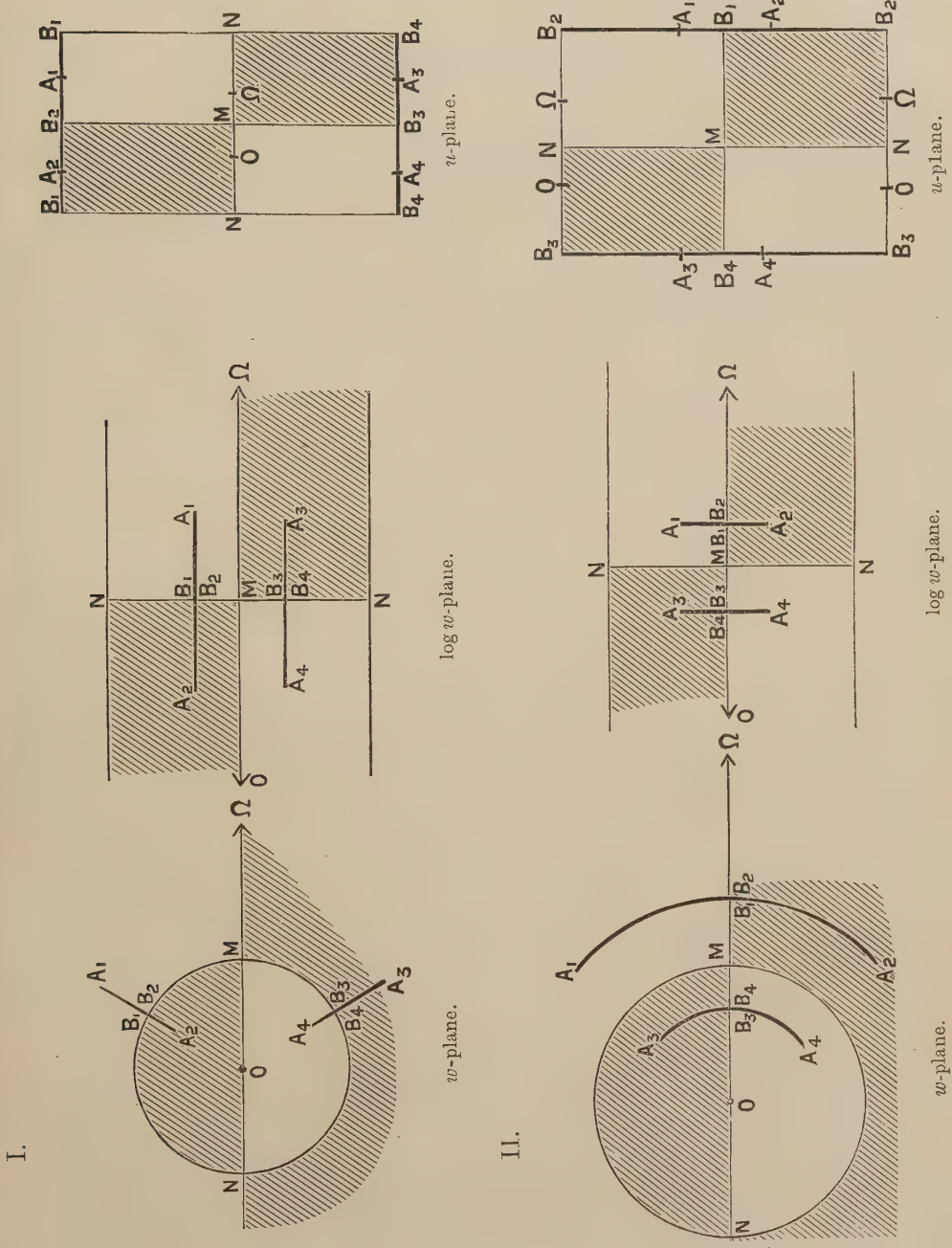
For the general classification of plane areas bounded by slits along arcs of two families of orthogonal circles, reference should be made to the work of Koebe.*

The Cases of Complete Symmetry, Problems I and II.

The lines of symmetry divide each figure into four regions obtainable from one another by inversion. If we consider $\log w$ with the restriction

* "Abhandlungen zür Theorie der Konformen Abbildung IV: Abbildung mehrfach zusammenhängender Bereiche auf Schlitzbereiche", *Acta Math.*, 41 (1918), 305–344.

$-\pi \leqslant \arg(w) \leqslant \pi$ the figure is represented on an infinite strip, the new figure being symmetrical about both axes.



Using the ordinary Schwarz-Christoffel formula, we transform the upper right-hand quarter of this figure into a half-plane of a variable t by means of a relation of the form

$$\frac{1}{w}\frac{dw}{dt} = A\,\frac{(t-a_1)}{(t-\omega)\sqrt{\{4(t-b_1)(t-b_2)(t-m)(t-n)\}}},$$

the notation being self-explanatory.

There is no loss of generality in taking m at infinity, and, with the substitution $t = \wp(u)$, we pass immediately to Weierstrassian elliptic functions with a real period $2\omega_1$ and a purely imaginary period $2\omega_3$ $(\omega_2 = \omega_1 + \omega_3)$.

The cyclical order of points on the boundaries differs in the two cases, but the analogy is best preserved by taking the correspondence :—

Problem I.—	M,	Ω,	N,	B_1,	A_1,	B_2,	M,
	$t = \infty$,	ω,	e_1,	e_2,	a,	e_3,	∞,
	$u = 0$,	ϖ,	ω_1,	ω_2,	a,	ω_3,	0.
Problem II.—	M,	B_1,	A_1,	B_2,	Ω,	N,	M,
	$t = \infty$,	e_1,	a,	e_2,	ω,	e_3,	∞,
	$u = 0$,	ω_1,	a,	ω_2,	ϖ,	ω_3,	0.

The determination of the constant factor A follows from the fact that Ω is a simple pole of w. Hence the residue of $d(\log w)/du$ at $u = \varpi$ is -1, and therefore

$$\frac{1}{w}\frac{dw}{du} = -\frac{\wp'(\varpi)}{\wp(u)-\wp(\varpi)} + B,$$

where B is written for brevity instead of

$$\wp'(\varpi)/\{\wp(a)-\wp(\varpi)\}.$$

This expression is real for all values of u lying on the slits or on the lines of symmetry of the fundamental parallelogram. In Problem I the slits correspond to the sides along which du is real, so that the value of $d(\log w)$ is real, and the amplitude is constant along the slits. In Problem II the slits correspond to the sides along which du is a pure imaginary, and hence $|w|$ is constant on the slits.

We now write the equation in the form

$$\frac{1}{w}\frac{dw}{du} = \zeta(\varpi - u) + \zeta(\varpi + u) - 2\zeta(\varpi) + B,$$

whence

$$w = Ce^{\mu u}\frac{\sigma(u+\varpi)}{\sigma(u-\varpi)},$$

where

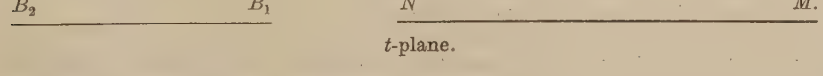
$$\mu = \frac{\wp'(\varpi)}{\wp(\alpha)-\wp(\varpi)} - 2\zeta(\varpi).$$

There is a further geometrical condition to be imposed on this formula, expressing the fact that B_1, B_2 coincide in the w-plane. In Problem I this means that $w(\omega_2) = w(\omega_3)$ and in Problem II that $w(\omega_2) = w(\omega_1)$. The results are respectively :

(Problem I) $\mu\omega_1 + 2\eta_1\varpi = 0$; (Problem II) $\mu\omega_3 + 2\eta_3\varpi = 0$.

These are to be regarded as equations determining $\wp(\alpha)$, and from this the positions of the points A in the w-plane.

The representation here obtained is *not* upon a plane of the variable t slit along portions of the axis of real numbers. Such a representation is obtained in the next section, but in doing so we lose the analytical simplification which complete symmetry affords. From the geometrical point of view, however, the present result is somewhat more complex. Consider only Problem I. The part of the figure corresponding to the upper right-hand quarter of the parallelogram in the plane of the variable u is represented on the lower half of the t-plane. Crossing the line B_2M, we get the representation of the upper left-hand quarter on the upper half-plane, so that the upper half-parallelogram is mapped on a t-plane slit from B_2 through B_1 and N to M. Now a point describing a circuit about B_1B_2 (or A_1A_2) in the w-plane in the positive direction, and returning to its starting point, crosses B_1N and MB_2 in succession. The description of the corresponding circuit in the plane of t must also end at the same point at which it began. The edges of the slit can therefore be joined from B_1 to N. We have now the representation of the upper half of the w-figure upon the t-plane slit along B_2B_1 and along NM.



Again, crossing NM in the right-hand half of the parallelogram is equivalent to crossing NM from the lower half of the slit t-plane, while crossing NM in the left-hand half of the parallelogram is equivalent to

crossing NM from the upper half of the slit t-plane, and each of these leads to points which correspond to portions of the w-figure which we have not yet considered. We must therefore adjoin a second plane, slit exactly like the first, as a second sheet of a Riemann surface, and make NM the line of junction of the two sheets. The slit B_3B_4 in the second sheet is coincident with the slit B_2B_1 in the first sheet, but there is no junction of the sheets along this line. The final result is that we have mapped the w-plane on a two-sheeted Riemann surface with a line of junction along a portion of the axis of real numbers, each sheet having a slit along another part of the same axis, the two slits being coincident with one another.

Problem III can be discussed in a similar manner.

It will be seen that the mapping of this section differs from that of succeeding sections by reason of the fact that the whole of the fundamental parallelogram in the plane of the variable u is used here, whereas in the later cases we use only one half of a parallelogram. This indicates that our present results are related to others of the later types by means of a quadratic transformation of the elliptic functions.

We may perhaps also remark that the introduction of the singularities at O, Ω by means of the transformation to the plane of $\log w$ is unimportant. When we obtain the representation of the rectilinear figure in this plane by the use of the Schwarz-Christoffel formula, t as a function of $\log w$ has a singularity of the same type, because there is an angle zero at the point Ω. When we discard the subsidiary variable $\log w$ as having fulfilled its purpose, this singularity automatically disappears. This point is perfectly familiar, occurring, for example, in the ordinary discussions of free stream-lines in any treatise on hydrodynamics.

The Cases of Partial Symmetry, Problems I and II.

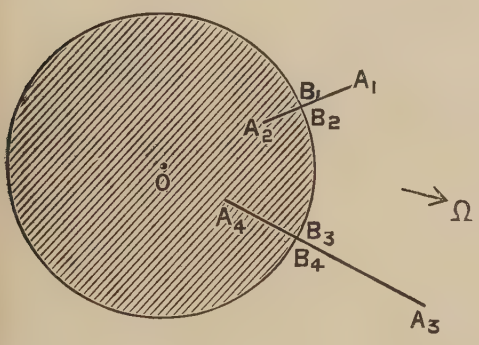
By drawing the line of symmetry we divide the w-plane into two symmetrical regions instead of four. We shall represent the w-region upon the plane of a variable t bounded by two slits along the real axis. The other parts of the real axis of the t-plane will correspond to the line of symmetry.

Consider the expression $Z = \{d(\log w)/dt\}^2$. Along the real axis of the t-plane, dt is real, and the corresponding point of the w-plane travels along arcs or along radii vectores of circles whose centre is the origin. It follows that Z is real in both cases, negative along an arc and positive along a radius vector. Now by the symmetrical continuation theorem of

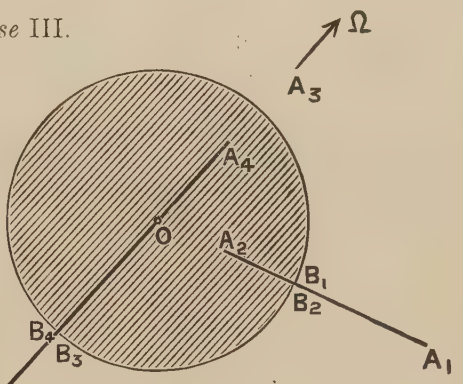
Schwarz, since Z is real for all real values of t, its poles and zeros lie symmetrically about the real axis.

In both problems at the points A_1, A_3 (in the upper half-plane) $w-w_A \sim C(t-t_A)^2$, while, at the points B_1, B_2, B_3, B_4, $t-t_B \sim C'(w-w_B)^2$. Hence the points A are double zeros and the points B are simple poles of Z, regarded as a function of t. There are no other singularities on the slits.

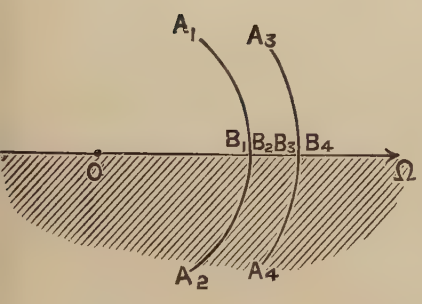
I. *Case* I.



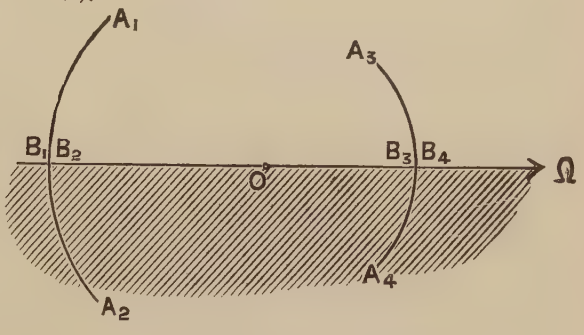
I. *Case* III.



II (*a*).



II (*b*).



The other points which require special consideration are O and Ω, which, however, play different parts in the various cases.

In Problem I, Case I, Ω lies in the interior of the region. It is a simple pole of w, so that $Z \sim (t-\varpi)^{-2}$, where ϖ is complex. The conjugate point ϖ' will be a similar singularity in the lower half-plane by the symmetric continuation theorem, and we have now

$$Z = \frac{A(t-a_1)^2(t-a_2)^2}{4(t-\varpi)^2(t-\varpi')^2(t-b_1)(t-b_2)(t-b_3)(t-b_4)}.$$

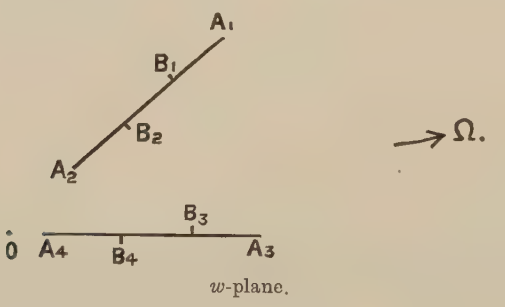
The value of A is determinate inasmuch as Z and, indeed, $Z^{\frac{1}{2}}$ are known (as regards the first term of their expansions) at O and Ω.

In the remaining cases the form of Z is the same, but the values of ϖ, ϖ' are real and distinct. In Problem I, Case III, two points Ω lie on the same slit; consequently, in the t-plane two points O coincide with them, but the points Ω are to be regarded as lying on one edge of the slit (on the unshaded side), while the points O lie on the other (on the shaded side). In Problem II(a) O, Ω lie in one of the two parts of the real axis which do not correspond to slits, while in Problem II(b) they lie one in each of these two parts of the real axis.

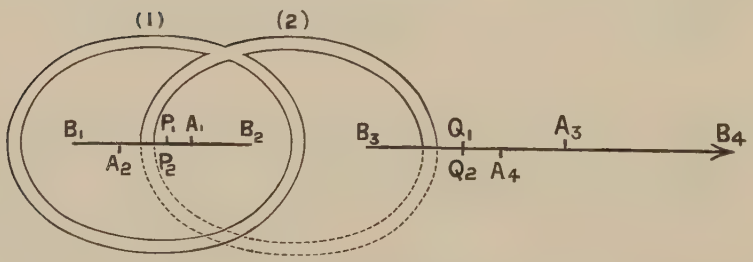
The above equation can obviously be solved by means of the elliptic functions. We delay no further, however, at this stage, but proceed to the consideration of the most general case, when the results required in this section will be apparent.

The General Cases.

Problem I. *Case* I.—



w-plane.



t-plane.

We seek the representation of the given area in the w-plane upon the plane of the variable t slit along portions of the axis of real numbers $B_1 B_2$, $B_3 B_4$. As we shall make use of the Weierstrassian Elliptic Func-

tions, we take B_3B_4 to be infinite in the positive direction, and B_1, B_2, B_3 to be e_3, e_2, e_1, in ascending order, where $e_1+e_2+e_3=0$.

Along either slit $d(\log w)/dt$, which we denote by W, is real, while w, and therefore W, is a uniform function of t in the slit plane. In the open plane, however, W cannot be uniform. Suppose we take a point P on the slit, P_1 denoting that we reach the point by a path leading to P from the upper half plane, P_2 by a path from the lower half-plane, and let w_1, w_2 be the corresponding values of w.

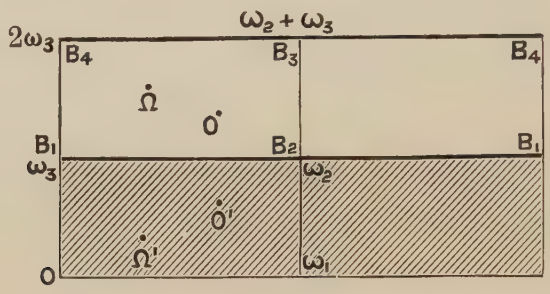
Then if
$$\frac{1}{w_1}\frac{dw_1}{dt}=\frac{1}{w_2}\frac{dw_2}{dt},$$
$$w_1=Cw_2.$$
But, at a point B,
$$w_1=w_2,$$
and, therefore,
$$C=1,$$
and so at all points,
$$w_1=w_2,$$

i.e. w itself is a uniform function of t in the open plane. Since w has only one pole and one zero in the plane of t, w and t are connected by a linear relation. This can only be the case for the trivial transformation of two collinear slits into two other collinear slits.

B_1, B_2, B_3, B_4 are then branch-points of W in the t plane, and, as in the cases of symmetry, the points B in the w-plane lie, in general, on the slits and not at their extremities. At each such point an angle π is transformed into an angle 2π. Hence, two branches of w, and so of W, must be interchanged by the description of a circuit about B. We regard the slit plane of t as one sheet of a Riemann surface with lines of junction along the slits. Let us join to it along B_1B_2 a second sheet lying below the first. We take a point Q on B_3B_4 calling it Q_1 in the upper sheet. We describe a path leading from Q_1, crossing B_1B_2 into the lower sheet and arriving at Q_2 (Q in the lower sheet on the other edge of the slit). Now since W is real along B_1B_2, the symmetric continuation theorem shows that the values of W at Q_1Q_2 are conjugate. But W is real at Q_1; hence the value at Q_2 is the same real number. Hence the same two sheets can be joined along B_3B_4, *i.e.* the Riemann surface is two-sheeted.

Further, W has two simple poles O, Ω in the upper sheet, with residues $+1$, -1 respectively. Hence there are simple poles at their reflexions in B_1B_2 (or B_3B_4), which we call O', Ω', also having residues $+1$, -1, according to the symmetric continuation theorem, these poles lying in the lower sheet. W then is a uniform function of position on the

surface with four simple poles, O, O' with residue $+1$, Ω, Ω' with residue -1. It follows that w is a simple case of a "Fonction à Multiplicateurs constants" associated with this surface; in fact, uniformization being obtained by means of the substitution $t = \wp(u)$, w is a doubly-periodic function of u of the second kind.



u-plane.

The upper half plane of the variable t is transformed by this substitution into a quarter of a period-parallelogram in the plane of u. The lower half-plane is then represented on the consecutive quarter-parallelogram obtained by continuation across B_2B_3. In this half-parallelogram occur one zero and one pole, say u_0, u_0' which correspond to O, Ω. The zero and pole corresponding to O', Ω' lie in the half-parallelogram obtained by continuation across B_1B_2, and are the reflexions of u_0, u_0' in that line, *i.e.* they are congruent to $\bar{u}_0$, $\bar{u}_0'$ where $\bar{u}$ as usual denotes the number conjugate to u.

We next determine the constant multipliers of w. We dissect the t surface in the conventional manner by two cross cuts. We take a point on one edge of cut (1), describe a path crossing B_1B_2, B_3B_4 in turn, and returning to the point of departure. By the principle of symmetric continuation we obtain, on the lower sheet of the t surface, the representation of the figure obtained by reflecting the original w figure in A_1A_2. A_3A_4 falls in a new position denoted by $A_3'A_4'$. We have to regard the w-plane as one sheet of a many-sheeted Riemann surface. The points A are branch-points of t as a function of w, and each sheet is joined to the consecutive one along one of the lines A_1A_2, A_3A_4 in the positions they take up in the sheets in succession. Thus the second sheet is joined to the first along A_1A_2, the third to the second along $A_3'A_4'$. When the t-point has returned to its point of departure on cut (1), it is now obvious that the w-point has been rotated through an angle 2α, where α is the angle between the slits. Hence the multiplier

corresponding to cut (1) is $e^{2i\alpha}$. In the u figure the description of the corresponding path is equivalent to changing u to $u-2\omega_3$.

The multiplier corresponding to cut (2) is obtained by the description of a circuit enclosing $B_1 B_2$, the corresponding circuit therefore enclosing $A_1 A_2$ in the w-plane. Since only one sheet of the t surface is used, w resumes its initial value; u, however, is increased by $2\omega_1$; w, then, is periodic with respect to $2\omega_1$.

Hence we may write

$$w = Ce^{\mu u} \frac{\sigma(u-u_0)\,\sigma(u-\bar{u}_0)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')}.$$

The condition that w is periodic with respect to $2\omega_1$ gives

$$\mu\omega_1 = \eta_1(u_0+\bar{u}_0-u_0'-\bar{u}_0')+ki\pi,$$

and in order that $d(\log w/du)$ may be real along the slits the value of k must be zero. Remembering that the multiplier corresponding to the period $2\omega_3$ is $e^{2i\alpha}$, we get

$$i\alpha = \eta_3(u_0+\bar{u}_0-u_0'-\bar{u}_0')-\mu\omega_3,$$

i.e.

$$\alpha = \frac{\pi}{2\omega_1}(u_0'+\bar{u}_0'-u_0-\bar{u}_0).$$

We observe a difference in character between the repetitions of the original figure. With respect to $2\omega_1$ the repetitions are periodic only. With respect to ω_3, however, symmetric repetitions of the original w figure are represented on alternate halves of the period-parallelograms.

Case II.—The modification required is not serious. The point O lies on one of the slits, say $A_3 A_4$. In tracing the boundary we must twice pass through O; there are therefore two points O on $B_3 B_4$ in the t plane, and we note that symmetric continuation provides no new poles of W. Accordingly we replace u_0, $\bar{u}_0$ by real numbers u_0, u_1.

Case III.—When both O, Ω lie on $A_3 A_4$, we also replace u_0', $\bar{u}_0'$ by real numbers u_0', u_1', and it is to be noted that the pairs of points u_0, u_1 and u_0', u_1' must not separate one another on the line $B_3 B_4$ in the u figure.

Case IV.—Two points O lie on one slit, say $B_1 B_2$, and two points Ω lie on the other, $B_3 B_4$. In this case we replace u_0, $\bar{u}_0$ by $v_0+\omega_3$, $v_1+\omega_3$, where v_0, v_1 are real, and the formula becomes

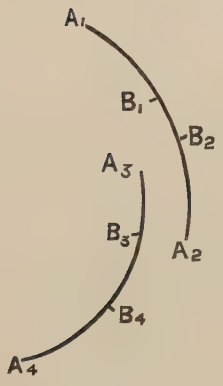
$$w = Ce^{\mu u} \frac{\sigma_3(u-v_0)\,\sigma_3(u-v_1)}{\sigma(u-u_0')\,\sigma(u-u_1')}.$$

Problem II.

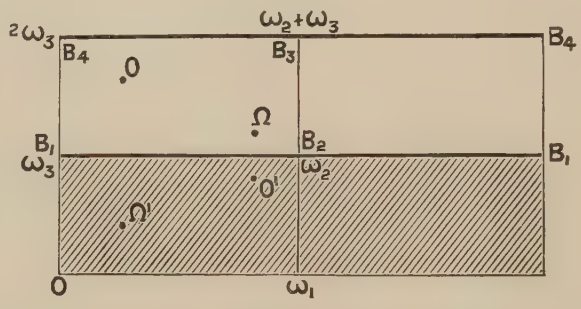
Taking the t-plane slit, as before, from e_3 to e_2 and from e_1 to $+\infty$, we now have that W is imaginary along the slits, and has poles O, Ω with residues $+1$, -1 in the upper sheet of the t-surface. Hence iW is real along the slits, its residues at the poles being $+i$, $-i$. Hence there are poles at the reflexions of O, Ω in B_1B_2, which lie in the lower sheet of the t-surface, the residues being $-i$, $+i$, *i.e.* the residues of W at the reflexions of O, Ω are -1, $+1$.

Hence w has simple zeros at O, O', and simple poles at Ω, Ω', where now O' is the reflexion of Ω in B_1B_2 and Ω' that of O.

We employ the same elliptic functions as in Problem I for the evaluation of w, but the correspondence of points in the u-plane is now as shown in the figure. w is again periodic with respect to $2\omega_1$; when we treat the w-plane as one sheet of a Riemann surface in order to



w-plane.



u-plane.

determine the multiplier belonging to the period $2\omega_3$, in place of the reflexions in the slits of Problem I we now employ inversions in the circular slits. The multiplier is accordingly m^2, where m is the ratio of the radii of the slits.

The formula consequently becomes

$$w = Ce^{\mu u}\frac{\sigma(u-u_0)\,\sigma(u-\bar{u}_0')}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0)},$$

and, in order that w may admit the period $2\omega_1$, we have

$$\mu\omega_1 = \eta_1(u_0-\bar{u}_0-u_0'+\bar{u}_0')+ki\pi.$$

The determination of k requires different considerations from those of Problem I. Suppose a point moves in the u-plane along the straight line

from a point u_1 to the point $u_1+2\omega_1$. The corresponding point in the w-plane describes a path along both edges of a slit returning to its initial position, and at the same time the amplitude of w also returns to its initial value, *i.e.* is not increased by a positive or negative multiple of 2π.

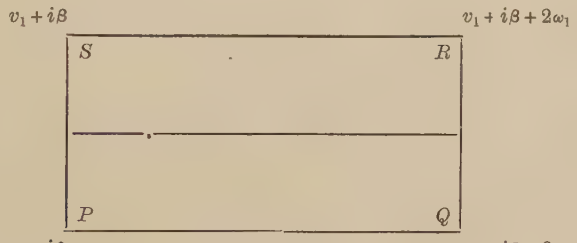
Consider the function

$$\log \frac{\sigma(u-u_0)}{\sigma(u-\bar{u}_0)}.$$

Write $u_0 = \alpha+i\beta$ and $v = u-\alpha$; then

$$\left[\log \frac{\sigma(u-u_0)}{\sigma(u-\bar{u}_0)}\right]_{u_1}^{u_1+2\omega_1} = \int_{v_1-i\beta}^{v_1-i\beta+2\omega_1} - \int_{v_1+i\beta}^{v_1+i\beta+2\omega_1} \xi(v)\,dv.$$

Now take the integral of $\xi(v)$ around the rectangle $PQRS$, the vertices of which are $v_1-i\beta$, $v_1-i\beta+2\omega_1$, $v_1+i\beta+2\omega_1$, $v_1+i\beta$. This rectangle contains one and only one pole of $\xi(v)$, for β is less than $2\omega_3/i$ and if, in the exceptional case where v_1 is a multiple of $2\omega_1$, it is necessary to deform the side SP to avoid a pole, the opposite side, QR, must as usual be deformed in exact congruence.



v-plane.

Then $$\frac{1}{2i\pi}\left[\int_P^Q + \int_Q^R + \int_R^S + \int_S^P \xi(v)\,dv\right] = 1.$$

But $$\int_Q^R \xi(v)\,dv = \int_P^S \xi(v+2\omega_1)\,dv.$$

Therefore $$\int_Q^R + \int_S^P \xi(v)\,dv = \int_P^S [\xi(v+2\omega_1)-\xi(v)]\,dv$$

$$= 4i\beta\eta_1.$$

Also $$\int_P^Q + \int_R^S \xi(v)\,dv = \int_{v_1-i\beta}^{v_1-i\beta+2\omega_1} - \int_{v_1+i\beta}^{v_1+i\beta+2\omega_1} \xi(v)\,dv.$$

Hence $$\left[\log \frac{\sigma(u-u_0)}{\sigma(u-\bar{u}_0)}\right]_{u_1}^{u_1+2\omega_1} = 2i\pi - 4i\beta\eta_1$$

$$= 2i\pi - 2\eta_1(u_0-\bar{u}_0),$$

whence, again,

$$\left[\log \left\{e^{\mu u} \frac{\sigma(u-u_0)}{\sigma(u-u_0')} \frac{\sigma(u-\bar{u}_0')}{\sigma(u-\bar{u}_0)}\right\}\right]_{u_1}^{u_1+2\omega_1} = 2\mu\omega_1 - 2\eta_1(u_0-\bar{u}_0) + 2\eta_1(u_0'-\bar{u}_0')$$

$$= 2ki\pi.$$

The change in amplitude of w as u is increased by $2\omega_1$ is thus seen to be $2k\pi$; as it must be zero, k also must be zero, *i.e.*

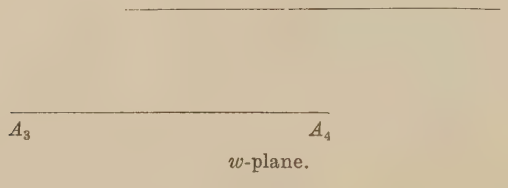
$$\mu\omega_1 = \eta_1(u_0-\bar{u}_0-u_0'+\bar{u}_0').$$

The multiplier corresponding to the period $2\omega_3$ being m^2, where m is the ratio of the radii of the slits, it then follows that

$$m = e^{\mu u_3 - \eta_3(u_0-\bar{u}_0-u_0'+\bar{u}_0')}$$

$$= e^{(i\pi/2\omega_1)(u_0-\bar{u}_0-u_0'+u_0')}.$$

Problem III. *Case* I.



We take the axis of real numbers parallel to the slits in the plane of the variable w, and slit the t-plane as in Problem I. In place of $d(\log w)/dt$ we have to consider the function dw/dt, so that w is periodic with respect to $2\omega_1$, and is increased by a constant when u is increased by $2\omega_3$. If the residue of w at Ω is $A+iB$, the residue at Ω' is $A-iB$.

Accordingly

$$w = (A+iB)\,\zeta(u-u_0')+(A-iB)\,\zeta(u-\bar{u}_0')+\mu u+C,$$

where $$\mu\omega_1+2A\eta_1 = 0,$$

and the distance between the slits is

$$(\mu\omega_3+2A\eta_3)/i.$$

In Case II, as in Problem I, Case II, we replace u_0', $\bar{u}_0'$ by real numbers u_0', u_1' and the two residues must be real, so that

$$w = A\zeta(u-u_0')+B\zeta(u-u_1')+\mu u+C,$$

where

$$\mu\omega_1+(A+B)\eta_1 = 0.$$

The Application of Dirichlet's Principle.

Our problems can be solved by a direct application of the harmonic analysis. We now seek the representation of the figure in the w-plane upon the annular area bounded by two concentric circles in the plane of a variable z. When w is expressed as a function of z, $\log w$ is an analytic function with known singularities and principal parts within this region, and for some of our problems its real part and for the others its imaginary part is constant on the boundaries, except that when some or all of the singularities lie on the boundaries there is a known discontinuity at each such point. We form the sum of the principal parts of $\log w$ at all the singularities, and obtain expansions in ascending or descending powers of z as may be valid for each boundary. Separating this into its real and imaginary parts and fixing our attention on the one concerned, we build up a complementary function in the form of a convergent Laurent series which, being added to the sum of the principal parts, reduces the real or imaginary part to a constant.

For this purpose we introduce the normal functions described by Picard, viz :—

$$\frac{r^n-q^{2n}r^{-n}}{1-q^{2n}}\cos(n\theta+\alpha), \quad \frac{q^n(-r^n+r^{-n})}{1-q^{2n}}\cos(n\theta+\alpha)$$

which reduce respectively to $\cos(n\theta+\alpha)$ and zero when $r=1$, and to zero and $\cos(n\theta+\alpha)$ when $r=q$, q being a real positive proper fraction. These being regular and harmonic within the ring $q \leqslant r \leqslant 1$, we are able to build up any harmonic function regular throughout the ring whose values on the boundaries can be expressed by means of convergent trigonometric series. The technique of the method has been fully elaborated by Villat, and is easily accessible.*

* Picard, *Traité d'Analyse*, 2 (1905), 105–108 (Gauthier-Villars, Paris); Villat, *Aperçus théoriques sur la résistance des fluides* (1920), 10–15 (Gauthier-Villars, Paris).

Problem I.

Case I.—Consider the function $w(z)$ by means of which the w-plane bounded by two radial slits is mapped upon the ring of the *z-plane* $q \leqslant |z| \leqslant 1$.

Since O, Ω belong to the interior of the area, each must correspond to a unique point in the interior of the ring; O is a simple zero, and Ω a simple pole of $w(z)$.

The boundary condition is that $\arg w$ is constant on each boundary.

Then, within the ring, $\log w$ has two logarithmic singularities, z_0, z_0', and is elsewhere regular, and the imaginary part, $I(\log w)$, is constant on each boundary.

Writing $z = re^{i\theta}$, $z_0 = \rho e^{i\alpha}$, $z_0' = \rho' e^{i\alpha'}$, we have, on the boundary $|z| = 1$,

$$I \log\left\{\frac{z-z_0}{z-z_0'}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} \{\rho^n \sin n(\theta-\alpha) - \rho'^n \sin n(\theta-\alpha')\}.$$

and, on the boundary, $|z| = q$,

$$I\left\{\log \frac{z-z_0}{z-z_0'}\right\} = I\left\{\log \frac{z_0}{z_0'}\right\} - \sum_{n=1}^{\infty} \frac{q^n}{n} \left\{\frac{\sin n(\theta-\alpha)}{\rho^n} - \frac{\sin n(\theta-\alpha')}{\rho'^n}\right\}.$$

Making use of the normal functions described in the last section, we consider the function

$$I\left\{\log \frac{z-z_0}{z-z_0'}\right\} - \sum_{n=1}^{\infty} \left[\frac{1}{n} \{\rho^n \sin n(\theta-\alpha) - \rho'^n \sin n(\theta-\alpha')\} \frac{r^n - q^{2n} r^{-n}}{1-q^{2n}}\right]$$

$$- \sum_{n=1}^{\infty} \left[\frac{q^{2n}}{n} \left\{\frac{\sin n(\theta-\alpha)}{\rho^n} - \frac{\sin n(\theta-\alpha')}{\rho'^n}\right\} \frac{r^n - r^{-n}}{1-q^{2n}}\right].$$

This function is harmonic within the ring, except at the points z_0, z_0', vanishes on the boundary $|z| = 1$, and reduces to a constant on the boundary $|z| = q$. We therefore add an arbitrary constant, and equate the result to $I\{\log w\}$.

But $$r^n \rho^n \sin n(\theta-\alpha) = I\left\{\frac{\rho^{2n} z^n}{z_0^n}\right\},$$

$$r^{-n} \rho^n \sin n(\theta-\alpha) = -I\left\{\frac{z_0^n}{z^n}\right\}, \quad \text{etc.},$$

so that we may write

$$\log w = \log \frac{z-z_0}{z-z_0'} - f\left(\frac{\rho^2 z}{z_0}\right) - f\left(\frac{q^2 z_0}{z}\right) - f\left(\frac{q^2 z}{z_0}\right) - f\left(\frac{q^2 z_0}{\rho^2 z}\right)$$
$$+ f\left(\frac{\rho'^2 z}{z_0'}\right) + f\left(\frac{q^2 z_0'}{z}\right) + f\left(\frac{q^2 z}{z_0'}\right) + f\left(\frac{q^2 z_0'}{\rho'^2 z}\right) + \text{const.},$$

where
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^{2n})}.$$

Provided $|x| < 1$, this series can be expanded in an absolutely convergent double series of powers of x and q^2. Rearranging this, we have

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^{2n})}$$
$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{q^{2mn} x^n}{n}$$
$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(q^{2m}x)^n}{n}$$
$$= -\sum_{m=0}^{\infty} \log(1-q^{2m}x).$$

Also
$$f(x) = -\log(1-x) + f(q^2 x),$$

so that

$$f\left(\frac{\rho^2 z}{z_0}\right) + f\left(\frac{q^2 z_0}{\rho^2 z}\right) = -\log\left[\left(1-\frac{\rho^2 z}{z_0}\right) \prod_{n=1}^{\infty} \left\{1-q^{2n}\left(\frac{\rho^2 z}{z_0} + \frac{z_0}{\rho^2 z}\right) + q^{4m}\right\}\right]$$

$$f\left(\frac{q^2 z}{z_0}\right) + f\left(\frac{q^2 z_0}{z}\right) = -\log\left[\prod_{n=1}^{\infty} \left\{1-q^{2m}\left(\frac{z}{z_0} + \frac{z_0}{z}\right) + q^{4m}\right\}\right],$$

whence

$$w = C \frac{z-z_0}{z-z_0'} \frac{z_0-\rho^2 z}{z_0'-\rho'^2 z}$$
$$\times \prod_{n=1}^{\infty} \left[\frac{\left\{1-q^{2m}\left(\frac{z}{z_0} + \frac{z_0}{z}\right) + q^{4m}\right\}\left\{1-q^{2m}\left(\frac{\rho^2 z}{z_0} + \frac{z_0}{\rho^2 z}\right) + q^{4m}\right\}}{\left\{1-q^{2m}\left(\frac{z}{z_0'} + \frac{z_0'}{z}\right) + q^{4m}\right\}\left\{1-q^{2m}\left(\frac{\rho'^2 z}{z_0'} + \frac{z_0'}{\rho'^2 z}\right) + q^{4m}\right\}}\right].$$

We now introduce the subsidiary variable u by writing

$$z = e^{(i\pi u)/\omega_1}, \quad q = e^{(i\pi\omega_3)/\omega_1}.$$

In this substitution ω_1 is at our disposal, so that if we take ω_1 to be real and positive, ω_3 is purely imaginary and $I(\omega_3)$ is positive. Further, we put

$$z_0 = e^{(i\pi u_0)/\omega_1}, \qquad z_0' = e^{(i\pi u_0')/\omega_1},$$

so that $$\rho^2/z_0 = e^{(-i\pi\bar{u}_0)/\omega_1}, \qquad \rho'^2/z_0' = e^{(-i\pi\bar{u}_0')/\omega_1}.$$

Since the points z, z_0, z_0' lie within the ring, it is necessary that $I(u)$, $I(u_0)$, $I(u_0')$ should be positive and not greater than $\omega_1\pi^{-1}\log(1/q)$.

Recalling the formula

$$\sigma(u) = \frac{2\omega_1}{\pi} e^{\eta_1 u^2/2\omega_1} \sin\frac{\pi u}{2\omega_1} \prod_{m=1}^{\infty} \frac{\{1-2q^{2m}\cos(\pi u/\omega_1)+q^{4m}\}}{(1-q^{2m})^2}$$

and, adjusting the constant C to our convenience, we get

$$w = Ce^{(\eta_1/\omega_1)(u_0+\bar{u}_0-u_0'-\bar{u}_0')u} \frac{\sigma(u-u_0)\,\sigma(u-\bar{u}_0)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')},$$

the same formula as before. The connexion of the two representations is then indicated by the relations

$$z = e^{(i\pi u)/\omega_1} \text{ of this section,}$$

$$t = \wp(u) \quad \text{of the earlier.}$$

Referring back to the value of $I\{\log w\}$, we see that the difference of the two constant values of this function on the boundaries is $I\{\log z_0/z_0'\}$. Hence the angle between the radial slits in the w-plane is the angle subtended at the origin in the z-plane by the points z_0, z_0', *i.e.* the points corresponding to O, Ω.

Case II.—In place of a single zero within the area, we have two simple zeros of w on a boundary (say the outer), so that $\log w$ has one singularity within the ring, corresponding to Ω, and two on the outer boundary corresponding to O. $I\{\log w\}$ is constant on each boundary except for discontinuities of amounts $\pm\pi$ at the two points O.

It is not necessary, however, to repeat the analysis with the modifications to meet the present requirements.

Suppose z_0 to be allowed to vary and to approach a point on the outer boundary.

Then $\rho \to 1$ and u_0 becomes real, and w tends to the value

$$Ce^{(\eta_1/\omega_1)(2u_0-u_0'-\bar{u}_0')u} \frac{\sigma^2(u-u_0)}{\sigma(u-u_0')\,\sigma(u-\bar{u})_0'}.$$

This function has a *double* zero on the outer boundary, a simple pole within the ring, and otherwise obeys the condition required.

We obtain a similar function with a double zero z_1 on the outer boundary and the same pole z_0' within the ring. From the pair, by taking the square root of their product, we derive the function

$$w = Ce^{(\eta_1/\omega_1)(u_0+u_1-u_0'-\bar{u}_0')u}\,\frac{\sigma(u-u_0)\,\sigma(u-u_1)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')}$$

as the function fulfilling our requirements.

Had we taken the zeros to lie on the inner boundary we should have had

$$\rho \to q \quad \text{so that} \quad u_0 \to v_0+\omega_3 \quad \text{where } v_0 \text{ is real},$$

$$\bar{u}_0 \to v_0-\omega_3,$$

so that the function with the double zero becomes

$$Ce^{(\eta_1/\omega_1)(2v_0-u_0'-\bar{u}_0')u}\,\frac{\sigma(u-v_0-\omega_3)\,\sigma(u-v_0+\omega_3)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')},$$

i.e.

$$C\sigma^2(\omega_3)\,e^{(\eta_1/\omega_1)(2v_0-u_0'-\bar{u}_0')u}\,\frac{\sigma_3^2(u-v_0)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')},$$

the final formula, with two simple zeros on the inner boundary, becoming

$$w = Ce^{(\eta_1/\omega_1)(v_0+v_1-u_0'-\bar{u}_0')u}\,\frac{\sigma_3(u-v_0)\,\sigma_3(u-v_1)}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0')}.$$

It is now obvious how we may reproduce our earlier results for Cases III and IV.

Problem II.—The solution is obtained in a similar manner by making the real part of $\log w$ constant on the boundaries. In fact, preserving the notation unaltered, we have

$$R\left\{\log\frac{z-z_0}{z-z_0'}\right\} = -\sum_{n=1}^{\infty}\frac{1}{n}\left\{\rho^n\cos n(\theta-\alpha)-\rho'^n\cos n(\theta-\alpha')\right\}$$

on the boundary $|z| = 1$, and, on the boundary $|z| = q$,

$$R\left\{\log\frac{z-z_0}{z-z_0'}\right\} = R\left\{\log\frac{z_0}{z_0'}\right\} - \sum_{n=1}^{\infty}\frac{q^n}{n}\left\{\frac{\cos n(\theta-\alpha)}{\rho^n}-\frac{\cos n(\theta-\alpha')}{\rho'^n}\right\},$$

whence we write

$$R\{\log w\} = R\left\{\log \frac{z-z_0}{z-z_0'}\right\}$$

$$+\sum_{n=1}^{\infty} \frac{1}{n}\{\rho^n \cos n(\theta-\alpha) - \rho'^n \cos n(\theta-\alpha')\} \frac{r^n - q^{2n} r^{-n}}{1-q^{2n}}$$

$$-\sum_{n=1}^{\infty} \frac{q^{2n}}{n}\left\{\frac{\cos n(\theta-\alpha)}{\rho^n} - \frac{\cos n(\theta-\alpha')}{\rho'^n}\right\} \frac{r^n - r^{-n}}{1-q^{2n}} + \text{const.}$$

or $$\log w = \log \frac{z-z_0}{z-z_0'} + f\left(\frac{\rho^2 z}{z_0}\right) - f\left(\frac{q^2 z_0}{z}\right) - f\left(\frac{q^2 z}{z_0}\right) + f\left(\frac{q^2 z_0}{\rho^2 z}\right)$$

$$-f\left(\frac{\rho'^2 z}{z_0'}\right) + f\left(\frac{q^2 z_0'}{z}\right) + f\left(\frac{q^2 z}{z_0'}\right) - f\left(\frac{q^2 z_0'}{\rho'^2 z}\right) + \text{const.}$$

$$w = C \frac{z-z_0}{z-z_0'} \frac{\rho'^2 z - z_0'}{\rho^2 z - z_0}$$

$$\times \prod_{m=1}^{\infty} \left[\frac{\left\{1-q^{2m}\left(\frac{z}{z_0}+\frac{z_0}{z}\right)+q^{4m}\right\}\left\{1-q^{2m}\left(\frac{\rho'^2 z}{z_0'}+\frac{z_0'}{\rho'^2 z}\right)+q^{4m}\right\}}{\left\{1-q^{2m}\left(\frac{z}{z_0'}+\frac{z_0'}{z}\right)+q^{4m}\right\}\left\{1-q^{2m}\left(\frac{\rho^2 z}{z_0}+\frac{z_0}{\rho^2 z}\right)+q^{4m}\right\}} \right]$$

$$= C e^{(\eta_1/\omega_1)(u_0-\bar{u}_0-u_0'+\bar{u}_0')u} \frac{\sigma(u-u_0)\,\sigma(u-\bar{u}_0')}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0)}.$$

Referring back to the value of $R\{\log w\}$ we see that the difference in the values of this function on the two boundaries is $R\{\log z_0/z_0'\}$, *i.e.* the ratio of the radii of the two slits in the w-plane is the ratio of the distances of the points z_0, z_0' from the origin of the z-plane.

Problem III, *Case* I.—The function $w(z)$ has one simple pole z_0' within the ring $q \leqslant |z| \leqslant 1$. We may take z_0' to be real. Then w is a uniform function of z, and we have $I(w)$ constant along the boundaries. We take the principal part of w to be $(a+ib)/(z-z_0')$ at Ω.

Then, when $|z| = 1$,

$$I\left\{\frac{a+ib}{z-z_0'}\right\} = \sum_{n=1}^{\infty} z_0'^{n-1}(b\cos n\theta - a \sin n\theta),$$

and, when $|z| = q$,

$$I\left\{\frac{a+ib}{z-z_0'}\right\} = -\frac{b}{z_0'} - \sum_{n=1}^{\infty} \frac{q^n(b\cos n\theta + a\sin n\theta)}{z_0'^{n+1}},$$

so that we take

$$I\{w\} = I\left(\frac{a+ib}{z-z_0'}\right) - \sum_{n=1}^{\infty} \frac{z_0'^{n-1}(b\cos n\theta - a\sin n\theta)(r^n - q^{2n}r^{-n})}{1-q^{2n}}$$

$$+ \frac{b}{z_0'} - \sum_{n=1}^{\infty} \frac{z_0'^{-n-1}q^{2n}(b\cos n\theta + a\sin n\theta)(r^n - r^{-n})}{1-q^{2n}} + \text{const.},$$

whence, after reduction,

$$w = \frac{(a+ib)}{z_0'}\left[\frac{z+z_0'}{2(z-z_0')} + \sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\left(\frac{z_0'^n}{z^n} - \frac{z^n}{z_0'^n}\right)\right]$$

$$+ \frac{a-ib}{z_0'}\left[-\frac{zz_0'+1}{2(zz_0'-1)} + \sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\left(z^n z_0'^n - \frac{1}{z_n z_0'^n}\right)\right] + \text{const.}$$

Using the same substitutions as before, we get

$$w = \frac{a+ib}{z_0'}\left[\frac{1}{2i}\cot\frac{\pi(u-u_0')}{2\omega_1} - 2i\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi(u-u_0')}{\omega_1}\right]$$

$$+ \frac{a-ib}{z_0'}\left[-\frac{1}{2i}\cot\frac{\pi(u+u_0')}{2\omega_1} + 2i\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi(u+u_0')}{\omega_1}\right] + \text{const.}$$

Recalling now the formula*

$$\xi(u) = \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{\eta_1 u}{\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1},$$

we are left with

$$w = \frac{(b-ia)\omega_1}{\pi z_0'}\xi(u-u_0') + \frac{(b+ia)\omega_1}{\pi z_0'}\xi(u+u_0') - \frac{2b\eta_1}{\pi z_0'}u + \text{const.}$$

The slight difference in form between this result and the previous one is readily explained by observing that the origin of the u-plane can be moved parallel to the axis of real numbers without affecting the validity of the result. Here, by making z_0' real, we have made u_0' purely imaginary, so that $\bar{u}_0' = -u_0'$.

Case II.—Introducing the shift of the origin, we write the formula of Case I in the general form

$$w = (A+iB)\xi(u-u_0') + (A-iB)\xi(u-\bar{u}_0') - \frac{2A\eta_1}{\omega_1}u + \text{const.}$$

* Harkness and Morley, *Theory of Functions*, 320.

If we allow z_0' to approach a point on the outer boundary of the ring, u_0' and $\bar{u}_0'$ approach a real value u_0', and we find

$$w = A\xi(u-u_0') - \frac{A\eta_1}{\omega_1}u + \text{const.}$$

and w is a function with one pole on the boundary of the ring, and $I\{w\}$ is constant on the two boundaries. Hence we deduce that

$$w = A\xi(u-u_0') + B\xi(u-u_1') - \frac{(A+B)\eta_1}{\omega_1}u + \text{const.}$$

defines a function w with two poles on the outer boundary, and with $I(w)$ constant on each boundary.

Special Cases.

Cases of partial symmetry arise as we have seen in Problem I, Cases I and II, in Problem II, and in Problem III, Cases I and II. While it would undoubtedly be simpler to deal with these, as problems on their own account, by the methods earlier indicated, and, in the case of Problem III, by the use of the Schwarz-Christoffel formula as carried out by Prof. Love, we can readily obtain the required results from the general cases.

The lines of symmetry are represented in the u-figures by the lines B_2B_3, B_1B_4. It follows that in Problem I, Case I, we replace u_0', $\bar{u}_0'$ by $-\bar{u}_0$, $-u_0$ and in Case III u_0'. u_1' by $-u_0$, $-u_1$.

In Problem II, in one case of partial symmetry, both O, Ω can be taken to lie on B_1B_4, *i.e.* u_0, u_0' are real, and O', Ω' therefore become $-u_0'$, $-u_0$. In the other case, we may take O to lie on B_2B_3 and Ω to lie on B_1B_4 so that $u_0+\omega_3$ and u_0' are real, and O', Ω' can be written $-u_0'$, $-u_0-\omega_3$, and the resultant formula is

$$w = Ce^{\mu u}\frac{\sigma_3(u-v_0)\,\sigma(u+u_0')}{\sigma_3(u+v_0)\,\sigma(u-u_0')}$$

where $v_0 = u_0-\omega_3$.

In Problem III, Case I, Ω lies on B_1B_4 in the case of partial symmetry, and we have

$$w = (A+iB)\,\zeta(u-i\alpha)+(A-iB)\,\zeta(u+i\alpha)+\mu u+C.$$

This, however, is insufficient, as the formula can be made to assume this form in the general case by a shift of the origin of the u-plane.

Suppose we approach Ω in the u-plane along B_1B_4. The corresponding point in the w-plane moves off to infinity along the imaginary axis. But, in these circumstances,

$$w \sim \frac{A+iB}{u+ia}.$$

It follows that $B = 0$, and the corrected formula is

$$w = A\{\zeta(u-ia)+\zeta(u+ia)\}+\mu u+C.$$

In Case II, somewhat more simply, we see that the two points Ω, Ω' coincide with B_3 and B_4, so that

$$w = A\zeta(u)+B\zeta(u+\omega_1)+\mu u+C.$$

We may note that, if we put, in Case I, $A = 0$, and, in Case II, $A = -B$, in the general formulae w reduces to an elliptic function which is, in fact, a linear transformation of $\wp(u)$, giving the trivial result that pairs of collinear slits in the planes of w and t are transformable into one another by a homographic transformation.

A special case of symmetry arises in Problem III, Case I, when the slits are of equal length and we have symmetry about a point. In this case

$$w = (A+iB)\,\zeta(u-\tfrac{1}{2}\omega_3)+(A-iB)\,\zeta(u+\tfrac{1}{2}\omega_3)+\mu u+C.$$

We obtain this by observing that the point in the u-plane corresponding to the centre of symmetry will be a centre of symmetry in that plane. It will therefore lie on the line midway between the lines which represent the slits, and it will then follow that the poles of w must lie on this line also at distances ω_1 from the centre of symmetry. Transforming the origin of the u-plane by a shift along the axis of real numbers we get the result given, and the centre of symmetry is then represented by the point $\omega_1+\frac{1}{2}\omega_3$.

Limiting cases occur when one of the slits in Problems I and III terminates at O or Ω. We obviously make the corresponding values of u coincident real values.

Again, in either of these problems one of the slits may be made doubly infinite without a gap. We may derive the solution in these cases from those of complete symmetry by using only one half of the parallelogram there employed in the representation; or, again, we may note that corresponding to O, Ω we only obtain one zero and one pole of w on the t-surface which leads analytically to the same result.

Again, in Problem II one of the slits, say the inner, may be made into a complete circle. There is then one pole and no zero of w in the upper sheet of the t-surface—or we may derive the same analytical results from the case of complete symmetry.

If both circles are made complete, w has neither pole nor zero on the Riemann surface, and we get the representation of the area between concentric circles by means of the formula $w = Ce^{\mu u}$, as is to be expected. This area is, of course, the one we took as our standard doubly-connected area in the second method of solution.

It may be noted that, if the angle between the straight slits in Problem I is $m\pi/n$, w becomes an elliptic function of u of periods $2\omega_1$, $2n\omega_3$. This appears to provide a means of passage to the special results of Thomae and Bähr to which we have referred.

A type of degeneration which we may notice is that obtained by contracting one of the slits to a single point. The corresponding slit on the t-plane is similarly contracted to a point. We now seek the representation of a plane bounded by a single slit and marked at a point P upon a t-plane slit along a straight line, and marked at a point collinear with the slit. As now there is only one line of passage from one sheet of the t-surface to the other, singly periodic functions suffice for the uniformisation.

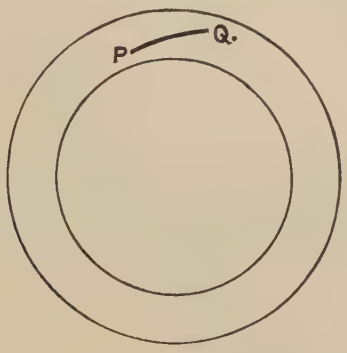
The Conformal Representation of some Riemann Surfaces.

We may note briefly that our methods have brought to light the formulae which effect the representations of some bounded Riemann surfaces. Thus in Problem II we have the formula

$$w = e^{\mu u} \frac{\sigma(u-u_0)\,\sigma(u-\bar{u}_0')}{\sigma(u-u_0')\,\sigma(u-\bar{u}_0)}.$$

We determined μ by the condition that the multiplier corresponding to the period $2\omega_1$ was unity, with zero amplitude. If in place of unity we had the multiplier $e^{2i\pi}$, *i.e.* in the notation adopted $k = 1$, our argument would remain unchanged except in this detail: that as the u-point traces out one of the sides of the half-parallelogram employed, the corresponding w-point traces out a complete circle. It appears then, that with $k = 1$ we get the representation of the two-sheeted figure, one sheet of which consists of the area exterior to the smaller circle, while

the other sheet is formed by the area interior to the larger circle, there being a line of junction PQ in the overlapping area.



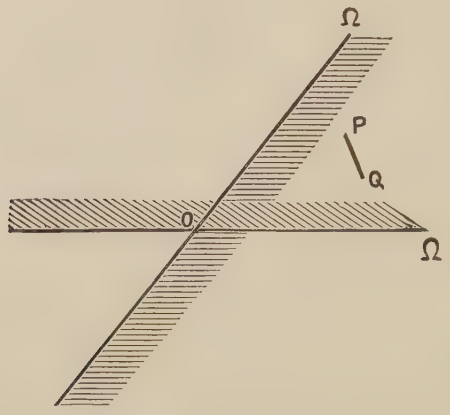
With higher values of k, the Riemann surface grows in complexity. Thus for $k = 2$ each circle is described completely twice over. The figure would appear to consist of three sheets—two are as described for the case $k = 1$; but we interpose a new sheet consisting of an annulus bounded by the two circles. Since $d(\log w)/du$ has only four zeros, and half of these must belong to the symmetric repetition of the figure obtained by inversion with respect to one of the bounding circles, there can only be two branch points on the bounded Riemann surface. As joining these by a branch line would lead only to the junction of two of the sheets, we are compelled to find a new mode of junction. This would appear to be provided as follows :—To the surface adjoin a symmetric repetition, say by inversion in the smaller circle. Then if P be a branch point in the two sheets which have this circle as boundary, and P' the repetition of P, a line of junction for these sheets can be drawn from P to P'.

It is not clear how the connexion can be obtained with higher values of k. Negative values of k provide no new problem, for by considering the function $1/w$ we have then a similar formula with a positive value of k.

Finally, it is interesting to note that there is a breakdown in the analogy of Problems I and II when we consider Riemann surfaces of this type whose boundaries are straight lines. Thus, if in Problem I k is not zero, $d(\log w)/du$ ceases to be real on the boundary, and the boundary cannot be a straight line.

If we seek the representation of a Riemann surface consisting of the two half-planes on the shaded sides of the boundary lines, with a line of junction PQ in the overlapping area, we obtain the appropriate formula

from the considerations that in this figure there is a point O, and similarly a point Ω on each of the boundaries. In this respect the figure differs



from all the figures of Problem I. We accordingly arrive at a formula

$$w = e^{\mu u} \frac{\sigma(u-u_0)\,\sigma_3(u-v_0)}{\sigma(u-u_0')\,\sigma_3(u-v_0')}$$

where u_0, v_0, u_0', v_0' are real numbers.

The corresponding Riemann surface based on Problem III can be treated analogously.

[*Note added September 5th*, 1924.—Since this paper went to press we have become acquainted with the work of Thiry and Villat on the conformal representation of doubly-connected areas. The former has given a formula for the case of an area with rectilinear boundaries; the latter considers the case when the boundaries consist of circular arcs. Our solutions obviously satisfy differential equations of the types given by these authors.*]

* R. Thiry, "Sur les solutions multiples des problèmes d'hydrodynamique relatifs aux mouvements glissants", Thèse (Gauthier-Villars, Paris, 1921); also published in *Annales Sci. de l'Ecole Normale* (3), 38 (1921), 229–339; H. Villat, *Comptes Rendus*, Feb. 11th, 1924.

NOTE ON THE REDUCTION OF A CERTAIN INVARIANT OF THREE QUATERNARY QUADRICS

By H. W. TURNBULL.

[Received and Read December 13th, 1923.]

IN the *Proc. London Math. Soc.* (2), 20 (1922), 483, I gave a list of forty-seven invariants of three quadrics. It was not apparent at the time that one of the included types, whose symbolic form is

$$(Abc)(B\gamma a)(AC)(CB)\,b_\gamma c_a, \tag{1}$$

could be reduced, although a simpler type,

$$(Abc)(B\gamma a)(AB)\,b_\gamma c_a, \tag{2}$$

reduces by squaring the identity

$$(Abc)(Ba\gamma)+(AB)\,b_\gamma c_a \equiv -(ABc\gamma)\,b_a+\Omega_{aa'}(aBb)\,c_a a'_\gamma \pmod{a_a,\ c_\gamma},$$

since (Bb) may be replaced by a single symbol β twice in each term where it appears in the squared identity.

In this, $\Omega_{aa'}$ denotes the two-term determinantal permutation $a, a' \ -a', a$, while the other symbols are explained in *loc. cit.*, 467, § 7, and 471, § 14.

The following procedure, however, reduces type (1) :—

Since the above identity is true without using the fact that $B\,[=(b'b'')]$ and b refer to the same quadric $b_x^2 = b_x'^2 = b_x''^2$, let these symbols B, b be replaced by arbitrary quadratic symbols $D[=(d'd'')]$ and d respectively, where d, d', d'' are not necessarily equivalent. The result of squaring the identity and transposing all square terms to the right is then

$$(Adc)(d'd''a\gamma)(Ad'd'')\,d_\gamma c_a \equiv -(Ad'd''c\gamma)\,d_a\Omega_{aa'}(ad'd'')\,c_a a'_\gamma+\Sigma RS \tag{3}$$

where the summation Σ consists of reducible terms, either because a_α, c_γ (and therefore the invariants a_α^2, c_γ^2) appear in a term, or because the term is the square of one of the four terms explicitly given in the first identity (with symbols D, d replacing B, b). These symbols D, occur explicitly twice in one factor, R, say, of RS.

If we form two more identities derived from (3) by interchanging d, d', d'' according to the substitutions

$$\text{(i)} \quad \begin{pmatrix} d\, d'\, d'' \\ d'\, d''\, d \end{pmatrix} \quad \text{and} \quad \text{(ii)} \quad \begin{pmatrix} d\, d'\, d'' \\ d''\, d\, d' \end{pmatrix},$$

and add the results, we obtain an identity

$$\left.\begin{aligned} &(Adc)(d'd''a\gamma)(Ad'd'')\,d_\gamma\,c_a \\ +&(Ad'c)(d''da\gamma)(Ad''d)\,d'_\gamma\,c_a \\ +&(Ad''c)(dd'a\gamma)(Add')\,d''_\gamma\,c_a \end{aligned}\right\} \equiv -[\Omega_{aa'}(ad'd''d)\,a'_\gamma]^2 c_a^2 + \Sigma R'S' + \Sigma', \quad (4)$$

where Σ' is a series of terms $(d'^2_\varpi\, d''^2_\rho + d''^2_\varpi\, d'^2_\rho)\,N$ and where R', S' are invariants of the same type as R, S. This first right-hand term is due to the effect of the substitutions on the first right-hand term in (3), other terms derived from the same source having a factor a_a, and therefore a_a^2. But this first term of (4) contains c_a^2 and may therefore be included in the series $\Sigma R'\,S'$. Σ' gives those terms where, owing to the substitutions, d' and d'' are no longer convolved. Replacing R' by $(d'\,d''K)(d'\,d''L)M$, since $d'\,d''$ is convolved twice in every factor R', the right member of (4) may be written

$$\Sigma(d'\,d''K)(d'\,d''L)M\,.\,S' + \Sigma'. \quad (5)$$

Now consider the effect of the six-term operator

$$\left(C\,\frac{\partial}{\partial D}\right) = C_{12}\frac{\partial}{\partial D_{12}} + C_{13}\frac{\partial}{\partial D_{13}} + C_{14}\frac{\partial}{\partial D_{14}} + C_{23}\frac{\partial}{\partial D_{23}} + C_{24}\frac{\partial}{\partial D_{24}} + C_{34}\frac{\partial}{\partial D_{34}}$$

$$= \Sigma(c'_1 c''_2 - c''_2 c'_1)\left(\frac{\partial^2}{\partial d'_1 \partial d''_2} - \frac{\partial^2}{\partial d'_2 \partial d''_1}\right), \qquad C = (c'\,c'')$$

operating on the identity (4). As far as the symbols $d'\,d''$ are concerned, all the terms of (4) are of two types

$$d'_\varpi\, d''_\rho\, d'_\sigma\, d''_\tau \quad \text{and} \quad (d'\,d''X)(d'\,d''Y).$$

Now
$$\left.\begin{aligned}\left(C\frac{\partial}{\partial D}\right)d'_\varpi d''_\rho d'_\sigma d''_\tau &= (C\varpi\rho)\,d'_\sigma d''_\tau+(C\sigma\tau)\,d'_\varpi d''_\rho\\ &\quad+(C\varpi\tau)\,d'_\sigma d''_\rho+(C\sigma\rho)\,d'_\varpi d''_\tau\\ &= \Omega_{c'c''}\left[c'_\varpi c''_\rho d'_\sigma d''_\tau+c'_\sigma c''_\tau d'_\varpi d''_\rho\right.\\ &\quad\left.+c'_\varpi c''_\tau d'_\sigma d''_\rho+c'_\sigma c''_\rho d'_\varpi d''_\tau\right]\end{aligned}\right\};\qquad(6)$$

Also
$$\left(C\frac{\partial}{\partial D}\right)(DK)(DL)=3(CK)(DL)+3(DK)(CL)-(CD)(KL).\qquad(7)$$

There is no difficulty in proving formula (6). Formula (7) is proved as follows :—

Let (DK) be written as $(d'd''K)$ or $-(d''d'K)$ or $\Sigma d'_i(d''K)_i$ or $-\Sigma d''_i(d'K)_i$; $i=1, 2, 3, 4$, then

$$\frac{\partial}{\partial d'_1}(DK)(DL)=(d''K)_1(d'd''L)+(d''L)_1(d'd''K);$$

hence

$$\frac{\partial^2}{\partial d'_1\partial d''_2}(DK)(DL)=K_{34}(DL)+L_{34}(DK)-(d''K)_1(d'L)_2-(d''L)_1(d'K)_2,$$

so that
$$\frac{\partial}{\partial D_{12}}(DK)(DL)=K_{34}(DL)+L_{34}(DK)+X+Y,\qquad(8)$$

where
$$X+Y=\begin{vmatrix}(d'K)_1 & (d'L)_2\\ (d''K)_1 & (d''L)_2\end{vmatrix}+\begin{vmatrix}(d'L)_1 & (d'K)_2\\ (d''L)_1 & (d''K)_2\end{vmatrix}.$$

Expanding the first determinant we have

$$-\begin{vmatrix}d'_2K_{34}+d'_3K_{42}+d'_4K_{23}, & d'_1L_{34}+d'_3L_{41}+d'_4L_{13}\\ d''_2K_{34}+d''_3K_{42}+d''_4K_{23}, & d''_1L_{34}+d''_3L_{41}+d''_4L_{13}\end{vmatrix}$$

$$=K_{34}L_{34}D_{12}+K_{34}L_{41}D_{32}+K_{34}L_{13}D_{42}+K_{42}L_{34}D_{13}$$
$$+K_{42}L_{13}D_{43}+K_{23}L_{34}D_{14}+K_{23}L_{41}D_{34}.$$

Similarly for the second determinant, with K, L interchanged. Thus

$$X+Y=K_{34}(LD)+L_{34}(KD)-D_{34}(KL).$$

Finally, substituting this last result in (8) leads to the required form (7). Accordingly we may utilize (6) and (7) to find the result of operating on both sides of identity (4) with $\left(C\frac{\partial}{\partial D}\right)$.

The effect of the operator on (5), which represents the right side of identity (4), leaves S' unchanged. Moreover, the result is a real invariant if we multiply throughout by the symbolic factor (CD). Also, by (6)

$$(CD)\left(C\frac{\partial}{\partial D}\right)\Sigma' = 4(Cd'd'')\,\Sigma(C\varpi\rho)(d'_\varpi d''_\rho + d''_\varpi d'_\rho)\,N$$

which vanishes identically when, after the operation, we replace d', d'' by equivalent symbols b', b''. Hence terms on the right are still reducible.

The effect of the operator on the left side of (4) is

$$\left.\begin{aligned}&[3(Adc)(C\alpha\gamma)(AD)d_\gamma c_\alpha + 3(Adc)(D\alpha\gamma)(AC)d_\gamma c_\alpha \\ &\qquad -(Adc)(A\alpha\gamma)(CD)d_\gamma c_\alpha] \\ &+\Omega_{c'c''}[(Ac'c)(c''d\alpha\gamma)(Ad''d)d'_\gamma c_\alpha + (Ad'c)(d''d\alpha\gamma)(Ac''d)c'_\gamma c_\alpha \\ &\qquad +(Ac'c)(d''d\alpha\gamma)(Ac''d)d_\gamma c_\alpha + (Ad'c)(c''d\alpha\gamma)(Ad''d)c'_\gamma c_\alpha] \\ &+\Omega_{c'c''}[(Ac''c)(c'd\alpha\gamma)(Ad'd)d''_\gamma c_\alpha + \dots \\ &\qquad +\dots \qquad\qquad +\dots \qquad\qquad]\end{aligned}\right\} \tag{9}$$

where the last four terms repeat the previous four with $'$ and $''$ completely interchanged. Of the eleven terms written in (9), six which contain $(A\alpha\gamma)$, $(C\alpha\gamma)$, c'_γ or c''_γ are reducible. In the remainder we may substitute the special symbols b, b', b'' for the arbitrary d, d', d'', and finally multiply throughout by (CB). The contents of the two last brackets [ ] are then equivalent, so that

$$3(Abc)(B\alpha\gamma)(AC)(CB)\,b_\gamma c_\alpha + 2\Omega_{c'c''}[(Ac'c)(c''b\alpha\gamma)(Ab''b)b'_\gamma c_\alpha + (Ac'c)(b''b\alpha\gamma)(Ac''b)\,b'_\gamma c_\alpha](CB) = 0.$$

But
$$(Ac'c)(c''b\alpha\gamma)c_\alpha(CB) \equiv \tfrac{1}{2}(Ac'c'')(cb\alpha\gamma)c_\alpha(CB) \pmod{cc'c''}$$
$$\equiv 0,$$

since
$$(cb\alpha\gamma)c_\alpha = (c_\alpha b_\gamma - c_\gamma b_\alpha)c_\alpha \equiv 0.$$

Also
$$\Omega_{c'c''}(Ac'c)(Ac''b)(CB) \equiv (Ac'c'')(Acb)(CB)$$
$$\equiv (AC)(Acb)(CB).$$

Hence the identity becomes

$$3(Abc)(B\alpha\gamma)(AC)(CB)\,b_\gamma c_\alpha + 2(Acb)(b''b\alpha\gamma)(AC)(CB)\,b'_\gamma c_\alpha \equiv 0,$$

or $$+2(Abc)(bb''\alpha\gamma)(AC)(CB)\,b'_\gamma c_\alpha \equiv 0,$$

or $$+(Abc)(b'b''\alpha\gamma)(AC)(CB)\,b_\gamma c_\alpha \equiv 0$$

(mod $bb'b''$),

since $B = (b'b'')$. Finally

$$4(Abc)(B\alpha\gamma)(AC)(CB)\,b_\gamma c_\alpha \equiv 0.$$

Thus the invariant type (1) is reducible, so that the six forms of this type for three quadrics may be deleted from the complete system, leaving at most forty-one invariants.

The process would have broken down if the operator $\left(C\,\frac{\partial}{\partial D}\right)$ were applied to (3) direct, without first permuting the symbols d, d', d''.

ON CERTAIN APPROXIMATE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

By HAROLD JEFFREYS.

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Revised April 17th, 1924.]

1.1. Many differential equations occurring in mathematical physics are reducible to the form

$$\frac{d^2y}{dx^2} - h^2\chi y = 0, \tag{1}$$

where χ is a function of x alone, and h is independent of x. The case where h is large is often of special interest. In this case an asymptotic solution can be obtained by Horn's method,† provided that χ does not vanish within the range of values of x considered. Horn does not appear, however, to have given the specially convenient form that the first term of the solution takes when the differential equation is of the form (1); it was discovered otherwise by Green, and has been obtained and used by R. H. Fowler and others and by the writer.‡ In the present paper the method of Horn is applied to a slightly more general equation than

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* This paper, with those entitled "On Certain Solutions of Mathieu's Equation", "On the Modified Mathieu's Equation", and "On the Free Oscillations of an Elliptic Lake", formed part of a dissertation highly commended by the Adjudicators for the Adams Prize of the University of Cambridge for the year 1923. Paragraphs are numbered decimally throughout all the papers.

† *Math. Annalen*, 52 (1899), 340–362. Mathieu functions are discussed for the case of R large, but h^2/R small, to use the notation of 2.1, but the specially important case of h^2 large, but R/h^2 moderate, is not considered.

‡ Cf. Lamb, *Hydrodynamics* (1906), 257; Jeffreys, *Mem. R.A.S.*, 60 (1915), 211–213; Fowler, Gallop, Lock, and Richmond, *Phil. Trans. Royal Soc.*, 221A (1920), 337–9.

(1), and a method is devised for adapting the solutions to the case where χ vanishes within the range.

1.2. Let the differential equation be

$$y''-(h^2\chi_0+h\chi_1+\chi_2)y = 0, \tag{1}$$

where accents denote differentiation with regard to x, χ_0, χ_1, and χ_2 are all functions of x, and h is a parameter independent of x, capable of assuming indefinitely great values. Let us assume that a solution exists which is expressible in the form

$$y = \phi e^{h\omega}\left(1+\frac{f_1}{h}+\frac{f_2}{h^2}+\ldots\right), \tag{2}$$

where ϕ, ω, f_1, f_2, ... are functions of x alone. Substituting in (1) from (2), and equating coefficients of h^2, h, and h^0, we find equations reducing to

$$\omega'^2 = \chi_0, \tag{3}$$

$$\frac{\phi'}{\phi} = \frac{\chi_1-\omega''}{2\omega'}, \tag{4}$$

$$2\omega' f_1' = \chi_2-\frac{\phi''}{\phi}. \tag{5}$$

We see from (4) that

$$\phi = \omega'^{-\frac{1}{2}} \exp\left(\int \frac{\chi_1}{2\omega'}\,dx\right), \tag{6}$$

and that, so long as a solution in the form (2) exists at all, it will reduce approximately to its first term when h is great enough. Thus approximate solutions are

$$y = \chi_0^{-\frac{1}{4}} \exp\left\{\pm\int^x\left(h\chi_0^{\frac{1}{2}}+\frac{\chi_1}{2\chi_0^{\frac{1}{2}}}\right)dx\right\}, \tag{7}$$

where the lower limits are arbitrary.

But if

$$\chi = h^2\chi_0+h\chi_1+\chi_2, \tag{8}$$

$$h\chi_0^{\frac{1}{2}}+\frac{\chi_1}{2\chi_0^{\frac{1}{2}}} = \chi^{\frac{1}{2}}+O(h^{-1}), \tag{9}$$

and the solutions may be replaced by

$$y = \chi^{-\frac{1}{4}} \exp\left(\pm \int^x \chi^{\frac{1}{2}}\, dx\right). \tag{10}$$

If χ is positive, this form is the most convenient; but if χ is negative and equal to $-\psi$, the solutions may be replaced by

$$y = \psi^{-\frac{1}{4}} \cos \int^x \psi^{\frac{1}{2}}\, dx \tag{11}$$

and

$$y = \psi^{-\frac{1}{4}} \sin \int^x \psi^{\frac{1}{2}}\, dx. \tag{11}$$

If ψ is an even function of x, it will usually be convenient to take the lower limit of both integrals equal to zero; if then $\int_0^x \psi^{\frac{1}{2}}\, dx$ be denoted by L, the solution will be

$$y = \psi^{-\frac{1}{4}} (A \cos L + B \sin L), \tag{12}$$

where A and B are arbitrary constants.

1 . 21. As an example, let us consider the function

$$y = \exp h\theta, \tag{1}$$

where θ is a function of x.

Then

$$y'' - (h^2\theta'^2 + h\theta'')y = 0. \tag{2}$$

An approximate solution of (2), by our formula, is

$$\theta'^{-\frac{1}{2}} \exp\left\{\int \left(h\theta' + \frac{\theta''}{2\theta'}\right) dx\right\}, \tag{3}$$

which is exactly equal to (1).

1 . 22. The solutions obtained are strictly only the limiting forms of the solutions as h tends to infinity; they must not, without due caution, be applied to cases where h is moderate. For instance, the other exact solution of 1 . 21 (2) is

$$y_2 = e^{h\theta} \int e^{-2h\theta}\, dx, \tag{1}$$

and we have by integration by parts

$$-2hy_2 = \frac{1}{\theta'}\exp(-h\theta) - \frac{1}{2h}\exp(-h\theta)\frac{\theta''}{\theta'^3} - O\left(\frac{1}{h^2}\right), \tag{2}$$

while the second solution of 1 . 21 (2) by our formula is simply

$$\frac{1}{\theta'}\exp(-h\theta).$$

Thus to use the approximate formula when h is finite would lead to serious error if $\theta''\theta'^{-3}$ was considerable. If $\theta = x^m$, where $m > 2$, $\theta''\theta'^{-3} \to 0$ as $x \to \infty$, and the approximation will be valid for large values of x even when h is moderate. But if $\theta = e^{-x}$, $\theta''\theta'^{-3} \to \infty$ with x, and the approximation breaks down when h is moderate but x large.

Again, although $\chi^{\frac{1}{2}}$ is approximately equal to $h\chi_0^{\frac{1}{2}} + (\chi_1/2\chi_0^{\frac{1}{2}})$ when h is large, this is not true when h is moderate but x large unless

$$\int\left(\chi_0\chi_2 - \tfrac{1}{4}\chi_1^2\right)\chi_0^{-\frac{3}{2}}\,dx$$

is then small. In the above example the approximation $\chi^{-\frac{1}{4}}\exp\int\chi^{\frac{1}{2}}\,dx$ breaks down for this reason if θ is $(\log x)^2$, although the approximation

$$\chi_0^{-\frac{1}{4}}\exp\left\{\int\left(h\chi_0^{\frac{1}{2}} + \frac{\chi_1}{2\chi_0^{\frac{1}{2}}}\right)dx\right\}$$

is correct.

1 . 3. The error in the solutions is in general of order h^{-1}; but we see from 1 . 2 (5) that, if χ_1 and χ_2 are zero,

$$f_1' = \tfrac{1}{8}\chi_0^{\frac{3}{2}}\chi_0'' - \tfrac{5}{32}\chi_0^{-\frac{5}{2}}\chi_0'^2,$$

so that f_1', and usually f_1, will tend to infinity near a zero of χ_0 and the solution we have obtained will break down.

1 . 4. Let us now consider the nature of the solutions in the neighbourhood of a zero of χ. It will be supposed that the terms in χ_1 and χ_2 are absent. Without loss of generality we may suppose that χ is positive when x is less than x_0, and negative when x is greater than x_0. The solutions are then of hyperbolic type up to x_0, and oscillatory beyond it. Special treatment is necessary to find what linear combinations of the solutions, when x is less than x_0, pass respectively into the two standard solutions valid when x is greater than x_0.

When x is nearly equal to x_0 we may write

$$x_0 - x = x_1, \quad x - x_0 = x', \tag{1}$$

and, provided that $\frac{1}{2}\chi'' x_1/\chi'$ is small, the differential equation reduces to

$$\frac{d^2y}{dx'^2} + \kappa x' y = 0, \tag{2}$$

where

$$\kappa = -\chi'(x_0) \tag{3}$$

and thus

$$\chi = \kappa x_1, \quad \psi = \kappa x'. \tag{4}$$

If $\int^x \chi^{\frac{1}{2}} dx = M$, and if $M(x_0) = M_0$, we have

$$\begin{aligned} M &= M_0 - \tfrac{2}{3}\kappa^{\frac{1}{2}} x_1^{\frac{3}{2}} \\ &= M_0 - \theta_1, \quad \text{say.} \end{aligned} \tag{5}$$

Then approximately

$$\chi^{-\frac{1}{4}} \exp(M) = \kappa^{-\frac{1}{4}} x_1^{-\frac{1}{4}} \exp(M_0 - \theta_1), \tag{6}$$

$$\chi^{-\frac{1}{4}} \exp(-M) = \kappa^{-\frac{1}{4}} x_1^{-\frac{1}{4}} \exp(-M_0 + \theta_1). \tag{7}$$

Now the solutions of (2) are $x_1^{\frac{1}{2}} I_{-\frac{1}{3}}(\theta_1)$ and $x_1^{\frac{1}{2}} I_{\frac{1}{3}}(\theta_1)$ when x_1 is positive, and continuations of these functions when x' is positive are $x'^{\frac{1}{2}} J_{-\frac{1}{3}}(\theta')$ and $-x'^{\frac{1}{2}} J_{\frac{1}{3}}(\theta')$ respectively, where

$$\theta' = \tfrac{2}{3}\kappa^{\frac{1}{2}} x'^{\frac{3}{2}}. \tag{8}$$

Applying the asymptotic expansions of the Bessel functions of order $\frac{1}{3}$ when the argument is large, we have

$$\begin{aligned} x_1^{-\frac{1}{4}} \exp(-\theta_1) &\sim \frac{2}{\alpha\sqrt{3}}\, x_1^{\frac{1}{2}} \{ I_{-\frac{1}{3}}(\theta_1) - I_{\frac{1}{3}}(\theta_1) \} \\ &= \frac{2}{\alpha\sqrt{3}}\, x'^{\frac{1}{2}} \{ J_{-\frac{1}{3}}(\theta') + J_{\frac{1}{3}}(\theta') \} \\ &\sim 2x'^{-\frac{1}{4}} \cos(\tfrac{1}{4}\pi - \theta'), \end{aligned} \tag{9}$$

where

$$\alpha = \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \kappa^{-\frac{1}{4}}, \tag{10}$$

and positive signs are to be taken for all roots. Similarly

$$x_1^{-\frac{1}{4}} \exp(\theta_1) \sim \frac{2}{\alpha} x_1^{\frac{1}{2}} I_{-\frac{1}{3}}(\theta_1)$$

$$= \frac{2}{\alpha} x'^{\frac{1}{2}} J_{-\frac{1}{3}}(\theta')$$

$$\sim 2x'^{-\frac{1}{4}} \sin(\tfrac{1}{12}\pi + \theta'). \tag{11}$$

Thus, by (6) and (7), remembering that when x' is small relations (4) hold,

$$\chi^{-\frac{1}{4}} \exp(M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(M_0) \cos(\tfrac{1}{4}\pi - \theta'), \tag{12}$$

$$\chi^{-\frac{1}{4}} \exp(-M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(-M_0) \sin(\tfrac{1}{12}\pi + \theta'), \tag{13}$$

still provided $\chi'' x'/2\chi'$ is small, where the sign $\longleftrightarrow$ is used to indicate that the functions it connects are asymptotic approximations to the same function in different circumstances. Now, when x is small,

$$\int_0^{x'} \psi^{\frac{1}{2}} dx' = \tfrac{2}{3}\kappa^{\frac{1}{2}} x'^{\frac{3}{2}}$$

$$= \theta', \tag{14}$$

add therefore if we take L equal to $\int_{x_0}^{x} \psi^{\frac{1}{2}} dx$, L approximates to θ' when x' is small. Now our solutions must be linear functions of $\psi^{-\frac{1}{4}} \cos L$ and $\psi^{-\frac{1}{4}} \sin L$, and equations (12) and (13) show what the multipliers must be. Hence finally

$$\chi^{-\frac{1}{4}} \exp(M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(M_0) \cos(\tfrac{1}{4}\pi - L), \tag{15}$$

$$\chi^{-\frac{1}{4}} \exp(-M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(-M_0) \sin(\tfrac{1}{12}\pi + L). \tag{16}$$

These solutions satisfy the differential equation approximately when $\chi' \chi^{-\frac{3}{2}}$ is small. The process of identification by means of the first terms of the asymptotic expansions of the Bessel functions is valid provided θ_1 or θ' is great. But when x' is small

$$\chi' \chi^{-\frac{3}{2}} = \kappa^{-\frac{1}{2}} x^{-\frac{3}{2}} = 2/(3\theta'),$$

so that the conditions for the accuracy of the solutions of 1 . 2 and of the first term in the asymptotic expansions of the Bessel functions are equivalent.

If the comparison is carried out where x_1 or x' is equal to τ, the relative error in the Bessel functions as solutions is of order $\chi''\tau/\chi'$, while the relative error in the asymptotic approximations is of order $1/\theta$, where θ may denote either θ_1 or θ'. Thus the total error is at most of order

$$\frac{\chi''\tau}{\chi'} + \frac{1}{\chi'^{\frac{1}{2}}\tau^{\frac{3}{2}}}.$$

If we choose τ to make this a minimum, we see that the relative error is of order $\chi''^{\frac{3}{5}}/\chi'^{\frac{4}{5}}$.

1.5. This method may be used to obtain approximate values for many functions. In comparison with the method of obtaining asymptotic expansions by means of definite integrals, it has the advantage of rapid application in general, and of practicability in many cases where no definite integral solution can be found. On the other hand, it is not usually easy to fix rigorously the limits of the error involved in adopting it.

1.6. As an application of the method to a problem whose solution is already known, let us consider Bessel's equation, where the order is large. If our equation is

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - n^2)y = 0, \tag{1}$$

put

$$x = ne^{\xi}. \tag{2}$$

Then

$$\frac{d^2y}{d\xi^2} + n^2(e^{2\xi} - 1)y = 0 \tag{3}$$

When n is large, approximate solutions when ξ is negative are $(1-e^{2\xi})^{-\frac{1}{4}}e^{M}$ and $(1-e^{2\xi})^{-\frac{1}{4}}e^{-M}$, where

$$M = n\int^{\xi}(1-e^{2\xi})^{\frac{1}{2}}d\xi$$

$$= \int^{x}(n^2-x^2)^{\frac{1}{2}}\frac{dx}{x}. \tag{4}$$

Putting $x = n \operatorname{sech} \theta$, we have

$$M = -\int^{\theta} n \tanh^2 \theta d\theta$$

$$= -n(\theta - \tanh \theta), \tag{5}$$

whence $$e^{-M} = \left\{\frac{n}{x} + \left(\frac{n^2}{x^2} - 1\right)^{\frac{1}{2}}\right\}^n e^{-\sqrt{(n^2-x^2)}}, \tag{6}$$

and the two solutions are

$$(n^2-x^2)^{-\frac{1}{4}} x^n \{n + (n^2-x^2)^{\frac{1}{2}}\}^{-n} \exp \sqrt{(n^2-x^2)} \tag{7}$$

and $$(n^2-x^2)^{-\frac{1}{4}} x^{-n} \{n + (n^2-x^2)^{\frac{1}{2}}\}^{n} \exp \{-\sqrt{(n^2-x^2)}\}. \tag{8}$$

From the behaviour of these functions when x is small we see that the first must be a constant multiple of $J_n(x)$; the second may involve both J_n and J_{-n}.

When x is greater than n these solutions are seen from 1.4 (15) and (16) to merge into

$$2(e^{2\xi}-1)^{-\frac{1}{4}} \cos(\tfrac{1}{4}\pi - L) \quad \text{and} \quad 2(e^{2\xi}-1)^{-\frac{1}{4}} \sin(\tfrac{1}{4}\pi + L)$$

respectively. Here

$$L = n \int_{\xi_0}^{\xi} (e^{2\xi}-1)^{\frac{1}{2}} d\xi$$

$$= \int_n^x (x^2-n^2)^{\frac{1}{2}} \frac{dx}{x}. \tag{9}$$

Putting $$x = n \sec \zeta, \tag{10}$$

$$L = n \int \tan^2 \zeta d\zeta$$

$$= n(\tan \zeta - \zeta), \tag{11}$$

so that the first solution is continued approximately by

$$\frac{2}{(n \tan \zeta)^{\frac{1}{2}}} \cos\{\tfrac{1}{4}\pi - n(\tan \zeta - \zeta)\} \tag{12}$$

and the second by

$$\frac{2}{(n \tan \zeta)^{\frac{1}{2}}} \sin\{\tfrac{1}{4}\pi + n(\tan \zeta - \zeta)\} n. \tag{13}$$

The first of these can easily be seen to be reducible to the first term of the Debye expansion of this function. The second differs from the first term of the Debye expansion of J_{-n} by a portion arising from the multiple of J_n involved in it. If x/n is great, (12) reduces to

$$2x^{-\frac{1}{2}} \cos\left(\frac{2n+1}{4}\pi - x\right)$$

which is the first term, save for a constant factor, of the Stokes expansion.

1.7. Again, we may consider the equation

$$\frac{d^2y}{dx^2} + (x^2 - a^2)y = 0, \tag{1}$$

where a is real and positive. When x is less than a, the solutions are $(a^2-x^2)^{-\frac{1}{4}}e^M$ and $(a^2-x^2)^{-\frac{1}{4}}e^{-M}$, where

$$M = \int_0^x (a^2-x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}x(a^2-x^2)^{\frac{1}{2}} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}. \tag{2}$$

When x is greater than a, these pass into $2(x^2-a^2)^{-\frac{1}{4}}e^{\frac{1}{4}\pi a}\cos(\frac{1}{4}\pi - L)$ and $2(x^2-a^2)^{-\frac{1}{4}}e^{-\frac{1}{4}\pi a}\sin(\frac{1}{12}\pi + L)$, where

$$L = \int_a^x (x^2-a^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}x(x^2-a^2)^{\frac{1}{2}} - \tfrac{1}{2}a^2 \cosh^{-1}(x/a). \tag{3}$$

ON CERTAIN SOLUTIONS OF MATHIEU'S EQUATION

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2.1. Mathieu's equation is

$$\frac{d^2Y}{d\eta^2}+(R-2h^2\cos 2\eta)Y=0,$$

where R and h are independent of η. In general the solutions are required only for real values of the variables. For any value of h an infinite number of values of R give periodic solutions; these periodic solutions are the Mathieu functions.

Expressions for the solution for moderate values of h have been obtained by several writers, especially Mathieu* and R. C. Maclaurin.† It will be seen that in general there will be two solutions, one an odd and the other an even function of η; but when h is different from zero the two solutions will not both be periodic for the same value of R. Mathieu denotes the odd periodic solution by P_1, and the even one by P_2. Both are functions of η and of h. Evidently if h is zero, R must be equal to n^2, where n is an integer; the solutions then reduce to $\cos n\eta$ and $\sin n\eta$. The solution that reduces to $\cos n\eta$ is now usually denoted by $ce_n(\eta)$, and that which reduces to $\sin n\eta$ by $se_n(\eta)$. The corresponding values of R will here be called C_n and S_n.

In the following account Mathieu's functions will be discussed by the method of the paper preceding this, which will in general be described as Paper I. It is found that $ce_n(\eta)$ and $se_n(\eta)$ probably have the same number of zeros as $\cos n\eta$ and $\sin n\eta$; the effect of increasing h is to alter the positions of the zeros without affecting their number.

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* *Journal de Math.* (2), 13 (1868), 137–203.

† *Trans. Camb. Phil. Soc.*, 17 (1899), 41–108.

Mathieu expresses the solutions in series of cosines and sines of integral multiples of η. Since the solutions are by hypothesis periodic continuous functions of η, they will possess convergent Fourier expansions; thus Mathieu's series must converge. For utility in actual computation, however, it is neither necessary nor sufficient that the series should converge; what is required is that the first few terms shall give a good approximation to the function represented.* Now if h is large, the coefficients in Mathieu's series commence by increasing rapidly; the terms therefore increase from the start up to a term much greater than the early ones. Mathieu's series are therefore useless for computation if h is large.

2.2. *Solution of Mathieu's equation when h is large.*

Mathieu's equation will be written in the form

$$\frac{d^2Y}{d\eta^2}+(R-2h^2\cos 2\eta)Y=0, \tag{1}$$

where we shall write ψ for $R-2h^2\cos 2\eta$. Several cases may arise. If R is positive and greater than $2h^2$, ψ is always positive, and the solutions of oscillatory type will be valid for all real values of η. R must be determined, however, so that $\int_0^{2\pi}\psi^{\frac{1}{2}}d\eta$ is an integral multiple of 2π; otherwise the solutions will not be periodic. It is clear that an infinite number of values of R will satisfy this condition. It will be noticed that in this case the same value of R will fit both the even and odd solutions, it would only be in a closer approximation than has been obtained in Paper I that any difference would arise.

If R is negative and $|R|$ is greater than $2h^2$, ψ is always negative, and therefore, if $\chi=-\psi$, $\chi^{\frac{1}{2}}$ has always the same sign. Hence, as η increases, $\cosh\int\chi^{\frac{1}{2}}d\eta$ and $\sinh\int\chi^{\frac{1}{2}}d\eta$ increase steadily, and ultimately exceed any preassigned limit. They are not periodic, and therefore no suitable solution exists for negative values of R less than $-2h^2$.

If R is positive but less than $2h^2$, ψ is negative when $\eta=0$ or π, and positive when η is $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. It vanishes for four values of η between

* A simple example of the insufficiency of convergence as a condition for usefulness in computation is provided by the exponential expansion of e^{-1000}.

0 and 2π, one in each quadrant. It is a matter for investigation whether any solution is possible in these circumstances. The same is true if R is negative but greater than $-2h^2$.

The solutions found involve elliptic integrals; and a method of approximation due to Legendre will be applied to them.

2.3. Let
$$G = \int_0^\phi (A + B\sin^2\phi)\frac{d\phi}{\Delta}, \tag{1}$$

where $\Delta^2 = 1 - k^2\sin^2\phi$, k is less than 1, and A, B, k are constants. A convenient method of evaluating this integral numerically is given by Legendre, based on Landen's transformation. Here it will only be required for the case where the upper limit is nearly zero or $\frac{1}{2}\pi$. If the upper limit is $\frac{1}{2}\pi$,
$$G = \tfrac{1}{2}\pi K A_n, \tag{2}$$

where
$$K = (1+k_1)(1+k_2)\ldots(1+k_n), \tag{3}$$

$$A_n = A + \tfrac{1}{2}B(1 + \tfrac{1}{2}k_1 + \tfrac{1}{4}k_1k_2 + \ldots + \frac{1}{2^n}k_1k_2\ldots k_n), \tag{4}$$

and each k is connected with the preceding one by the relation
$$k_n = \frac{2k_{n+1}^{\frac{1}{2}}}{1+k_{n+1}}. \tag{5}$$

If $k_n = \sin\alpha_n$, $k_{n+1} = \tan^2\frac{1}{2}\alpha_n$. Thus the k's are all positive, and tend rapidly towards zero as n increases. If, for instance,

$k = 0{\cdot}9$, $\alpha_0 = 64°$, $k_1 = \frac{1}{3}$, $\alpha_1 = 20°$, $k_2 = \frac{1}{35}$, $\alpha_2 = 1°{\cdot}7$, $k_3 = \frac{1}{40000}$.

In (2), (3), and (4), n is to be taken great enough to make k_{n+1} insensible.

Evidently, if ϕ is small, we have
$$G = A\phi + \tfrac{1}{3}(B + \tfrac{1}{2}k^2)\phi^3 + \tfrac{1}{10}k^2B\phi^5 + \ldots, \tag{6}$$

and if $\phi = \frac{1}{2}\pi - \lambda$, where λ is small,
$$G = \tfrac{1}{2}\pi K A_n - \frac{1}{k'}\left\{(A+B)\lambda - \tfrac{1}{3}\left(B + \tfrac{1}{2}\frac{k^2}{k'^2}\right)\lambda^3 + \ldots\right\}, \tag{7}$$

where $k'^2 = 1 - k^2$.

2.4. *Solution of Mathieu's equation with h large and R greater than $2h^2$.*

In this case ψ is always positive, and the solutions of Paper I are applicable. In order that the solution may be periodic, however, $\int_0^{2\pi} \psi^{\frac{1}{2}} d\eta$ must be a multiple of 2π. Now if we write

$$4h^2 = (R+2h^2)k^2, \tag{1}$$

we have $$L = \int_0^\eta \psi^{\frac{1}{2}} d\eta = (R+2h^2)^{\frac{1}{2}} \int_0^\eta (1-k^2 \cos^2 \eta)^{\frac{1}{2}} d\eta. \tag{2}$$

If $$\eta = \tfrac{1}{2}\pi - \eta', \tag{3}$$

this becomes

$$= (R+2h^2)^{\frac{1}{2}} \left[\int_0^{\frac{1}{2}\pi} (1-k^2 \sin^2 \eta')^{\frac{1}{2}} d\eta' - \int_0^{\eta'} (1-k^2 \sin^2 \eta')^{\frac{1}{2}} d\eta' \right], \tag{4}$$

and these integrals are of the form just discussed, where A is put equal to 1 and B to $-k^2$. Hence the results of 2.3 can be applied.

If $\eta = 2\pi$, $\eta' = -\frac{3}{2}\pi$ at upper limit, and $\phi_n = -2^n \cdot \frac{3}{2}\pi$. Then

$$(R+2h^2)^{-\frac{1}{2}} \int_0^{2\pi} \psi^{\frac{1}{2}} d\eta = 2\pi K A_n. \tag{5}$$

If then $$\int_0^{2\pi} \psi^{\frac{1}{2}} d\eta = 2m\pi, \tag{6}$$

we have the following equation to determine R.

$$m = K A_n (R+2h^2)^{\frac{1}{2}}$$

$$= (1+k_1)(1+k_2)\ldots(1+k_n)$$

$$\times \left\{1 - \tfrac{1}{2}h^2 \left(1 + \tfrac{1}{2}k_1 + \tfrac{1}{4}k_1 k_2 + \ldots + \frac{1}{2^n} k_1 k_2 \ldots k_n\right)\right\} (R+2h^2)^{\frac{1}{2}} \tag{7}$$

$$= f(R), \quad \text{say.}$$

If h retains a fixed value, but $R \to \infty$, $k, k_1, \ldots$ all tend continuously to 0, and $f(R)$ is a continuous function of R, tending to infinity like $R^{\frac{1}{2}}$. Hence, however great m may be, greater than some definite value, it

will be possible to choose R so as to satisfy this equation. For very great values of m, R will be approximately equal to m^2.

The least value of R covered by this discussion is $2h^2$. Then $k = 1$, and the method of approximation breaks down. But we have, when the upper limit is $\frac{1}{2}\pi$.

$$L = (R+2h^2)^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} (1-k^2\cos^2\eta)^{\frac{1}{2}} d\eta, \tag{8}$$

and if $k^2 = 1-\alpha^2$ where α is small,

$$L = \frac{2h}{(1-\alpha^2)^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} (\sin^2\eta+\alpha^2\cos^2\eta)^{\frac{1}{2}} d\eta, \tag{9}$$

which is evidently an increasing function of α, and therefore of R. When $k = 1$, the integral becomes $2h$, and hence

$$f(R) = 4h/\pi. \tag{10}$$

At the same time $R = 2h^2$. Thus the lowest value of m that satisfies our conditions is the integer next greater than $4h/\pi$, and for any greater value of m there will be one and only one value of R that satisfies the conditions.

In this discussion $\psi^{\frac{1}{2}}$ has always been positive. Hence L is a continuously increasing function of η, and therefore, as η increases from 0 to 2π and L simultaneously increases from 0 to $2m\pi$, $\cos L$ and $\sin L$ will execute m complete oscillations. The intervals between consecutive zeros will evidently be approximately proportional to $\psi^{\frac{1}{2}}$, so that these intervals will be greatest when η is an even multiple, and least when η is an odd multiple, of $\frac{1}{2}\pi$.

The solutions of the equation are $\psi^{-\frac{1}{4}}\cos L$ and $\psi^{-\frac{1}{4}}\sin L$. Hence the extent of the oscillations is proportional to $\psi^{-\frac{1}{4}}$, and therefore is greatest when η is an even multiple, and least when η is an odd multiple, of $\frac{1}{2}\pi$.

2.5. *A property of periodic solutions of Mathieu's equation.*

It can be proved that any periodic solution of Mathieu's equation must be either an even or an odd function of η, and also either an even or an odd function of $\frac{1}{2}\pi-\eta$. For if $f(\eta)$ and $g(\eta)$ are the even and odd functions of η that satisfy the equation, then, since the equation is un-

altered when $\eta+\pi$ is written for η, $f(\eta+\pi)$ and $g(\eta+\pi)$ will also be solutions. They must be connected with $f(\eta)$ and $g(\eta)$ by relations

$$f(\eta+\pi) = Af(\eta)+Bg(\eta), \tag{1}$$

$$g(\eta+\pi) = Cf(\eta)+Dg(\eta). \tag{2}$$

But since $f(\eta)$ is an even function,

$$f(\eta+\pi) = f(-\eta-\pi), \tag{3}$$

and if $f(\eta)$ has period 2π,

$$\begin{aligned} f(-\eta-\pi) &= f(-\eta+\pi) \\ &= Af(-\eta)+Bg(-\eta) \\ &= Af(\eta)-Bg(\eta), \end{aligned} \tag{4}$$

and by comparison with (1) we see that $B = 0$. Further,

$$f(\eta+2\pi) = Af(\eta+\pi) = A^2 f(\eta), \tag{5}$$

whence $A = \pm 1$. Thus

$$f(\pi-\eta) = \pm f(\eta), \tag{6}$$

and $f(\eta)$ is either an odd or an even function of $\frac{1}{2}\pi-\eta$.

Similarly $g(\eta)$, if it is periodic, is either an even or an odd function of $\frac{1}{2}\pi-\eta$.

The above argument is applicable not only to Mathieu's equation, but also to any differential equation of the form

$$\frac{d^2Y}{d\eta^2}+\psi Y = 0,$$

where ψ is an even function of η with period π.

2.51. *Solution of Mathieu's equation when h is large and $|R|$ is less than $2h^2$.*

In this case ψ is negative when η is 0 or π, and positive when η is $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. It will vanish for some value η_0 in the first quadrant, and at corresponding points $\pi-\eta_0$, $\pi+\eta_0$, $2\pi-\eta_0$ in the others. Thus the

solution will be of hyperbolic type near 0 and π, and oscillatory around $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. It will be periodic if, when continued across the critical value η_0 by the method of 1 . 4, it is found to satisfy the condition of 2 . 5.

2 . 52. *Approximation valid from* $\eta = 0$ *to* η_0.

The even and odd solutions are approximately $\chi^{-\frac{1}{4}} \cosh M$ and $\chi^{-\frac{1}{4}} \sinh M$ where

$$M = \int_0^\eta \chi^{\frac{1}{2}} d\eta. \tag{1}$$

We have $$R = 2h^2 \cos 2\eta_0. \tag{2}$$

If we put $$\sin \eta_0 = k, \tag{3}$$

$$M = (2h^2 - R)^{\frac{1}{2}} \int_0^\theta \frac{k \cos^2 \theta d\theta}{(1 - k^2 \sin^2 \theta)^{\frac{1}{2}}}, \tag{4}$$

where $$\sin \eta = k \sin \theta. \tag{5}$$

We can approximate to this integral by Legendre's formula.

It is noticed at once, however, that M increases steadily with η, and must be of order h when η is near η_0. Hence the solutions increase exponentially, and must have large values near η_0 in comparison with their values near zero.

When η is nearly η_0, put

$$\eta = \eta_0 - \nu; \quad \theta = \tfrac{1}{2}\pi - \lambda, \tag{6}$$

where λ and ν are small, ν^2 being negligible, and are connected by the relation

$$\lambda^2 = 2\nu \cot \eta_0. \tag{7}$$

Applying the formula 2 . 3 (7), and remembering that

$$k_1 = \tan^2 \tfrac{1}{2}\eta_0, \tag{8}$$

we have approximately

$$M = \tfrac{1}{2}\pi \frac{2h^2 - R}{4h} (1 + k_1)(1 + k_2)(1 - \tfrac{1}{2}k_1) - \tfrac{1}{3} \frac{2h^2 - R}{(2h^2 + R)^{\frac{1}{2}}} \lambda^3 \tag{9}$$

$$= 2\pi h \sin^2 \tfrac{1}{2}\eta_0 (1 - \tfrac{1}{2} \tan^2 \tfrac{1}{2}\eta_0)(1 + k_2) - \tfrac{4}{3} h \nu^{\frac{3}{2}} \sin^{\frac{1}{2}} 2\eta_0 \tag{10}$$

$$= G - \tfrac{4}{3} h \sin^{\frac{1}{2}} 2\eta_0 . \nu^{\frac{3}{2}}, \text{ say.} \tag{11}$$

2.53. *Approximation valid from* $\eta = \eta_0$ to $\eta = \frac{1}{2}\pi$.

The solution is approximately $\psi^{-\frac{1}{4}} \cos L$ or $\psi^{-\frac{1}{4}} \sin L$, where

$$L = \int_{\eta}^{\frac{1}{2}\pi} (R - 2h^2 \cos 2\eta)^{\frac{1}{2}} d\eta. \tag{1}$$

Put
$$\cos\eta = \cos\eta_0 \sin\theta = k' \sin\theta, \tag{2}$$

$$(2h^2 + R)/2h^{\frac{1}{2}} = k'. \tag{3}$$

Then
$$L = \int_0^{\theta} 2h \cos^2\eta_0 \frac{\cos^2\theta\, d\theta}{(1 - k'^2 \sin^2\theta)^{\frac{1}{2}}}, \tag{4}$$

and if we write
$$\tfrac{1}{2}\pi - \eta_0 = \eta', \tag{5}$$

$$\theta = \tfrac{1}{2}\pi - \lambda', \tag{6}$$

$$\eta = \eta_0 + \nu', \tag{7}$$

where λ' and ν' are small, we have

$$\lambda'^2 = 2\nu' \tan\eta_0 = 2\nu' \cot\eta', \tag{8}$$

$$L = 2\pi h \sin^2 \tfrac{1}{2}\eta' (1 - \tfrac{1}{2}\tan^2 \tfrac{1}{2}\eta')(1 + k_2') - \tfrac{4}{3} h \sin^{\frac{1}{2}} 2\eta' \cdot \nu'^{\frac{3}{2}} \tag{9}$$

$$= G' - \tfrac{4}{3} h \sin^{\frac{1}{2}} 2\eta' \nu'^{\frac{3}{2}}, \tag{10}$$

say. In this equation k_2' denotes the value of k_2 obtained from 2.3 (5) corresponding to $k_1 = \tan^2 \frac{1}{2}\eta'$, and is not equal to $(1 - k_2^2)^{\frac{1}{2}}$.

2.54. *Identification of the two solutions.*

The two approximations so far obtained are valid over different ranges, namely, that where ψ is large and negative, and that where ψ is large and positive. They do not overlap, or even meet, at any point, and it is therefore necessary to employ the method of 1.4 to connect them together. We had

$$\chi^{-\frac{1}{4}} \exp(M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(M_0) \cos(\tfrac{1}{4}\pi - L),$$

$$\chi^{-\frac{1}{4}} \exp(-M) \longleftrightarrow 2\psi^{-\frac{1}{4}} \exp(-M_0) \sin(\tfrac{1}{12}\pi + L),$$

which therefore effect the required continuations.

2 . 55. *Determination of R.*

The even and odd solutions of Mathieu's equation valid when η is between $\pm\eta_0$ are respectively $\chi^{-\frac{1}{4}}\cosh M$ and $\chi^{-\frac{1}{4}}\sinh M$. By what has just been said, the continuation of the first of these is

$$\psi^{-\frac{1}{4}}\{e^G\cos(\tfrac{1}{4}\pi-L)+e^{-G}\sin(\tfrac{1}{12}\pi+L)\}.$$

The solution which is an even function of $\frac{1}{2}\pi-\eta$ is $A\psi^{-\frac{1}{4}}\cos(G'-L)$, where A is a constant. Putting

$$\tfrac{1}{4}\pi-L=N, \tag{1}$$

and equating coefficients of $\cos N$ and $\sin N$, we find

$$A\cos(G'-\tfrac{1}{4}\pi)=e^G+e^{-G}\sin\tfrac{1}{3}\pi, \tag{2}$$

$$A\sin(G'-\tfrac{1}{4}\pi)=e^{-G}\cos\tfrac{1}{3}\pi. \tag{3}$$

Hence
$$\cot(G'-\tfrac{1}{4}\pi)=2e^{2G}. \tag{4}$$

Thus if
$$G'=(m+\tfrac{1}{4})\pi+\mu, \tag{5}$$

where m is an integer and is small, we shall have

$$\mu=\tfrac{1}{2}e^{-2G}. \tag{6}$$

In general G will be great; then μ can be neglected.

If m be given in succession the values 0, 1, 2, ..., (5) determines G', and then the expression for G' in 2 . 53 (9) and (10) determines η' and hence R. With sufficient accuracy

$$(m+\tfrac{1}{4})\pi=2h\sin^2\tfrac{1}{2}\eta'(1-\tfrac{1}{2}\tan^2\tfrac{1}{2}\eta')(1+k_2'). \tag{7}$$

Now we had

$$G'=2h\cos^2\eta_0\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta\,d\theta}{(1-\cos^2\eta_0\sin^2\theta)^{\frac{1}{2}}}, \tag{8}$$

which is an increasing function of $\cos\eta_0$, and therefore is a maximum when η_0 is 0, giving

$$R=2h^2. \tag{9}$$

In this case
$$G'=2h. \tag{10}$$

When η_0 is $\frac{1}{2}\pi$, $G' = 0$ and $R = -2h^2$. (11)

Hence for every value of m between 0 and $2h/\pi$ there is one possible value of η_0 and one of R.

The number of complete oscillations executed by the solution as η goes from $\frac{1}{2}\pi$ to η_0 is the integral part of $G'/2\pi$ or $\frac{1}{2}m$. The interval between consecutive zeros is proportional to $\chi^{-\frac{1}{2}}$, and is therefore longest when η is nearly η_0, and shortest near $\frac{1}{2}\pi$. The amplitude of the oscillation is greatest near η_0. Between 0 and η_0 there are no oscillations; the solution rises steadily to a maximum near η_0.

The solution that is an odd function of η, but an even one of $\frac{1}{2}\pi - \eta$, may be treated similarly. The result will evidently be

$$G' = (m+\tfrac{1}{4})\pi - \mu, \tag{12}$$

where μ has the same value as before.

To find a solution that is an even function of η, but an odd one of $\frac{1}{2}\pi - \eta$, we have, as before,

$$A \sin(G'-x') = e^{G} \cos(\tfrac{1}{4}\pi - x') + e^{-G} \sin(\tfrac{1}{1\,2}\pi + x'). \tag{13}$$

Whence $$G' = (m+\tfrac{3}{4})\pi - \mu, \tag{14}$$

where m is an integer and μ has the same value as before.

The solutions that are odd functions of both η and $\frac{1}{2}\pi - \eta$ are given by

$$G' = (m+\tfrac{3}{4})\pi + \mu. \tag{15}$$

The above general remarks apply to all these solutions.

2.6. *Summary.*

The lowest value of R that gives a truly periodic solution of Mathieu's equation when h is large is rather greater than $-2h^2$. The corresponding solution has a sharp maximum at $\eta = \frac{1}{2}\pi$, never changes sign, and dies down steadily to an insignificant value when $\eta = 0$. The next possible value of R gives a solution that is even in $\frac{1}{2}\pi - \eta$, but odd in η, and performs a quarter oscillation in each quadrant. When R has greater values, the solutions between $\frac{1}{2}\pi$ and η_0 are oscillating functions whose amplitude

increases considerably near η_0. In passing from η_0 to 0 they die down steadily to an imperceptible amount. As R increases, η_0 tends to zero, and the solutions are oscillatory over a greater and greater range of values of η, the amplitude in each case being greatest near η_0. When R is nearly equal to $2h^2$, η_0 is nearly zero, and the number of oscillations performed in the quadrant is the integer nearest to h/π. In this case the amplitude is greatest when η is 0 or π, and least when it is $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. When R increases still further, the solution is oscillatory for all values of η, the amplitudes continuing to be greatest at 0 and π, and least at $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ All solutions are even or odd functions of η and of $\frac{1}{2}\pi - \eta$. The greater the value of R, the greater the number of oscillations as η goes from 0 to 2π, approximating to $R^{\frac{1}{2}}$ when R is great.

2.7. *Identification of the present forms with Mathieu's.*

The solutions of Mathieu's equation are functions both of η and of h, and it may be of interest to ascertain which of the solutions for large values of h will reduce to $ce_n(\eta)$ or $se_n(\eta)$, when h is small. The following argument suggests an answer, although it does not amount to a proof. Let us consider first the even solutions separately. They may, for any value of h, be ordered according to the values of R corresponding to them. Consider two different integers n and q, where q is the greater, with the corresponding solutions ce_n and ce_q, and the parameters C_n and C_q. Then, when h is small, C_q is greater than C_n. If h increases, C_n and C_q increase continuously. If possible, suppose that C_n ever becomes greater than C_q. Then there must be some value of h for which C_n and C_q are equal, and therefore $ce_n(\eta)$ and $ce_q(\eta)$ coalesce.

Now the condition that the solution of Mathieu's equation shall be periodic may be regarded as an equation whose roots are the possible values of R. If any two values of R become equal, they will become imaginary for greater values of h, and there will be no corresponding real solutions. But when R is great compared with $2h^2$, and the number of times that the solution changes from positive to negative as η increases from 0 to 2π is m, we have in the limit $R = m^2$. For every value of m there is a corresponding solution; and whatever h may be, m can be chosen great enough for the corresponding value of R to lie between $m^2 \pm 1$.

Thus for any range of values of h we can choose a number so great that the number of solutions for which R is less than this number is the same for all the values of h considered. Therefore no solutions can

cease to exist, unless whenever a pair disappear they are immediately replaced by a new pair that have hitherto been imaginary. Neglecting this alternative, it is probable that for all values of h the order of the solution is equal to m. Thus $m = n$. The same argument may be applied to the odd solutions.

It appears, therefore, that, when h increases, the number of zeros of any one solution as η increases from 0 to 2π does not vary. The effect of increasing h is not to alter the number of zeros and maxima, but to shift them towards the odd multiples of $\frac{1}{2}\pi$.

The above argument gives no indication of the ratio of the solutions of 2 . 3 to 2 . 5 to $ce_n(\eta)$ and $se_n(\eta)$ when h is large. This ratio is, of course, a function of h and n alone, and it will be convenient to apply the notation $ce_n(\eta)$ and $se_n(\eta)$ to these solutions as they stand, without regard to any possible future determination of the value of the ratio.

ON THE MODIFIED MATHIEU'S EQUATION

By Harold Jeffreys.

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3 . 1. The equation

$$\frac{d^2X}{d\xi^2}+(2h^2\cosh 2\xi-R)X=0$$

will here be called the "modified Mathieu's equation." It is obtained from the typical Mathieu equation already considered when $\iota\xi$ is substituted for η. R is usually so chosen that one solution of the typical equation is periodic. Then all the solutions of the latter are applicable to the modified equation, $\iota\xi$ being written for η in them, subject only to the condition that the approximations used in finding them are still valid in the modified equation. In accordance with the course already adopted, $R-2h^2\cosh 2\xi$ will be denoted by χ; when it is negative it will also be called $-\psi$. In general the solutions found will have to hold between 0 and α, where α is some positive constant.

In connexion with the typical equation we saw that R was necessarily algebraically greater than $-2h^2$, but could exceed any preassigned number, even if h was fixed beforehand. Further, if h remained constant, R was an increasing function of m, the number of times the sign of the solution changed from positive to negative as η increased from 0 to 2π; and when h was large enough, $R/2h^2$ tended to become a function of m alone.

3 . 2. In dealing with this equation it will often be necessary to make use of the integral

$$L=\int^{\xi}\psi^{\frac{1}{2}}d\xi.$$

The lower limit is to be zero if R is less than $2h^2$; if R is greater than $2h^2$, the lower limit is to be ξ_0, where

$$2h^2 \cosh 2\xi_0 = R.$$

3.21. Consider first the case where R is less than $2h^2$. In this case it can be shown* that

$$L = \int_0^\xi (2h^2 \cosh 2\xi - R)^{\frac{1}{2}} d\xi \tag{1}$$

$$= 2h \left\{ \Delta \tan \phi - \int_0^\phi \Delta d\phi + (1-k^2) \int_0^\phi \frac{d\phi}{\Delta} \right\} \tag{2}$$

where

$$2h^2 + R = 4h^2 k^2, \tag{3}$$

$$\tan \phi = (1-k^2)^{-\frac{1}{2}} \sinh \xi, \tag{4}$$

$$\Delta^2 = 1 - k^2 \sin^2 \phi. \tag{5}$$

3.22. If R is greater than $2h^2$, put

$$R = 2h^2 \cosh 2\xi_0, \tag{1}$$

$$\cosh \xi = \cosh \xi_0 \operatorname{cosec} \theta, \tag{2}$$

$$\operatorname{sech} \xi_0 = k, \tag{3}$$

$$1 - k^2 \sin^2 \theta = \Delta^2. \tag{4}$$

Then

$$L = \int_{\xi_0}^\xi (2h^2 \cosh 2\xi - R)^{\frac{1}{2}} d\xi \tag{5}$$

$$= 2h \cosh \xi_0 \left\{ \Delta \cot \theta - \int_0^{\frac{1}{2}\pi} \Delta d\theta \right\}. \tag{6}$$

The integrated part may be written $2h \tanh \xi (\cosh^2 \xi - \cosh^2 \xi_0)^{\frac{1}{2}}$. It evidently increases indefinitely with ξ. The part involving the integral cannot exceed $\pi h \cosh \xi_0$.

* Legendre, *Fonctions elliptiques*, 1 (1825), 16.

3 . 3. Let us consider now the modified equation with $-2h^2<R<2h^2$. Then if h is great we know that $\psi^{-\frac{1}{2}}\cos L$ and $\psi^{-\frac{1}{2}}\sin L$ are approximate solutions of the equation, the former being even and the latter odd. The first factor shows that the maxima decrease as ξ increases. The value of L is expressed by elliptic integrals in 3 . 21.

The condition satisfied by R is that which makes the solutions of the typical equation oscillatory when η is in the neighbourhood of the odd multiples of $\frac{1}{2}\pi$, but of hyperbolic type near the multiples of π.

3 . 31. If $R>2h^2\cosh 2\alpha$, χ is always positive. Hence $d^2X/d\xi^2$ and X always have the same sign. Accordingly, whether the solution considered is an even or an odd function of ξ, it is impossible for either the function or its first derivative to vanish for any value of ξ other than zero. In most cases one of these will be the condition to be satisfied at the outer boundary.

3 . 32. If $2h^2<R<2h^2\cosh 2\alpha$, χ will be positive when ξ is zero, and negative when ξ is equal to α, vanishing for some intermediate value of ξ, say ξ_0. The solution may be obtained by the method of Paper I. We have

$$\kappa = 4h^2\sinh 2\xi_0. \tag{1}$$

When ξ is less than ξ_0, the even and odd solutions of the equation are $\chi^{-\frac{1}{2}}\cosh M$ and $\chi^{-\frac{1}{2}}\sinh M$ respectively, where

$$M = 2h\int_0^{\xi}(\cosh^2\xi_0-\cosh^2\xi)^{\frac{1}{2}}d\xi. \tag{2}$$

When ξ is greater than ξ_0, these solutions are replaced by

$$\psi^{-\frac{1}{2}}\{\exp(M_0)\,.\,\cos(\tfrac{1}{4}\pi-L)+\exp(-M_0)\,.\,\sin(\tfrac{1}{1\,2}\pi+L)\} \tag{3}$$

and

$$\psi^{-\frac{1}{2}}\{\exp(M_0)\,.\,\cos(\tfrac{1}{4}\pi-L)-\exp(-M_0)\,.\,\sin(\tfrac{1}{1\,2}\pi+L)\}\,. \tag{4}$$

In general $\exp(-2M_0)$ will be very small, and the second terms can accordingly be dropped. Hence both solutions are approximately equivalent to $\exp(M_0)\,.\,\psi^{-\frac{1}{2}}\cos(\frac{1}{4}\pi-L)$, where L is to be found from 3 . 22.

3.4. Another asymptotic approximation sometimes applicable was found by Maclaurin. He puts

$$\cosh\xi = x, \quad 4h^2 = \lambda^2, \quad \lambda x = z, \quad R+2h^2 = p^2, \quad t = iz, \quad X = ue^t \tag{1}$$

where

$$u = t^m\left(1+\frac{a_1}{t}+\frac{a_2}{t^2}+\ldots\right). \tag{2}$$

He finds that m is equal to $-\frac{1}{2}$, and obtains the expansion term by term. [It may be remarked that in his equation connecting a_{n+1}, a_n, a_{n-1}, and a_{n-2}, $(2n-1)(2n+3)\lambda^2 a_{n-2}$ appears to be a misprint for

$$(2n-1)(2n-3)\lambda^2 a_{n-2}.]$$

When λ is small, p^2 remaining finite, the solution reduces to the asymptotic expansion of $J_p(z)$, which is valid when $z > \frac{1}{4}\pi(2p+3)$. Maclaurin's expansion is then useful if h is a small number and

$$2h\cosh\xi > \tfrac{1}{2}\pi(2p+3).$$

3.5. An alternative asymptotic solution may be found by putting

$$he^{\xi} = \theta. \tag{1}$$

The differential equation becomes

$$\theta\frac{d}{d\theta}\left(\theta\frac{dX}{d\theta}\right)+(\theta^2+h^4\theta^{-2}-R)X = 0. \tag{2}$$

Apart from the term in $h^4\theta^{-2}$ this is a Bessel equation. If we put

$$i\theta = t, \quad X = ue^t, \tag{3}$$

$$u = t^m\left(1+\frac{a_1}{t}+\frac{a_2}{t^2}+\ldots+\frac{a_n}{t^n}+\ldots\right), \tag{4}$$

we find

$$t^2\frac{d^2u}{dt^2}+(2t^2+t)\frac{du}{dt}+\left(t^2-\frac{h^4}{t^2}-R\right)u = 0, \tag{5}$$

$$m = -\tfrac{1}{2},$$

and in general

$$2(n+1)a_{n+1} = \{(n+\tfrac{1}{2})^2-R\}a_n-h^4a_{n-2}. \tag{6}$$

Evaluating the coefficients, we have

$$a_1 = \tfrac{1}{2}(\tfrac{1}{4}-R), \tag{7}$$

$$a_2 = \frac{1}{2.4}(\tfrac{1}{4}-R)(\tfrac{9}{4}-R), \tag{8}$$

$$a_3 = \frac{1}{2.4.6}(\tfrac{1}{4}-R)(\tfrac{9}{4}-R)(\tfrac{25}{4}-R)-\tfrac{1}{6}h^4, \tag{9}$$

$$a_4 = \frac{1}{2.4.6.8}(\tfrac{1}{4}-R)(\tfrac{9}{4}-R)(\tfrac{25}{4}-R)(\tfrac{49}{4}-R)-\tfrac{1}{48}h^4(13-4R). \tag{10}$$

The advantages of this expansion over Maclaurin's are seen in the greater compactness of the expressions for the coefficients, with a corresponding diminution in the labour of computation. In addition, it proceeds by powers of h^4 instead of h^2, and therefore approximates much more rapidly when h is small; and even if h is about 2, the smaller coefficients of its powers in this expansion make it more convenient. Its range of validity is determined by the conditions that 8θ must be greater than $4R$, and that $h^4\theta^{-4}$ shall be small; in other words, $he^{\xi} > \frac{1}{2}R$, $e^{-4\xi}$ is small.

4.1. If α is small and $2h^2-R$ is large compared with $4h^2\sinh^2\alpha$, ψ will always be positive, and its variation will be small compared with its whole amount. Hence to a first approximation we may treat it as a constant, and the solutions will be $\cos(2h\sin\eta_0 . \xi)$ and $\sin(2h\sin\eta_0 . \xi)$.

4.2. If α is small, R cannot considerably exceed $2h^2$, but $2h^2-R$ may be small and either positive or negative. If, then, h is moderate, χ will be small, and the variation in X from the centre to the boundary must be small. In this case we may solve by continued approximation, whether $R-2h^2$ is moderate or small, positive or negative.

Thus

$$\psi = -(R-2h^2\cosh 2\xi) = 4h^2(\cosh 2\xi-1-S) \tag{1}$$

where

$$S = (R-2h^2)/4h^2. \tag{2}$$

If R is less than $2h^2$, S is equal to $\sin^2\eta_0$

Then

$$\psi = 8h^2(\xi^2+\tfrac{1}{3}\xi^4+\tfrac{2}{45}\xi^6-\tfrac{1}{2}S), \tag{3}$$

if ξ^8 be neglected.

If Q denote the operation of integrating from 0 to ξ, solutions of this equation are

$$X_1 = 1 - Q^2\psi + Q^2\psi Q^2\psi - Q^2\psi Q^2\psi Q^2\psi + \ldots, \tag{4}$$

$$X_2 = \xi - Q^2\psi\xi + Q^2\psi Q^2\psi\xi - Q^2\psi Q^2\psi Q^2\psi\xi + \ldots. \tag{5}$$

To order ξ^9 we have

$$\left.\begin{aligned} X_1 = 1 &- \tfrac{2}{3}h^2(\xi^4 + \tfrac{2}{15}\xi^6 + \tfrac{1}{105}\xi^8 - 3S\xi^2) \\ &+ \tfrac{2}{3}h^4(\tfrac{1}{7}\xi^8 - \tfrac{14}{15}S\xi^6 - \tfrac{16}{105}S\xi^8 + S^2\xi^4) \\ &- \tfrac{4}{45}h^6S^2(\tfrac{11}{7}\xi^8 - S\xi^6) \\ &+ \tfrac{2}{315}h^8S^4\xi^8 \end{aligned}\right\}, \tag{6}$$

$$\left.\begin{aligned} X_2 = \xi &- \tfrac{2}{3}h^2(\tfrac{3}{5}\xi^5 + \tfrac{2}{21}\xi^7 + \tfrac{1}{135}\xi^9 - S^2\xi^5) \\ &+ \tfrac{2}{15}h^4(\tfrac{1}{3}\xi^9 - \tfrac{26}{21}S\xi^7 - \tfrac{40}{189}S\xi^9 + S^2\xi^5) \\ &- \tfrac{4}{315}h^6S^2(\tfrac{17}{9}\xi^9 - S\xi^7) \\ &+ \tfrac{2}{2835}h^8S^4\xi^9 \end{aligned}\right\}, \tag{7}$$

4 . 3. If α is small but h large, and $2h^2 - R$ small compared with $4h^2$, the last approximation will be satisfactory provided $h^2S\alpha^2$ and $h^2\alpha^4$ are small or moderate. If the former is great, but not the latter, the approximation of 4 . 1 will succeed. If the latter is great, it will be necessary to employ a method similar to those suitable for great values of α.

THE FREE OSCILLATIONS OF WATER IN AN ELLIPTICAL LAKE

By HAROLD JEFFREYS.

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5 . 1. Let the horizontal coordinates of any point, referred to the major and minor axes of the lake, be x and y; u and v the corresponding components of velocity; ζ the elevation of the surface above its equilibrium position; ω the component about the vertical of the earth's angular velocity of rotation; D the depth of the lake, supposed uniform; g the acceleration of gravity; t the time.

If the displacements are small enough for their squares to be neglected, we can prove that

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\sigma^2-4\omega^2}{gD}\zeta=0 \tag{1}$$

and that

$$u=\frac{g}{\sigma^2-4\omega^2}\left(i\sigma\frac{\partial\zeta}{\partial x}+2\omega\frac{\partial\zeta}{\partial y}\right), \tag{2}$$

$$v=\frac{g}{\sigma^2-4\omega^2}\left(i\sigma\frac{\partial\zeta}{\partial y}-2\omega\frac{\partial\zeta}{\partial x}\right). \tag{3}$$

If ds and dn denote elements of length, such that ds is at right angles to dn and in the positive sense from it, and if u_1 and v_1 denote velocity components in the directions of dn and ds, we shall have, by resolution of (2) and (3) in these directions,

$$u_1=\frac{g}{\sigma^2-4\omega^2}\left(i\sigma\frac{\partial\zeta}{\partial n}+2\omega\frac{\partial\zeta}{\partial s}\right), \tag{4}$$

$$v_1=\frac{g}{\sigma^2-4\omega^2}\left(i\sigma\frac{\partial\zeta}{\partial s}-2\omega\frac{\partial\zeta}{\partial n}\right). \tag{5}$$

Now let us put

$$x = c \cosh \xi \cos \eta, \tag{6}$$

$$y = c \sinh \xi \sin \eta, \tag{7}$$

where c is a constant, and ξ, η new variables. Let us also put

$$\frac{\sigma^2 - 4\omega^2}{gD} = \frac{4h^2}{c^2}. \tag{8}$$

Then (4) becomes

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} + 2h^2(\cosh 2\xi - \cos 2\eta)\zeta = 0. \tag{9}$$

If again we suppose ζ to be of the form XY, where X is a function of ξ only and Y one of η only, this equation gives

$$\frac{1}{X}\frac{d^2X}{d\xi^2} + 2h^2 \cosh 2\xi = -\frac{1}{Y}\frac{d^2Y}{d\eta^2} + 2h^2 \cos 2\eta \tag{10}$$

$$= R,$$

where R is independent of both ξ and η. Hence

$$\frac{d^2X}{d\xi^2} + (2h^2 \cosh 2\xi - R)X = 0, \tag{11}$$

$$\frac{d^2Y}{d\eta^2} + (R - 2h^2 \cos 2\eta)Y = 0. \tag{12}$$

The general solution of (9) will be expressible as a linear function of solutions of the form XY, where X and Y respectively satisfy equations (11) and (12). Equation (12) is Mathieu's equation; (11) has here been called the modified Mathieu's equation.

5 . 11. The boundary conditions are obtained as follows. If $2c$ be the distance between the foci of the ellipse, the curves $\xi =$ constant are confocal ellipses, and $\eta =$ constant are confocal hyperbolas. The curve $\xi = 0$ reduces to the two sides of the line joining the foci. The tide height and the velocity components must all be continuous across this line. Now for any value of R, 5 . 1 (11) possesses an even and an odd solution in ξ; call these X_0 and X_1. Similarly 5 . 1 (12) possesses an even and an odd solution in η; call these Y_0 and Y_1. Then ζ may contain constituents of the form $AX_0Y_0 + BX_1Y_0 + CX_0Y_1 + DX_1Y_1$.

5 . 12. When ξ and y are small, we have, in the limit,

$$\frac{\partial}{\partial x} = -\frac{1}{c \sin \eta} \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y} = \frac{1}{c \sin \eta} \frac{\partial}{\partial \xi}. \tag{1}$$

Now when we cross the line joining the foci, η changes to $2\pi - \eta$, and therefore $\sin \eta$ changes its sign. But u, v, $\partial\xi/\partial x$, and $\partial\xi/\partial y$ cannot change sign, unless they vanish, and their continuity and equations (1) therefore show that $\partial\xi/\partial\xi$ and $\partial\xi/\partial\eta$ must change sign.

Now
$$\frac{\partial \xi}{\partial \xi} = AX_0'Y_0 + BX_1'Y_0 + CX_0'Y_1 + DX_1'Y_1. \tag{2}$$

When ξ is zero the two terms in X_0' vanish, since X_0' is an odd function. Y_0 is unaltered when $2\pi - \eta$ is put for η, and X_1' does not vanish; hence B is zero.

Again,
$$\frac{\partial \xi}{\partial \eta} = AX_0Y_0' + BX_1Y_0' + CX_0Y_1' + DX_1Y_1'. \tag{3}$$

When ξ is zero the terms in X_1 vanish. Y_1' does not change when $2\pi - \eta$ is put for η and X_0 does not vanish; hence C is zero.

The continuity of ξ requires again that C shall be zero. Thus any solution of 5 . 1 (9) must be either even in both ξ and η, or odd in both ξ and η.

These results hold completely if ω is zero as well as when it is finite.

5 . 13. The condition at the outer boundary is that the normal velocity is zero. This shows that $i\sigma(\partial\xi/\partial n) + 2\omega(\partial\xi/\partial s)$ vanishes on the boundary, if dn be measured normally outwards and ds along the boundary in the direction of η increasing. In these circumstances

$$\frac{dn}{d\xi} = \frac{ds}{d\eta} = c(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}. \tag{1}$$

Hence
$$i\sigma \frac{\partial \xi}{\partial \xi} + 2\omega \frac{\partial \xi}{\partial \eta} = 0 \tag{2}$$

on the boundary. If ω is zero, this implies that

$$\frac{dX}{d\xi} = 0, \tag{3}$$

for every solution of 5 . 1 (11) separately.

Let the value of ξ on the boundary be α. Then the major semi-axis is $c \cosh \alpha$, the minor semi-axis $c \sinh \alpha$, and the eccentricity $\operatorname{sech} \alpha$.

5 . 14. We notice that, for every solution, if ω is zero, the lines $X = 0$ give nodal ellipses, and the lines $Y = 0$ nodal hyperbolas. The number of nodal ellipses between the line of foci and the boundary will be called the class of the solution, and the number of times the elevation changes from positive to negative as η increases from 0 to 2π will be called the order of the solution. Evidently there is no solution of class zero and order zero, for it would violate the equation of continuity.

If the ellipse degenerates into a circle, the nodal ellipses become circles, and the nodal hyperbolas become radii.

Henceforward only the case of no rotation will be considered.

6. *Lake of small eccentricity.*

No method of solution is applicable to every mode of vibration of an approximately circular lake. When the lake is exactly circular, the function X reduces to $J_m(\kappa r)$, and the boundary condition is $J'_m(\kappa a) = 0$, corresponding to the condition in the present problem $X'(a) = 0$. A natural plan would therefore be to assume $J_m(\theta)$ as a first approximation to the solution in the elliptic case, and to construct closer approximations with this as a basis. The correction, however, is not in a convenient form for computation, and the method has not been pursued.

6 . 1. An alternative is to use the asymptotic solution of 3 . 5. We had

$$\theta = he^{\xi}, \tag{1}$$

and if the value of ξ on the outer boundary is a, we shall write

$$f = e^{-a}. \tag{2}$$

Then f will be a small quantity, and

$$h = f\theta_0, \tag{3}$$

where θ_0 is the value of θ on the outer boundary. Now the conditions for the utility of the approximation of 3 . 5 were that $R/2\theta_0$ and $h^4\theta_0^{-4}$ should be small.

If h is moderate, R will not differ greatly from m^2, and θ_0 will be approximately equal to the corresponding zero of $J'_m(\theta)$. The first condition is fulfilled only by the larger zeros of this function; thus the method gives only the modes of high class. For any one lake $h^4\theta_0^{-4} = f^4$ and is an absolute constant. This imposes a definite limitation on the accuracy of the solution.

If h is great the approximations of 3 . 2 or 3 . 3 will be applicable.

6 . 2. The most convenient way of determining the slowest modes appears to be by adapting Mathieu's solutions of the typical equation mentioned in 2 . 1. If in $ce_n(\eta)$ or $se_n(\eta)$ we replace η by $i\xi$, we obtain at once the corresponding solution of the modified equation with the same values of h and R; further, our physical conditions given in 5 . 12 require that the even solution of the ξ equation shall be associated with the even solution of the η equation, and the odd solution with the odd one. Hence these solutions are those required. It will be convenient to denote $ce_n(i\xi)$ by $Ce_n(\xi)$, and $se_n(i\xi)$ by $iSe_n(\xi)$. Then $Ce_n(\xi)$ and $Se_n(\xi)$ are series of hyperbolic functions of multiples of ξ.

6 . 21. Consider first the mode in which the elevation of the water is proportional to $Ce_1(\xi)\, ce_1(\eta)$. This mode is symmetrical about the major axis; the only nodal hyperbola is the minor axis, with regard to which the surface is antisymmetrical. We have

$$\begin{aligned} 2Ce_1'(\xi) = {} & e^{\xi}-e^{-\xi}-\frac{3h^2}{8}(e^{3\xi}-e^{-3\xi})+\frac{5h^4}{192}(e^{5\xi}-e^{-5\xi})-\frac{3h^4}{64}(e^{3\xi}-e^{-3\xi}) \\ & -\frac{7h^6}{9216}e^{7\xi}+\frac{5h^6}{1152}(e^{5\xi}-e^{-5\xi})-\frac{3h^6}{1536}(e^{3\xi}-e^{-3\xi}) \\ & +\frac{9h^8}{737280}e^{9\xi}-\frac{7h^8}{49152}e^{7\xi}+\frac{5h^8}{24576}(e^{5\xi}-e^{-5\xi})+\frac{33}{36864}(e^{3\xi}-e^{-3\xi}) \end{aligned} \tag{1}$$

where $e^{-7\xi}$ has been neglected.

When $$\xi = \alpha, \quad he^{\xi} = \theta_0, \quad e^{-\alpha} = f, \quad h = \theta_0 e^{-\alpha}. \tag{2}$$

Substituting these values and retaining f^8, we have

$$\begin{aligned} 2hCe_1'(\alpha) = {} & \theta_0(1-f^2)-\frac{3}{8}\theta_0^3(1-f^6)+\theta_0^5\left\{\frac{5}{192}-\frac{3}{64}(f^2-f^8)\right\} \\ & -\theta_0^7\left(\frac{7}{9216}-\frac{5}{1152}f^2+\frac{1}{512}f^4\right) \\ & +\theta_0^9\left(\frac{1}{81920}-\frac{7}{49152}f^2+\frac{5}{24576}f^4+\frac{11}{12288}f^6\right). \end{aligned} \tag{3}$$

Take $f = \frac{1}{2}$. If a and b are the major and minor semi-axes of the ellipse,

$$\frac{b}{a} = \tanh\alpha = \frac{1-f^2}{1+f^2} = \frac{3}{5} \tag{4}$$

and
$$\frac{a}{c} = \cosh a = \tfrac{1}{2}(f^{-1}+f) = \frac{5}{4}. \tag{5}$$

Then

$$2hCe_1'(a) = \frac{3}{4}\theta_0 - \frac{3}{8}\cdot\frac{63}{64}\theta_0^3 + \theta_0^5\left(\frac{5}{192} - \frac{3}{64}\cdot\frac{1}{4}\cdot\frac{63}{64}\right)$$
$$-\theta_0^7\left(\frac{7}{9216} - \frac{5}{4608} + \frac{1}{8192}\right)$$
$$+\theta_0^9\left(\frac{1}{81920} - \frac{1}{28000} + \frac{5}{400000} + \frac{11}{780000}\right)$$
$$= 0{\cdot}7500\theta_0 - 0{\cdot}3692\theta_0^3 + 0{\cdot}0144\theta_0^5 + 0{\cdot}000257\theta_0^7 - 0{\cdot}0000025\theta_0^9. \tag{6}$$

Putting this equal to zero, we have a quartic equation for θ_0^2, whose lowest root is
$$\theta_0^2 = 2{\cdot}23. \tag{7}$$

Hence
$$h = f\theta_0 = 0{\cdot}748. \tag{8}$$

If V be the velocity of a free wave in water whose depth is equal to that of the lake, we have
$$\sigma = \frac{2hV}{c} = \frac{5}{2}\frac{hV}{a} = 1{\cdot}87\,\frac{V}{a}. \tag{9}$$

6 . 211. In the case of the corresponding mode of a circular lake of radius a, the period equation is
$$J_1'\left(\frac{\sigma a}{V}\right) = 0,$$

whence
$$\sigma = 1{\cdot}84\,\frac{V}{a}.$$

Thus the considerable eccentricity indicated by the ratio of the axes being 0·6 causes the slowest period of the lake to differ by only 2 per cent. from that of the corresponding mode in a circle of the same major axis.

6 . 22. Consider next the mode in which the elevation of the surface is proportional to $Se_1(\xi)\,se_1(\eta)$. This mode is antisymmetrical about the major axis and symmetrical about the minor axis. It consists of an

oscillation of the water from side to side of the lake, the mode considered in 6 . 21 being an oscillation from end to end. We have

$$2Se_1'(\xi) = e^{\xi}+e^{-\xi}-\frac{3}{8}(e^{3\xi}+e^{-3\xi})+\frac{5}{192}h^4e^{5\xi}+\frac{3h^4}{64}(e^{3\xi}+e^{-3\xi})$$
$$-\frac{7h^6}{9216}e^{7\xi}-\frac{5h^6}{1152}e^{5\xi}-\frac{3h^6}{1536}(e^{3\xi}+e^{-3\xi})$$
$$+h^8\left(\frac{9}{737280}e^{9\xi}+\frac{7}{49152}e^{7\xi}+\frac{5}{24576}e^{5\xi}-\frac{33}{36864}e^{3\xi}\right) \quad (1)$$

to our previous order. Hence the period equation is

$$0 = \theta_0(1+f^2)-\frac{3}{8}\theta_0^3(1+f^6)+\theta_0^5\left\{\frac{5}{192}+\frac{3}{64}f^2(1+f^6)\right\}$$
$$-\theta_0^7\left(\frac{7}{9216}+\frac{5f^2}{1152}+\frac{1}{512}f^4\right)$$
$$+\theta_0^9\left(\frac{1}{81920}+\frac{7f^2}{49152}+\frac{5f^4}{24576}-\frac{11}{12288}f^6\right) \quad (2)$$

and if $f = \frac{1}{2}$, this becomes

$$0 = 1{\cdot}2500\theta_0-0{\cdot}3808\theta_0^3+0{\cdot}0376\theta_0^5-0{\cdot}001913\theta_0^7+0{\cdot}000046\theta_0^9. \quad (3)$$

The lowest root of this is given by

$$\theta_0^2 = 5{\cdot}66, \quad (4)$$

whence

$$h = f\theta_0 = 1{\cdot}19, \quad (5)$$

$$\sigma = \frac{5}{2}\frac{hV}{a} = 2{\cdot}97\frac{V}{a} = 1{\cdot}78\frac{V}{b}. \quad (6)$$

It is seen that this mode is decidedly faster than the one last considered, as might have been anticipated from the shorter distance the wave has to travel in each half-period. The period is about 3 per cent. longer than the slowest period in a circle whose radius is equal to the minor semi-axis of the ellipse.

6 . 23. It appears that the modes just considered are the only ones such that the adapted Mathieu solutions afford a practicable method of calculating the periods. Attempts have been made with the modes

depending on $Ce_2(\xi)\,ce_2(\eta)$ and $Ce_0(\xi)\,ce_0(\eta)$; but the first three terms of the former indicate a value of θ_0^2 of about 12, and with this value the later terms show no sign of converging at the stage to which they have been calculated by Mathieu. The latter is still slower in converging. This is not surprising, since it is necessarily a mode of class at least 1, whereas the others are of class zero.

7. *Lake of great eccentricity.*

7.1. In this section it will be supposed that α is small enough to make the expansion of $\cosh 2\xi$ as far as ξ^6 sufficiently accurate at all points of the region. Then the approximations of 4.2 will be sufficiently close. If we adopt the even solution, the condition that $dX/d\xi$ shall be zero when ξ is equal to α gives

$$-\left(2\alpha^2+\frac{2}{5}\,\alpha^4+\frac{4}{105}\,\alpha^6-3S\right)$$

$$+h^2\left(\frac{4}{7}\,\alpha^6-\frac{14}{5}\,S\alpha^4-\frac{64}{105}\,S\alpha^6+2S^2\alpha^2\right)$$

$$-\frac{2}{15}\,h^4S^2\left(\frac{44}{7}\,\alpha^6-3S\alpha^4\right)+\frac{4}{105}\,h^6S^4\alpha^6=0. \qquad (1)$$

So long as h is not so great as to make $h^2\alpha^4$ considerable, this shows that one value of S, and only one, must be small; and this will be approximately equal to $\frac{2}{3}\alpha^2$. Now we had

$$S=(R-2h^2)/4h^2. \qquad (2)$$

But R is known as a function of h and m, and therefore the equation

$$R-2h^2=\frac{8}{3}\,h^2\alpha^2 \qquad (3)$$

is our period equation. The smaller α is, the larger h may be without this approximation failing. The smallness and positive sign of S indicate that the waves reach the ends of the lake, and that the amplitudes are greater there than elsewhere.

The value of X is unity on the line joining the foci, while equation 4.2 (6) shows that on the boundary it can only differ from unity by a quantity of order $h^2\alpha^4$, which by our conditions is small. Hence the elevation is almost the same right across the lake at any section along a

confocal hyperbola, and there is no nodal ellipse. These modes are therefore of class zero. They correspond to those modes of vibration of a canal in which the elevation is almost the same right across the canal at any section; in other words, to the longitudinal modes. Hence if we start with a circular lake, and let one axis shorten gradually, the lake remaining always an ellipse, until it is reduced to a straight canal of variable width, the major axis always remaining parallel to a fixed diameter of the original circle, the modes of the circular lake symmetrical about this diameter, and with no nodal circle, will gradually pass through the modes of zero class of the ellipses and ultimately become the longitudinal modes of the canal. The section along a nodal hyperbola is concave upwards near the line of foci, but has an inflexion where χ is zero, or $\xi = a\sqrt{\frac{2}{3}}$, and for greater values of ξ is convex upwards, becoming flat again when $\xi = a$. Thus these modes have what may be called an ellipse of inflexion.

7.11. We may consider the slowest longitudinal mode of a very narrow ellipse, so that our period equation is

$$R = 2h^2, \tag{1}$$

where the order is unity. This equation, as far as h^{10}, is

$$1 - h^2 - \frac{1}{8}h^4 - \frac{1}{64}h^6 - \frac{1}{1536}h^8 + \frac{11}{36864}h^{10} = 0. \tag{2}$$

The lowest root of this is found by Horner's method to be given by

$$10h^2 = 8{\cdot}89,$$

whence

$$h = 0{\cdot}943. \tag{3}$$

In the case of a straight canal the major axis $2a$ is equal to the distance $2c$ between the foci. Hence

$$\sigma = \frac{2hV}{c} = \frac{2hV}{a} = 1{\cdot}886\,\frac{V}{a}. \tag{4}$$

This may be compared with the speeds $1{\cdot}84\ V/a$ for the corresponding mode of a circle, and $1{\cdot}87\ V/a$ for an ellipse whose axes are in the ratio 3/5. Thus the period varies by only a few per cent. in the whole transition from a circle to a straight canal.

The amplitude of the vibration at any point is proportional to

$$ce_1(\eta) = \cos\eta - \frac{1}{8}h^2\cos 3\eta + \frac{1}{64}h^4\left(\frac{1}{3}\cos 5\eta - \cos 3\eta\right) - \ldots$$

$$= \cos\eta - \frac{1}{9}\cos 3\eta + \frac{1}{81}\left(\frac{1}{3}\cos 5\eta - \cos 3\eta\right) - \ldots. \quad (5)$$

In a uniform canal the amplitude of the slowest mode at distance x from the middle is proportional to $\sin x/a$. If we put x/a equal to $\cos\eta$ in this case we shall have

$$\xi \propto \sin(\cos\eta) = 2J_1(1)\cos\eta - 2J_3(1)\cos 3\eta + 2J_5(1)\cos 5\eta - \ldots$$

$$= 0{\cdot}880\cos\eta - 0{\cdot}039\cos 3\eta + 0{\cdot}0005\cos 5\eta - \ldots. \quad (6)$$

A comparison with (5) shows that the chief difference between the uniform and elliptic canals rests in the distinctly larger coefficient of $\cos 3\eta$ in the elliptic case, leading to a fall in the amplitude as we recede from the end, and then an increase again above the corresponding value in the case of the uniform canal before the centre is reached.

7 . 12. The Mathieu expansions converge too slowly for any other mode to be calculated by this method. The above mode and the lowest symmetrical mode have, however, been calculated by a method specially constructed for a canal of elliptic plan. If u denote the velocity along the canal, $\kappa = \sigma/V$, and b is the breadth at any point, the velocity satisfies the equation

$$\frac{\partial}{\partial x}\frac{1}{b}\frac{\partial}{\partial x}(bu) + \kappa^2 u = 0, \quad (1)$$

or, if we write $bu = U$, $x = a\cos\eta$, $b = 2B\sin\eta$, where B is now a constant :

$$0 = \kappa^2 U + b\frac{\partial}{\partial x}\frac{1}{b}\frac{\partial U}{\partial x} \quad (2)$$

$$= \kappa^2 U + \frac{1}{a^2}\frac{\partial}{\partial\eta}\operatorname{cosec}^2\eta\frac{\partial U}{\partial\eta}. \quad (3)$$

Let us write $$\lambda = a^2\kappa^2. \quad (4)$$

The equation (3) can be solved by the method of Baker* and Cowley and Levy.

* H. F. Baker, *Phil. Trans. Royal Soc.*, 216 A (1916), 129–186. W. L. Cowley and H. Levy, *Phil. Mag.*, 41 (1921), 584–607.

If Q denote the operation of integrating with regard to η from 0 to η, the series

$$U = U_0 - Q\sin^2\eta Q\lambda U_0 + Q\sin^2\eta Q\lambda Q\sin^2\eta Q\lambda U_0 - \ldots \quad (5)$$

is a formal solution of the equation, provided that

$$\frac{d}{d\eta}\operatorname{cosec}^2\eta\frac{dU_0}{d\eta} = 0, \quad (6)$$

whence U_0 is a linear function of 1 and $Q\sin^2\eta$.

Now the velocity must not be infinite at the ends, and therefore the only solution admissible is derived from

$$U_0 = Q\sin^2\eta. \quad (7)$$

Put $\phi = 2\eta/\pi$, so that ϕ is unity when $\eta = \frac{1}{2}\pi$, and write

$$RX = \int_0^\phi X d\phi = \frac{2}{\pi}QX. \quad (8)$$

Put also

$$-\frac{\pi^2\lambda}{16} = \mu, \quad (9)$$

$$2\operatorname{versin}2\eta = M, \quad (10)$$

$$U = RM + \mu RMR^2M + \mu^2 RMR^2MR^2M + \ldots \quad (11)$$

$$= U_0 + \mu U_1 + \mu^2 U_2 + \ldots, \quad (12)$$

a constant factor having been applied.

This is in a form suitable for direct numerical integration, which has been carried out by dividing the range of integration into ten equal intervals and always treating the average value of the integrand in any interval as equal to the mean of the extreme values. This method leads to some systematic error, but this is evidently relatively smallest when ϕ is nearly unity.

If U vanishes when ϕ is 1, or $\eta = \frac{1}{2}\pi$, the velocity vanishes in the centre, and we have a symmetrical mode; in other words, a mode of even order. If ζ vanishes when $\eta = \frac{1}{2}\pi$, we have an asymmetrical or odd mode. Now the amplitude of ζ is proportional to $d(bu)/dx$. Hence the odd modes are determined by the vanishing of $R^{-1}U$ when ϕ is unity.

The following scheme indicates the process of integration :—

ϕ	M	U_0	RU_0	MRU_0	$U_1 = RMRU_0$	RU_1	MRU_1	$RMRU_1 = U_2$	RU_2	MRU_2	$RMRU_2 = U_3$
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0	0	0	0	0
0·1	0·0979	0·0052	0·0003	0·0000	0·0000	0·0000	0	0	0	0	0
0·2	0·3820	0·0405	0·0025	0·0010	0·0000	0·0000	0	0	0	0	0
0·3	0·8244	0·1335	0·0102	0·0084	0·0005	0·0000	0	0	0	0	0
0·4	1·382	0·2706	0·0315	0·0435	0·0031	0·0002	0·0003	0	0	0	0
0·5	2·000	0·5708	0·0735	0·1470	0·0126	0·0010	0·0020	0·0001	0	0	0
0·6	2·618	0·969	0·1505	0·394	0·0397	0·0036	0·0094	0·0007	0	0	0
0·7	3·176	1·390	0·2685	0·853	0·1020	0·0107	0·0348	0·0029	0·0002	0·0006	0
0·8	3·618	1·925	0·4342	1·571	0·2231	0·0269	0·0973	0·0095	0·0008	0·0029	0·0002
0·9	3·902	2·518	0·6564	2·562	0·4298	0·0596	0·2326	0·0260	0·0026	0·0104	0·0009
1·0	4·000	3·142	0·9394	3·7576	0·7458	0·1184	0·4736	0·0613	0·0070	0·0280	0·0028

When $\theta = \frac{1}{2}\pi$, $U = 3{\cdot}142 + 0{\cdot}7458\mu + 0{\cdot}0613\mu^2 + 0{\cdot}0028\mu^3$,

$R^{-1}U = MRU = 6{\cdot}283 + 3{\cdot}7576\mu + 0{\cdot}4736\mu^2 + 0{\cdot}0280\mu^3$.

7 . 121. The condition that $R^{-1}U = 0$ when $\phi = 1$ gives the equation

$$6{\cdot}283 + 3{\cdot}7576\mu + 0{\cdot}4736\mu^2 + 0{\cdot}0280\mu^3 = 0 \qquad (13)$$

to determine μ for the asymmetrical modes. The arithmetically lowest root of this is $-2{\cdot}23$. Now

$$\mu = \frac{\pi^2\lambda}{16} = -\frac{\pi^2 a^2\kappa^2}{16} = -\frac{\pi^2 a^2\sigma^2}{16V^2}. \qquad (14)$$

Hence

$$\frac{\sigma a}{V} = \frac{4}{\pi}(-\mu)^{\frac{1}{2}}, \qquad (15)$$

and in this case

$$\frac{\sigma a}{V} = 1{\cdot}94, \qquad (16)$$

corresponding to the value 1·89 obtained by another method in 7 . 11. The difference arises from the errors introduced in the numerical integration.

In a uniform canal of the same length we should have

$$\frac{\sigma a}{V} = \tfrac{1}{2}\pi = 1{\cdot}57, \qquad (17)$$

so that the variation in the width involves a considerable shortening of the period. Apparently the energy tends to become confined to the centre on account of continual reflexion all the way along the sides, so that the effective length is distinctly shorter than the true length. This suggestion is confirmed when the nature of the actual motion is examined.

We can readily compute the expansions of u and ζ in powers of μ from the table in 7 . 12, using the formulae

$$u = \frac{U}{\sin\theta}, \tag{18}$$

$$\zeta = \frac{R^{-1}U}{4\sin^2\theta}, \tag{19}$$

omitting constant factors. We find the following results :—

ϕ	$\pi x/2a$	u	u_1	ζ	ζ_1
0·0	90°	0	0	1·571	1·57
0·1	89°	·033	·03	1·59	1·57
0·2	86°	·131	·12	1·60	1·57
0·3	80°	·292	·30	1·55	1·55
0·4	73°	·450	·51	1·50	1·50
0·5	64°	·770	·76	1·41	1·42
0·6	53°	1·098	1·06	1·24	1·25
0·7	41°	1·316	1·32	1·02	1·03
0·8	28°	1·55	1·54	0·72	0·74
0·9	14°	1·70	1·70	0·38	0·38
1·0	0°	1·75	1·75	0·00	0·00

Here $\pi x/2a$ is proportional to the distance from the centre, and u_1 and ζ_1 are proportional respectively to the velocity range and tide range at the corresponding position in a uniform canal. It is seen that the difference between the uniform and elliptic canals is surprisingly slight. The greatest difference in the velocities occurs when $\pi x/2a$ is 73°; this corresponds to reflexion of the tidal current by the narrowing of the canal. There is little difference in ζ anywhere; the increase in ζ for small values of ϕ is probably spurious, arising from slight inaccuracies in computation.

7 . 122. In the even modes U vanishes in the centre, and therefore

$$3{\cdot}142+0{\cdot}7458\mu+0{\cdot}0613\mu^2+0{\cdot}0028\mu^3 = 0. \tag{1}$$

The smallest root of this equation is $-6{\cdot}88$. This gives

$$\frac{\sigma a}{V} = 3{\cdot}24, \tag{2}$$

corresponding to

$$\frac{\sigma a}{V} = 3{\cdot}14 \tag{3}$$

for a uniform canal. Thus the change in the period is much less than in the lowest asymmetrical mode. In the following table the results are tabulated as for the latter :—

Φ	$\pi x/a$	u	u_1	ζ	ζ_1
0·0	180°	0	0	1·57	1·57
0·1	178°	·033	·033	1·57	1·57
0·2	172°	·131	·129	1·59	1·55
0·3	160°	·286	·316	1·50	1·48
0·4	146°	·424	·520	1·34	1·30
0·5	128°	·692	·73	1·02	·97
0·6	106°	·902	·89	·70	·43
0·7	82°	·928	·92	·18	−·22
0·8	56°	·816	·77	−·36	−·88
0·9	28°	·506	·44	−·90	−1·38
1·0	0°	0	0	−1·25 ?	−1·57

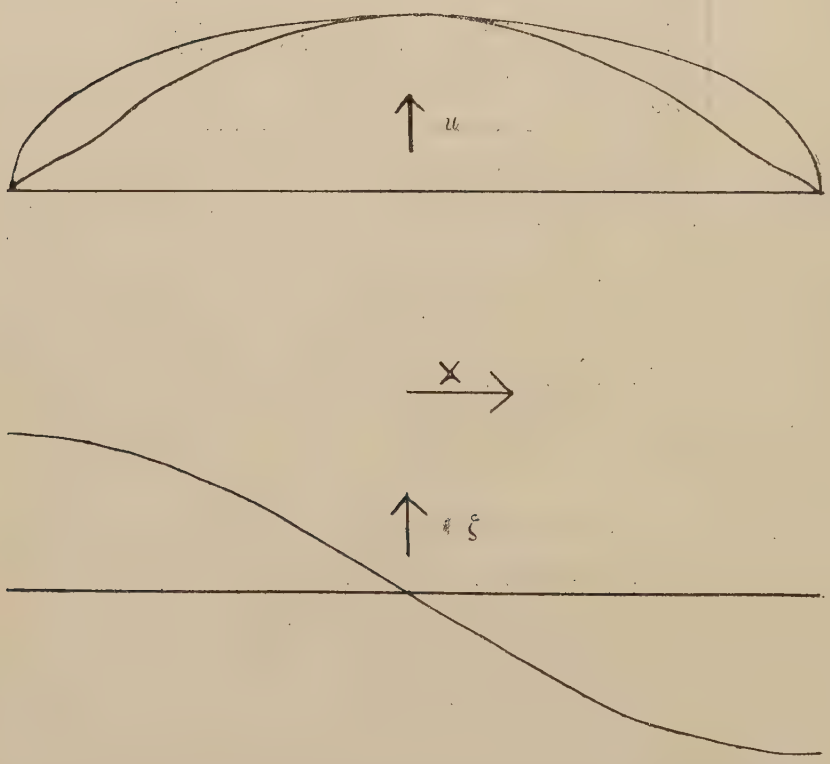

Fig. 1. Lowest asymmetrical mode of elliptic canal.
The upper curve in the u diagram shows the variation in width along the canal.

The velocities therefore tend to be greater towards the middle and less towards the ends than in the uniform canal. The amplitude follows much the same law near the ends, but is much smaller in the middle, corresponding to the greater width and the equation of continuity. The nodes are shifted towards the centre.

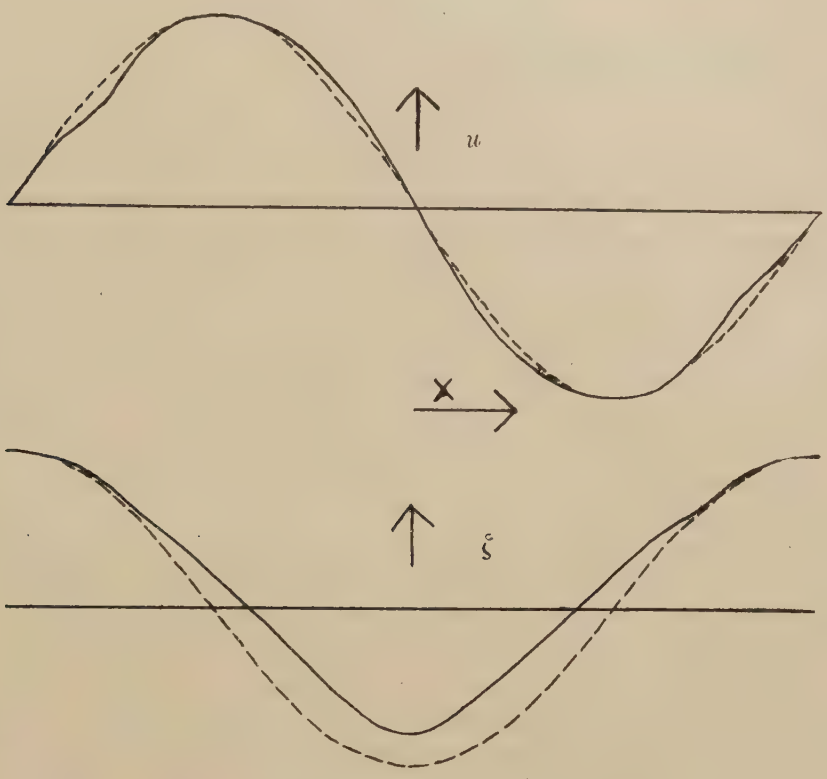

Fig. 2. Lowest symmetrical mode of elliptic canal.
The dotted curves represent the results for a uniform canal.

7.13. To find the higher longitudinal modes a different method is required, since none of the series so far used converges sufficiently rapidly. In these $(R-2h^2)/4h^2$ will still be approximately $\frac{2}{3}a^2$, but h will be great, and therefore we can use the approximation of 2.4. If

$$(R+2h^2)/4h^2 = 1-\beta^2, \tag{1}$$

so that

$$S = \beta^2, \tag{2}$$

we had $$2m\pi = \frac{8h}{(1-\beta^2)^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} (\sin^2\eta + \beta^2\cos^2\eta)^{\frac{1}{2}}\, d\eta, \tag{3}$$

where m is the order of the mode. If β^2 be neglected, we have

$$m = 4h/\pi, \tag{4}$$

so that $$\sigma = 2hV/a = \tfrac{1}{2}\pi mV/a. \tag{5}$$

In the case of a uniform canal we have also

$$\sigma = \tfrac{1}{2}\pi mV/a, \tag{6}$$

so that the periods of high modes are the same in the two cases. Now any such modes may be regarded as the resultant of two waves travelling in opposite directions with velocity V. Thus whatever the form of the canal, if reflexion takes place at the ends, the period of a mode with the same number of maxima must be the same. Our result therefore indicates that most of the energy in the higher longitudinal modes is reflected at the ends. The fact that the periods of the second mode in the two cases differ by only a few per cent. suggests that the approximation becomes very accurate for quite low modes.

The ratio of the amplitudes at the middle and the ends depends on the values of χ. Now χ at the middle is $R+2h^2$, and at the ends is $R-2h^2$. Hence the ratio of the values of χ is practically S or $\frac{2}{3}a^2$, and the ratio of the amplitudes is $(\frac{2}{3})^{-\frac{1}{4}}a^{-\frac{1}{2}}$, the greatest occurring at the ends. It is thus the same for all modes such that this approximation is satisfactory.

In a circular lake the speed of a high mode of zero class is given approximately by $\sigma = \frac{1}{4}\pi(2m-3)V/a$, which scarcely differs from the results just obtained. Hence the period of any mode of zero class, symmetrical about the major axis, is probably never very different in a circle, an ellipse, or an elliptic canal.

The mode may be of such high order that h^2a^4 is no longer small. In this case S will still be small, and the equation $R = 2h^2$ will still give an approximate determination of the periods; but the ratio of the amplitudes will cease to be $(\frac{2}{3})^{-\frac{1}{4}}a^{-\frac{1}{2}}$. Special treatment will be necessary in this case.

7.2. So far only the modes of class zero, asymmetrical about the major axis, have been considered. There remain the modes of higher classes and the modes of class zero but asymmetrical about the major axis. If the class of any mode is not zero, the value of X must be zero

or stationary when ξ is zero, be zero for at least one other value of ξ, and finally be stationary when $\xi = \alpha$. Hence $d^2X/Xd\xi^2$ must be of order α^{-2}, and if α is small this shows that $2h^2 \cosh 2\xi - R$ is large of order α^{-2}; but $\cosh 2\xi$ is never far from unity, and R is greater than $-2h^2$. Hence h is large of order $1/2\alpha$. Thus all these modes are rapid. It is easy to see also that in the modes of class zero, but antisymmetrical about the major axis, h is again of this order.

When ξ is small we can treat $\cosh 2\xi$ as equal to unity. Then the solutions of Mathieu's modified equation are $\cos(2h^2-R)^{\frac{1}{2}}\xi$ and $\sin(2h^2-R)^{\frac{1}{2}}\xi$. When ξ is equal to α the solution must be stationary; hence for the modes involving cosines we have

$$(2h^2-R)^{\frac{1}{2}}\alpha = r\pi,$$

and for those involving sines

$$(2h^2-R)^{\frac{1}{2}}\alpha = (r+\tfrac{1}{2})\pi.$$

Here r is the class of mode. Now from 2 . 52 (2) we have

$$(2h^2-R)^{\frac{1}{2}} = 2h \sin \eta_0$$

$$= \sigma c \sin \eta_0 / V.$$

Hence for a mode symmetrical about the major axis,

$$\sigma c\alpha \sin \eta_0 = r\pi V,$$

and for an antisymmetrical mode

$$\sigma c\alpha \sin \eta_0 = (r+\tfrac{1}{2})\pi V.$$

Now $c\alpha \sin \eta_0$ is half the breadth of the ellipse at the point of greatest amplitude. If we call this $\frac{1}{2}b$, the normal distance from one nodal ellipse to the next is $\frac{1}{2}b/r$ in a symmetrical mode, and $\frac{1}{2}b/(r+\frac{1}{2})$ in an antisymmetrical mode. If then the normal distance between consecutive nodal ellipses is ϖ, these equations can all be expressed in the form

$$\sigma\varpi = \pi V.$$

In other words, the period is double the time it would take a plane wave to travel normally from one nodal ellipse to the next in the neighbourhood of the critical points. Further, for modes of the same class, the period is evidently greater the more closely η_0 approximates to $\frac{1}{2}\pi$. The

mode of zero order will therefore be the slowest of any class, that of order unity the next, and so on. The speed of the mode of zero order and class r is given by

$$\sigma_0 c/V = r\pi/a.$$

This completes the discussion of the modes of a long, narrow ellipse, such that $4h^2a^2$ is not so large as to be comparable with $2h^2-R$. Even if h is great of order a^{-2}, $2h^2-R$ will in general be of order a^{-4}, and therefore still great in comparison with $4h^2a^2$; and the approximations just given will still be satisfactory. If, however, the mode is such that $2h^2-R$ and $4h^2a^2$ are of the same order, $(2h^2-R)/4h^2$ is small and therefore

$$m = 4h/\pi$$

approximately. This determines the periods of the modes such that the waves reach the ends of the lake. The modified Mathieu's equation in this case reduces to

$$\frac{d^2X}{d\xi^2} + \{(2h^2-R)+4h^2\xi^2\}X = 0,$$

which is of the type discussed in 1.4. As h increases, the number of zeros of either solution of this equation, less than a, can be made to increase indefinitely, and therefore, however great the order and class of the mode may be, there will be a possible value of h. Thus the possible modes constitute a doubly infinite set.

8. *Note on G. I. Taylor's theory of tides in a rotating rectangular basin.*

In a recent paper by G. I. Taylor* several curious results are stated, some of which he suggests are probably very general. They are as follows.

(1) The currents across the lake are exceedingly small. For a gulf of infinite length, with parallel sides and a rectangular end, the transverse velocities in the middle, at a distance from the end about equal to one and a half times the breadth, are about 5×10^{-6} of the longitudinal velocities.

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* *Proc. London Math. Soc.* (2), 20 (1922), 148–181.

(2) There is only a single infinity of modes of oscillation.

(3) All modes consist of waves moving round the lake, and in every mode in the same direction, namely that of the rotation of the earth.

In the lowest mode of the rotating ellipse the first of these results does not hold. On the outer boundary we have

$$\frac{\partial^2 \xi}{\partial \xi \partial t} + 2\omega \frac{\partial \xi}{\partial \eta} = 0. \tag{1}$$

If
$$\xi = \xi_0 + \xi_1, \tag{2}$$

where ξ_0 is even in ξ and η and ξ_1 is odd, we have

$$\frac{\partial^2 \xi_1}{\partial \xi \partial t} + 2\omega \frac{\partial \xi_0}{\partial \eta} = 0, \tag{3}$$

$$\frac{\partial^2 \xi_0}{\partial \xi \partial t} + 2\omega \frac{\partial \xi_1}{\partial \eta} = 0. \tag{4}$$

If the lake is narrow, so that a is small, ξ_0 changes by only a fraction of itself of order $h^2 a^4$ in passing from the outside to the line joining the foci. The same therefore holds in general of $\partial \xi_0 / \partial \eta$. Hence the change in the motion will be small, and the period of the nearest longitudinal mode will not be much affected by the rotation. Now, while R may be nearly equal to $2h^2$ for the function that enters with the largest coefficient, it will differ from it by a finite amount for every other term. Hence the change in $\partial \xi_1 / \partial \xi$ in passing from the outside to the line joining the foci will be of order a^2 of the mean value. Since (3) holds when ξ is equal to a, it follows that, when $\xi = 0$,

$$\frac{\partial^2 \xi_1}{\partial \xi \partial t} + 2\omega \frac{\partial \xi_0}{\partial \eta}$$

is of order $\quad a^2 \dfrac{\partial^2 \xi_1}{\partial \xi \partial t}$ or $2\omega a^2 m \xi_0$.

The velocity along the confocal ellipses is proportional to

$$i\sigma \frac{\partial \xi}{\partial \eta} - 2\omega \frac{\partial \xi}{\partial \xi},$$

which is comparable with $\sigma m \xi_0$. Hence the ratio of the amplitudes of the transverse and longitudinal velocities near the centre is of order

$2\omega a^2/\sigma$. Hence Taylor's first result does not hold for an ellipse, the transverse component being a much larger fraction of the longitudinal one than he obtains. The explanation appears to be that in his case transverse velocities are generated only in the reflexion at the ends and die down exponentially as the wave recedes from the end. In the ellipse waves are being reflected at all points of the boundary, on account of the continuous variation in width, so that effectively the "end" includes the whole margin.

With regard to the second result, and also the third, quoted above, they had already been shown by Lamb not to hold for a circular lake, but Taylor suggested that this form was probably exceptional. He has not, however, proved them even in the rectangular case. The speeds of his modes arise as the solutions of an infinite determinantal equation, given on page 174 of his paper. A function of σ occurs in every row and column of this determinant, and therefore a single infinity of roots would be expected on this ground alone. But, in addition, each of these functions has itself an infinite number of zeros, and therefore there is a close analogy between this determinant and the corresponding one for a circle. Consequently a double infinity of roots is to be expected.

It may be remarked that the question whether an infinite set is to be regarded as singly or doubly infinite is largely a matter of the method of approach to the problem. Any double infinity can be rearranged so as to constitute a single infinity, and *vice versa*. In the case of a rectangular lake with no rotation and with the length and breadth incommensurable, for instance, the modes could be ordered according to increasing speed, and would then constitute a single infinity. On the other hand, they may be ordered according to the multiples of x and y occurring in the elevation, and then form a double infinity. In some sense a double infinity of functions is required to express an arbitrary function of position on a plane surface; and since a lake, whether rotating or not, can be started off in any manner, the subsequent motion must require a doubly infinite set of modes to express it.

Taylor shows that his infinite determinant is a function of σ^2, and that therefore to every value of σ corresponds an equal and opposite one. He then shows that equal and opposite values of σ correspond to oscillations of the same type, but in opposite phases; he infers that the waves corresponding to positive and negative values of σ move round in the same direction. But it is easily seen that up to this point the circle is precisely analogous. If in this case we have

$$\zeta e^{-i\sigma t} = \Sigma A_m J_m(\kappa r) \cos m\phi + \Sigma D_m J_m(\kappa r) \sin m\phi,$$

and we apply the condition that

$$i\sigma \frac{\partial \xi}{\partial r} + 2\omega \frac{\partial \xi}{r \partial \phi} = 0,$$

when $r = a$, we get a series of equations

$$i\sigma\kappa a A_m J'_m(\kappa a) + 2\omega m D_m J_m(\kappa a) = 0,$$

$$i\sigma\kappa a D_m J'_m(\kappa a) - 2\omega m A_m J_m(\kappa a) = 0.$$

When we eliminate the A's and D's we have an infinite determinant corresponding to Taylor's; but it breaks up into an infinite number of factors, each giving modes of one order. A typical factor is

$$\kappa^2 a^2 \sigma^2 \{J_m(\kappa a)\}^2 - 4\omega^2 m^2 \{J_m(\kappa a)\}^2 = 0,$$

which may be written

$$\kappa a \sigma J_m(\kappa a) \pm 2\omega m J_m(\kappa a) = 0.$$

This equation has an infinite number of real roots, corresponding to modes of the same order but different classes. Thus the circle has a double infinity of modes, though it possesses all the features pointed out by Taylor in his argument for the rectangle.

Indeed, to show that all the modes travelled round in the same sense, it would be necessary to show that modes with different absolute values of σ travel round in the same way; and Taylor has not considered this point. In the circle, indeed, waves may travel around in either direction, but in general no two waves have the same period. It appears probable that the analogy with the circle will hold for lakes of most forms, including the rectangle, but this has not yet been proved. The suggested difference between the circular and rectangular cases appears to rest on a misinterpretation of Lamb's results.* Lamb has arranged his work so that the roots of the period equation do not occur in pairs, as would be necessary to render possible a comparison of the type contemplated.

Taylor remarks that, in all known cases, waves in gulfs and lakes move round in the direction of the earth's rotation, and considers that this fact confirms the result just quoted. It is probable, however, that the true explanation is that in most systems the easiest mode to excite is the slowest, which in this case is one of zero class, and therefore moves round in this direction.

* *Hydrodynamics* (1906), 306.

9. *Summary.*

The slowest modes of certain ellipses have been calculated; it appears that the periods depend on the major axis, the most extreme variation of the minor axis producing changes of only a few per cent. in the period. In the higher modes, however, the nodal hyperbolas crowd towards the minor axis, and in one large set of modes of vibration the motion consists of a series of standing waves around the minor axis, increasing with amplitude and wave-length with distance from this axis, and then suddenly dying down; there is practically undisturbed water from this point to the end of the lake. However narrow the lake, a mode of this type must have at least one nodal ellipse, or else be anti-symmetrical with regard to the major axis. Other modes consist of waves extending all round the ellipse. Any numbers of nodal ellipses and hyperbolas may occur. The longitudinal modes of a narrow lake are limiting cases of modes with no nodal ellipse; in all of them the motion reaches the ends.

CYLINDRICAL SOLUTIONS OF EINSTEIN'S GRAVITATION EQUATIONS

By H. E. J. CURZON.

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1. *The Equations* $G_{mn} = 0$.

If the line element is given by

$$ds^2 = -e^{\lambda}(dx_1^2+dx_2^2)-x_2^2\,e^{-\rho}\,dx_3^2+e^{\rho}\,dx_4^2,$$

where λ and ρ are functions of x_1 and x_2 only, then the contracted Riemann-Christoffel tensors are given by

$$G_{11} = \tfrac{1}{2}(\lambda_{11}+\lambda_{22}+\rho_1^2+\lambda_2/x_2),$$

$$G_{12} = \tfrac{1}{2}(\rho_1\rho_2-\rho_1/x_2-\lambda_1/x_2),$$

$$G_{22} = \tfrac{1}{2}(\lambda_{11}+\lambda_{22}+\rho_2^2-\lambda_2/x_2-2\rho_2/x_2),$$

$$G_{33} = -\tfrac{1}{2}e^{-\lambda-\rho}x_2^2(\rho_{11}+\rho_{22}+\rho_2/x_2),$$

$$G_{44} = -\tfrac{1}{2}e^{-\lambda+\rho}(\rho_{11}+\rho_{22}+\rho_2/x_2),$$

where λ_1 denotes $\partial\lambda/\partial x_1$; and so on.

The equation $G_{44} = 0$ is $\nabla^2\rho = 0$ in cylindrical coordinates symmetrical about the axis of x_1. Hence a solution of $G_{44} = 0$ is given by

$$\rho = -2m_1/r_1-2m_2/r_2,$$

where r_1, r_2 are bipolar coordinates of a point P referred to two origins A and B on the x_1 axis, m_1 and m_2 being constants of integration.

From $G_{12}=0$ and from $G_{11}=0=G_{22}$ it follows that

$$\mu_1=\lambda_1+\rho_1=x_2\rho_1\rho_2,$$

$$\mu_2=\lambda_2+\rho_2=x_2(\rho_2^2-\rho_1^2)/2.$$

Transforming by $r_1^2=(x_1-a)^2+x_2^2$, $r_2^2=(x_1+a)^2+x_2^2$, these give

$$\begin{aligned}d\mu=&[2x_2\{x_2^2-(x_1-a)^2\}dx_2+4x_2^2(x_1-a)dx_1]m_1^2/r_1^6\\&+[2x_2\{x_2^2-(x_1+a)^2\}dx_2+4x_2^2(x_1+a)dx_1]m_2^2/r_2^6\\&+4[2x_2^2x_1dx_1-x_2(x_1^2-a^2-x_2^2)dx_2]m_1m_2/r_1^3r_2^3.\end{aligned}$$

Whence

$$\mu=-x_2^2m_1^2/r_1^4-x_2^2m_2^2/r_2^4+\{(x_1^2+x_2^2-a^2)/r_1r_2-1\}m_1m_2/a^2,$$

choosing the constant of integration so that μ vanishes at infinity.

Hence

$$\mu=-m_1^2\sin^2\theta_1/r_1^2-m_2^2\sin^2\theta_2/r_2^2-2m_1m_2\sin^2\{(\theta_1-\theta_2)/2\}/a^2,$$

where θ_1 and θ_2 are the angles made by AP and BP with the axis of x_1.

This solution for two singularities on the axis of x_1 may be generalized for any number of collinear singularities, so that in general

$$\rho=-2\Sigma(m_p/r_p),$$

$$\lambda=2\Sigma(m_p/r_p)-\Sigma(m_p\sin\theta_p)^2/r_p^2-2\Sigma m_pm_q\sin^2\{(\theta_p-\theta_q)/2\}/a^2.$$

The field is Euclidean at infinity.

2. *The Equations* $G_{mn}=-8\pi E_{mn}$.

Using the notation in Eddington's Theory of Relativity, and denoting κ_4 for the moment by θ, we have

$$F_{41}=-F_{14}=\theta_1, \qquad F_{42}=-F_{24}=\theta_2,$$

$$F^{41}=-F^{14}=-e^{-\lambda-\rho}\theta_1, \qquad F^{42}=-F^{24}=-e^{-\lambda-\rho}\theta_2.$$

Since $\sqrt{(-g)}=x_2e^{\lambda}$, therefore $\mathfrak{F}^{41}=-x_2e^{-\rho}\theta_1$, $\mathfrak{F}^{42}=-x_2e^{-\rho}\theta_2$. Hence the equation $\partial\mathfrak{F}^{4\nu}/\partial x_\nu=0$ gives

$$\theta_{11}+\theta_{22}+\theta_2/x_2=\rho_1\theta_1+\rho_2\theta_2. \qquad (1)$$

From the formula

$$E_\mu^\nu = -F^{\nu\alpha}F_{\mu\alpha}+\tfrac{1}{4}g_\nu^\mu F^{\alpha\beta}F_{\alpha\beta},$$

we have

$$E_1^1 = \tfrac{1}{2}e^{-\lambda-\rho}(\theta_1^2-\theta_2^2) = -E_2^2,$$

$$E_1^2 = e^{-\lambda-\rho}\theta_1\theta_2,$$

$$E_3^3 = -\tfrac{1}{2}e^{-\lambda-\rho}(\theta_1^2+\theta_2^2) = -E_4^4.$$

Hence the equations $G_{mn} = -8\pi E_{mn}$ become

$$\lambda_{11}+\lambda_{22}+\rho_1^2+\lambda_2/x_2 = 8\pi e^{-\rho}(\theta_1^2-\theta_2^2) \tag{2}$$

$$\rho_1\rho_2-\rho_1/x_2-\lambda_1/x_2 = 16\pi e^{-\rho}\theta_1\theta_2, \tag{3}$$

$$\lambda_{11}+\lambda_{22}+\rho_2^2-\lambda_2/x_2-2\rho_2/x_2 = -8\pi e^{-\rho}(\theta_1^2-\theta_2^2) \tag{4}$$

$$\rho_{11}+\rho_{22}+\rho_2/x_2 = 8\pi e^{-\rho}(\theta_1^2+\theta_2^2). \tag{5}$$

From (2) and (4)

$$\lambda_2+\rho_2 = -x_2(\rho_1^2-\rho_2^2)/2+8\pi x_2 e^{-\rho}(\theta_1^2-\theta_2^2). \tag{6}$$

Out of equations (1) and (5) the following value of θ in terms of ρ arises as a solution :

$$\theta = e^{\rho/2}/(4\pi)^{\frac{1}{2}}-1/(4\pi)^{\frac{1}{2}}, \tag{7}$$

the constant of integration being such that $\theta = 0$ when $\rho = 0$.

Substituting from (7) in (3) and (6), it follows that

$$\lambda+\rho = 0, \tag{8}$$

taking $\lambda = 0$ when $\rho = 0$.

Substituting from (7) in (5), the equation becomes

$$\rho_{11}+\rho_{22}+\rho_2/x_2 = (\rho_1^2+\rho_2^2)/2. \tag{9}$$

Putting $\rho = -2\log\sigma$ this becomes

$$\sigma_{11}+\sigma_{22}+\sigma_2/x_2 = 0. \tag{10}$$

Hence

$$\lambda = -\rho = 2\log\{1+\Sigma 2\sqrt{\pi\epsilon_p}/r_p\} \tag{11}$$

is a solution of $G_{mn} = -8\pi E_{mn}$, if κ_4 be taken so that

$$\theta = \kappa_4 = \{(1+\Sigma 2\sqrt{\pi\epsilon_p}/r_p)^{-1}-1\}/(2\sqrt{\pi}), \tag{12}$$

where the ϵ_p are constants of integration, and r_p has the same meaning as in § 1.

For an isolated electron this reduces to the form

$$ds^2 = -(1+2\sqrt{\pi\epsilon}/r_1)^2(dr_1^2+r_1^2 d\theta^2+r_1^2\sin^2\theta d\phi^2)+(1+2\sqrt{\pi\epsilon}/r_1)^{-2}dt^2,$$

$$\kappa_4 = -\epsilon/(r_1+2\sqrt{\pi\epsilon}).$$

Making the substitution $r_1+2\sqrt{\pi\epsilon} = r$, we have

$$ds^2 = -\gamma^{-1}dr^2+r^2 d\theta^2+r^2\sin^2\theta d\phi^2+\gamma dt^2,$$

$$\kappa_4 = -\epsilon/r, \quad \text{where} \quad \gamma = 1-4\sqrt{\pi\epsilon}/r+4\pi\epsilon^2/r^2,$$

which may be reconciled with the Jeffery-Nordström solution, where

$$\gamma = 1-2m/r+4\pi\epsilon^2/r^2, \quad \kappa_4 = -\epsilon/r.$$

In conclusion I have to thank Prof. Jeffery for valuable suggestions which have led to a closer examination of the singularities in the field, and to much saving of labour in the algebraic part of the paper.

ON INEQUALITIES IN THE THEORY OF FUNCTIONS

By J. E. LITTLEWOOD.

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1. *General theorems.*

1.1. Suppose that $$w = \bar{f}(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n,$$

regular in $|z| < R$, gives the conformal representation of $|z| < R$ on a domain W of a Riemann surface, the point P, or $w = \bar{a}_0$ in some particular sheet, corresponding to $z = 0$. Suppose, further, that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a function regular in $|z| < R$, that $a_0 = \bar{a}_0$, and that, as z describes any contour whatever in $|z| < R$, beginning and ending at $z = 0$, the point $w = f(z)$ (supposed to start at P) describes a contour lying in W, and beginning and ending at P. The important conception, originating in a well known paper of Lindelöf,* of a pair of functions, f, $\bar{f}$ related in this way, demands a name. We shall say that "f is subordinate to $\bar{f}$ in $|z| < R$." We may also define the idea "f is subordinate to $\bar{f}$ in $|z| \leqslant R$"; we have only to replace the sign "$<$" everywhere by "$\leqslant$".†

If $\bar{f}$ is a "schlicht" function in $|z| < R$, *i.e.* takes no value more than once, the domain W (considered on the simple w-plane) is "schlicht", *i.e.* does not overlap itself. In this case the conditions to be satisfied by f are that $f(0) = \bar{f}(0)$ and that f takes no value not taken somewhere by $\bar{f}$. This state of things is easy to grasp intuitively; the general con-

* E. Lindelöf, *Acta Soc. Sci. Fenn.*, 35 (1908), No. 7. See Appendix 2 below.

† And interpret "domain" to mean "closed domain."

ception is less so, but we are naturally led to it if we are to remove the restriction that f is "schlicht".

We shall denote the unit circle $|z|<1$ always by γ. We generally use ρ for $|z|$, and in any case ρ always denotes the modulus of some variable in a unit circle. To avoid incessant repetition we lay down once for all that where ρ occurs it satisfies $\rho < 1$.

THEOREM 1.—*If f is subordinate to $\bar{f}$ in γ, then*

$$f(z) = \bar{f}\{\omega(z)\},$$

where $\omega(z)$ is regular, and $|\omega(z)| \leqslant |z|$, in γ.

Important special cases of this theorem have been considered by a number of writers; but the general statement was first given quite recently by Rogosinski.* I add the very simple proof for completeness.

Let $Z(w)$ be that function, inverse to $w = \bar{f}(z)$ and so regular in W, for which $Z(\bar{a}_0) = 0$. Clearly $\omega(z) = Z\{f(z)\}$ is regular, and

$$|\omega(z)| = |Z(f)| < 1,$$

in γ. Also $\omega(0) = Z\{f(0)\} = Z(\bar{a}_0) = 0$. By Schwarz's lemma

$$\left|\frac{\omega(z)}{z}\right| \leqslant 1,$$

and this proves the theorem.

It follows at once, since the maximum modulus occurs on a boundary, that

$$\operatorname*{Max}_{|z|=\rho} |f(z)| \leqslant \operatorname*{Max}_{|z|=\rho} |\bar{f}(z)|. \tag{1.11}$$

Further, making $\rho \to 0$ in (1.11), and remembering that $a_0 = \bar{a}_0$, we have

$$|a_1| \leqslant |\bar{a}_1|. \tag{1.12}$$

1.2. I propose, in this section of the paper, to discuss some new types of inequality between functions f and $\bar{f}$. I begin by collecting, for convenience, some simple corollaries of Theorem 1.

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* W. Rogosinski, *Math. Zeitschrift*, 17 (1923), 260. See Appendix 2.

LEMMA 1.—*If $g(z)$ is regular in $|z|<R$, and $|\omega(z)|\leqslant|z$ in $|z|<R$, then $g\{\omega(z)\}$ is subordinate to $g(z)$ in $|z|<R$.*

This is evident. A similar result holds if we write "$\leqslant R$" for "$<R$" throughout.

LEMMA 2.—*If f is subordinate to $\bar{f}$ in $|z|<R$, and $R'<R$, then f is subordinate to $\bar{f}$ in $|z|\leqslant R'$.*

By Theorem 1 $$f(R\xi)=\bar{f}\{R\omega(\xi)\} \quad (|\xi|<1).$$

Lemma 2 follows from the facts $\omega(0)=0$ and

$$|R\omega(\xi)|\leqslant R' \quad (|R\xi|\leqslant R').$$

LEMMA 3.—*If f is subordinate to $\bar{f}$ in $|z|<R$, and $\bar{\chi}(z)=\psi\{\bar{f}(z)\}$ is regular in $|z|<R$, then $\chi(z)=\psi(f)$ is (regular and) subordinate to $\bar{\chi}$ in $|z|<R$.*

For $$\chi(z)=\psi\{\bar{f}(\omega)\}=\bar{\chi}(\omega),$$

and the result follows from Lemma 1.

1.3. If f is regular in γ and $\lambda>0$, we define

$$M_\lambda(\rho)=M_\lambda(\rho,f)=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(\rho e^{i\theta})|^\lambda\,d\theta\right\}^{1/\lambda}. \tag{1.31}$$

It may be shown that $M_\lambda(\rho)$ tends to a limit as $\lambda\to0$ and again as $\lambda\to\infty$; in fact that

$$M_0(\rho)=\lim_{\lambda\to0}M_\lambda(\rho)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|\,d\theta\right\} \tag{1.32}$$

unless f is identically zero; and

$$M_\infty(\rho)=\lim_{\lambda\to\infty}M_\lambda(\rho)=M(\rho)=\operatorname*{Max}_{|z|=\rho}|f(z)|. \tag{3}$$

Suppose $\lambda\to0$ and that $k\lambda<\frac{1}{2}$, where k is the greatest multiplicity of a zero of f lying on $|z|=\rho$. We have, for $z=\rho e^{i\theta}$,

$$\left||f|^\lambda-(1+\lambda\log|f|)\right|\leqslant\tfrac{1}{2}(\lambda\log|f|)^2\exp\left(\lambda\left|\log|f|\right|\right)$$

$$\leqslant\lambda^2(\log|f|)^2\exp\left(\frac{1}{2k}\left|\log|f|\right|\right)\leqslant\lambda^2(\log|f|)^2(|f|^{1/(2k)}+|f|^{-1/(2k)})=\lambda^2\psi(\rho,\theta),$$

say. Now $$\frac{1}{2\pi}\int_0^{2\pi}\psi\,d\theta < K(\rho),$$

a number independent of λ. Hence

$$M_\lambda(\rho) = \{1+\lambda L+\vartheta\lambda^2 K(\rho)\}^{1/\lambda},$$

where $L = \frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|\,d\theta$, and $|\vartheta|\leqslant 1$. Evidently

$$\lim_{\lambda\to 0} M_\lambda(\rho) = e^L,$$

and this is (1.32). To prove (1.33) we observe first that $M_\lambda(\rho)\leqslant M(\rho)$ and so $\overline{\lim_{\lambda\to\infty}} M_\lambda(\rho)\leqslant M(\rho)$. On the other hand, if $z_0 = \rho e^{i\theta_0}$ is a value of z giving $|f(z_0)| = M = M(\rho)$, we have

$$|f(\rho e^{i\theta})| > M-\epsilon \quad \{|\theta-\theta_0|\leqslant\delta=\delta(\epsilon)\},$$

$$M_\lambda(\rho)\geqslant\left\{\frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta}|f(\rho e^{i\theta})|^\lambda\right\}^{1/\lambda}\geqslant(M-\epsilon)\left(\frac{\delta}{\pi}\right)^{1/\lambda},$$

$$\lim_{\lambda\to\infty} M_\lambda(\rho)\geqslant M-\epsilon,$$

$$\underline{\lim}\, M_\lambda(\rho)\geqslant M(\rho).$$

Hence $$\lim_{\lambda\to\infty} M_\lambda(\rho) = M(\rho),$$

which is (1.33).

The generalized inequality of Schwarz shows further that $M_\lambda(\rho)$ is an increasing function of λ.

1.4. The principal theorem of the present section is

THEOREM 2.—*Let f be subordinate to $\bar f$ in γ (or only in $|z|\leqslant\rho$). Then*

(1.41) $$M_\lambda(\rho, f)\leqslant M_\lambda(\rho,\bar f)\quad(0\leqslant\lambda\leqslant\infty).$$

We prove this first in the special case $\lambda = 2$. Let

$$\bar f(z) = \Sigma\bar a_n z^n = \bar a_0+zg_1(z),$$

$$g_1(z) = \bar a_1+zg_2(z),\quad g_2(z) = \bar a_2+zg_3(z),\quad \ldots.$$

We have, by Theorem 1,

$$f(z) = \bar f\{\omega(z)\},$$

where $\omega(z) = z\chi(z)$, χ is regular in $|z| \leqslant \rho$, and

$$\operatorname*{Max}_{|z|=\rho} |\chi(z)| = k \leqslant 1.$$

Then

$$\int_0^{2\pi} |f(z)|^2 \, d\theta = \int_0^{2\pi} |\bar{f}(\omega)|^2 \, d\theta = \int_0^{2\pi} |\bar{a}_0 + \omega g_1(\omega)|^2 \, d\theta$$

$$= \int_0^{2\pi} |\bar{a}_0|^2 \, d\theta + \int_0^{2\pi} |\omega g_1(\omega)|^2 \, d\theta + 2\bar{a}_0' \int_0^{2\pi} \omega g_1(\omega) \, d\theta,$$

where $\bar{a}_0'$ is the conjugate of $\bar{a}_0$. The last integral is

$$-i \int \chi(z) \, g_1(\omega) \, dz$$

taken round $|z| = \rho$, and this is zero, by Cauchy's theorem.

Hence, since $|\omega| \leqslant k\rho$

$$\int_0^{2\pi} |\bar{f}(\omega)|^2 \, d\theta \leqslant 2\pi |\bar{a}_0|^2 + k^2\rho^2 \int_0^{2\pi} |g_1(\omega)|^2 \, d\theta.$$

Similarly

$$\int_0^{2\pi} |g_1(\omega)|^2 \, d\theta \leqslant 2\pi |a_1|^2 + k^2\rho^2 \int_0^{2\pi} |g_2(\omega)|^2 \, d\theta,$$

and so on, and finally

$$\int_0^{2\pi} |f(z)|^2 \, d\theta = \int_0^{2\pi} |\bar{f}(\omega)|^2 \leqslant 2\pi (|\bar{a}_0|^2 + |\bar{a}_1|^2 k^2\rho^2 + |\bar{a}_2|^2 k^4\rho^4 + \ldots)^*$$

$$\leqslant 2\pi \Sigma |\bar{a}_n|^2 \rho^{2n} = \int_0^{2\pi} |\bar{f}(z)|^2 \, d\theta.$$

This proves the theorem in the special case.

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* To justify the limit operation involved here we have only to show that

$$k^{2n}\rho^{2n} \int_0^{2\pi} |g_n(\omega)|^2 \, d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

But, since $|\omega| \leqslant \rho$, we have

$$|g_n(\omega)| = \left| \sum_{s=0}^{\infty} \bar{a}_{n+s} \omega^s \right| \leqslant \sum_{s=0}^{\infty} |\bar{a}_{n+s}| \rho^s,$$

$$k^{2n}\rho^{2n} \int_0^{2\pi} \leqslant 2\pi \left(\sum_{s=0}^{\infty} |\bar{a}_{n+s}| \rho^{n+s} \right)^2,$$

and this tends to 0 as $n \to \infty$.

1.5. Consider now the general case. It is clearly enough to prove (1.41) when $0 < \lambda < \infty$.

Let now $z_1, z_2, \ldots, z_\nu$ be the zeros of $\bar{f}$ in $|z| < \rho$, and let z_n' be the conjugate of z_n. Then

$$(\bar{f})^{\frac{1}{2}\lambda} = F_1(z)\, F_2(z),$$

where

$$F_1(z) = \prod_{n=1}^{\nu} \left\{ \frac{z - z_n}{\rho^2 - z z_n'} \right\}^{\frac{1}{2}\lambda},$$

and $F_2(z)$ is regular in $|z| < \rho$, and continuous in $|z| \leqslant \rho$. Further

$$|f(z)|^{\frac{1}{2}\lambda} = |F_1(\omega)\, F_2(\omega)|.$$

Now $|(z - z_n)/(\rho^2 - z z_n')|$ is equal to $K(\rho)$, independent of θ, on $|z| = \rho$, *and has a value not greater than* $K(\rho)$ *for* $|z| \leqslant \rho$.* It follows that $|F_1(z)| = T(\rho)$ say, independent of θ, on $|z| = \rho$; and, since $|\omega| \leqslant \rho$, that $|F_1(\omega)| \leqslant T(\rho)$. We have now

$$\int_0^{2\pi} |f(z)|^\lambda\, d\theta = \int_0^{2\pi} |F_1(\omega)\, F_2(\omega)|^2\, d\theta \leqslant T^2 \int_0^{2\pi} |F_2(\omega)|^2\, d\theta$$

$$\leqslant T^2 \int_0^{2\pi} |F_2(z)|^2\, d\theta$$

by the special case of the theorem.† Hence

$$\int_0^{2\pi} |f(z)|^\lambda\, d\theta \leqslant \int_0^{2\pi} |TF_2(z)|^2\, d\theta = \int_0^{2\pi} |F_1(z)\, F_2(z)|^2\, d\theta = \int_0^{2\pi} |\bar{f}(z)|^\lambda\, d\theta,$$

$$M_\lambda(\rho, f) \leqslant M_\lambda(\rho, \bar{f}).$$

We note explicitly the special cases $\lambda = 0, 2, \infty$. They give respectively

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})|\, d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log |\bar{f}(\rho e^{i\theta})|\, d\theta, \tag{1.51}$$

$$\Sigma |a_n|^2 \rho^{2n} \leqslant \Sigma |\bar{a}_n|^2 \rho^{2n}, \tag{1.52}$$

$$\operatorname*{Max}_{|z|=\rho} |f| \leqslant \operatorname*{Max}_{|z|=\rho} |\bar{f}|, \tag{1.53}$$

of which the last is (1.33).

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* Since the maximum occurs on the boundary.

† Strictly speaking we appeal to the result: "if f is subordinate to $\bar{f}$ in $|z| < \rho$, and f, $\bar{f}$ are continuous in $|z| \leqslant \rho$, then $M_2(\rho, f) \leqslant M_2(\rho, \bar{f})$." This inequality is true with ρ replaced by $\rho' < \rho$, and is therefore true as it stands by continuity.

1 . 6. The case $\lambda = 1$ is also important as providing inequalities for the coefficients of f.

THEOREM 3.—*If f is subordinate to $\bar{f}$ in γ and is not identically zero, then, for $n > 1$,*

$$|a_n| < e\left[\frac{M_1(\rho, \bar{f})}{\rho}\right]_{\rho=(n-1)/n}.$$

For

$$|a_n| = \frac{\rho^{-n}}{2\pi}\left|\int_0^{2\pi} f(\rho e^{i\theta})\, e^{-ni\theta}\, d\theta\right| \leqslant \frac{\rho^{-n}}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|\, d\theta \leqslant \rho^{-(n-1)}\frac{M_1(\rho, \bar{f})}{\rho}.$$

In this we take $\rho = \dfrac{n-1}{n}$, and observe that $\rho^{-(n-1)} = \left(1+\dfrac{1}{n-1}\right)^{n-1} < e$.

There is an element of ugliness in this theorem, in that it cannot, by the nature of the case, give best possible results. At the same time there are important classes of functions for which it gives results that are both new and very nearly best possible.

1 . 7. THEOREM 4.—*Let f be subordinate to $\bar{f}$ in γ, and $f(0) = \bar{f}(0) = 0$. Let $z_n(a)$, $\bar{z}_n(a)$ be the n-th non-zero roots of $f-a = 0$ and $\bar{f}-a = 0$, arranged in order of increasing moduli; $|z_n(a)| = \rho_n(a)$, $|\bar{z}_n(a)| = \bar{\rho}_n(a)$. Then*

$$(1.71) \qquad \prod_{\rho_n(a)<\rho}\left(\frac{\rho}{\rho_n(a)}\right) \leqslant \prod_{\bar{\rho}_n(a)<\rho}\left(\frac{\rho}{\bar{\rho}_n(a)}\right) \quad (a \neq 0),$$

$$(1.72) \qquad |a_\nu|\rho^{\nu-1}\prod_{\rho_n(0)<\rho}\left(\frac{\rho}{\rho_n(0)}\right) \leqslant |\bar{a}_\mu|\rho^{\mu-1}\prod_{\bar{\rho}_n(0)<\rho}\left(\frac{\rho}{\bar{\rho}_n(0)}\right),$$

where a_ν is the first non-zero coefficient of f, and $\bar{a}_\mu$ that of $\bar{f}$. If a product contains no factors it is interpreted to mean unity.

Since $f-a$ is subordinate to $\bar{f}-a$, we have

$$(1.73) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|f-a|\, d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|\bar{f}-a|\, d\theta.$$

By Jensen's theorem, if $\phi = \sum_\lambda^\infty b_n z^n$ is regular in γ, $b_\lambda \neq 0$, and ξ_n is the n-th non-zero root of $\phi = 0$, we have

$$(1.74) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|\phi(\rho e^{i\theta})|\, d\theta = \log\left|b_\lambda \rho^\lambda \prod_{|\xi_n|<\rho}\left(\frac{\rho}{\xi_n}\right)\right|.$$

From (1 . 73) and (1 . 74) we obtain Theorem 4.

1.8. This completes our general theory. In what follows we apply it to various classes of functions, the functions of each class being, speaking very roughly, subordinate to some particular function. It is evident that, to obtain a particular application of the general theory, we must make a special study of the relevant superordinate function. The rest of the paper is therefore partly occupied with investigations of this kind.

It will be convenient to have a systematic notation. We shall be concerned with functions of types denoted respectively by $P(z)$, $Q(z)$, $R(z)$, $S(z)$, $B(z)$, $C(z)$, each regular in γ. When, *e.g.*, P and $\bar{P}$ occur together, or S and $\bar{S}$, it is to be understood that P is subordinate to $\bar{P}$, S to $\bar{S}$, and so on. Further P, for example, will denote a function of a definite class; the notation is meant to imply that $\bar{P}$ also belongs to this class—$\bar{P}$ is a P. Finally we denote the coefficient of z^n in $P(z)$ by p_n, and so for Q, R, ...

We use the symbol $A(x, y, \ldots)$ to denote a positive constant depending *only* on the parameters shown explicitly, viz. x, y, ..., but not always the same at different occurrences. In particular A always denotes a positive absolute constant, but not always the same one. When we wish to preserve the identity of an A we write it A_1 or A_2, ...

2. *The functions P, Q, R.*

2.1. We denote by $P(z)$ any ("Picard") function which is regular and never takes the values 0 or 1 in γ; by $Q(z)$ any function which is regular and never takes any value $\pm 2n\pi i$ $(n = 0, 1, 2, \ldots)$ in γ; by $R(z)$ any function which is regular and never takes any value

$$-4\pi^2 n^2 \quad (n = 0, 1, 2, \ldots)$$

in γ. It is evident that the necessary and sufficient condition for a function to be a Q is that it should be of the form $\log P$, and that the necessary and sufficient condition for a function to be an R is that it should be of the form Q^2.

2.2. THEOREM 5.—*$P(z)$ is subordinate in γ to the function*

$$\bar{P}(z) = \bar{P}(z, p_0)$$

uniquely determined, for given p_0, *by the relations*

$$\bar{P}(z) = k^2(\tau), \qquad z = \frac{\tau - \tau_0}{\tau - \tau_0'},$$

$$k^2(\tau_0) = p_0,$$

$$(2.1) \qquad -1 < \Re\tau_0 \leqslant 1, \quad |\tau_0 - \tfrac{1}{2}| \geqslant \tfrac{1}{2}, \quad |\tau_0 + \tfrac{1}{2}| > \tfrac{1}{2};$$

where $k^2(\tau)$ *is the elliptic modular function, and* τ_0' *is the conjugate of* τ_0.

The substance of this theorem is well known;* we may, however, recall the proof in outline. The equation $w = k^2(\tau)$ represents the curvilinear triangle (with zero angles and one corner at $i\infty$) bounded by $\Re\tau = 1$, $I\tau \geqslant 0$; $|\tau - \frac{1}{2}| = \frac{1}{2}$, $I\tau \geqslant 0$; $\Re\tau = 0$, $I\tau \geqslant 0$, on the upper half-plane of w; hence $w = P(z)$ represents a curvilinear triangle, of arcs normal to $|z| = 1$ at the corners of the triangle, on the half w-plane. The corners of the τ and z triangles correspond, further, to the points $w = 0, 1, \infty$. It follows from Schwarz's inversion (or "symmetry") principle of continuation that $w = \bar{P}(z)$ represents γ on that Riemann surface W which has winding points of infinite order at $w = 0, 1, \infty$ (in each sheet). W has as boundary only the set of winding points. It is intuitive that, as z describes a closed contour c in γ, $P(z)$ describes a path c' lying in W and ending with the w with which it began. But since the total changes in $\arg P(z)$ and $\arg\{P(z) - 1\}$ are zero, $P(z)$ must end in the *sheet* in which it began.† It follows that $P(z)$ is subordinate to $\bar{P}(z)$. We observe that $\bar{P}$ is a P, as is required by our conventions.

THEOREM 6.—(i) $Q(z)$ *is subordinate in* γ *to the function*

$$\bar{Q}(z) = \bar{Q}(z, q_0),$$

determined by the relations

$$\bar{Q}(z) = \log k^2(\tau), \qquad z = \frac{\tau - \tau_0}{\tau - \tau_0'},$$

$$(2.22) \qquad \log k^2(\tau_0) = q_0,$$

where τ_0 *is subject to* (2.21), *and the determination of the logarithm is fixed by* (2.22).

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* Cp. *e.g.* Lindelöf, *loc. cit.*, § 12.

† This follows, more intuitively, also from the fact that c' must shrink to a point when c does.

(ii) *$R(z)$ is subordinate in γ to*

$$\bar{R}(z) = \bar{R}(z, r_0) = \{\bar{Q}(z, \sqrt{r_0})\}^2.$$

For if $p_0 = e^{q_0}$ we have, with appropriate logarithms throughout, $\bar{Q} = \log \bar{P}$ and $\bar{Q}(0) = q_0$. By Lemma 3, $Q = \log P$ is subordinate to $\bar{Q} = \log \bar{P}$. This proves the first part; the second is very similar.

2.3. Turning now to the study of the functions $\bar{P}$, $\bar{Q}$, $\bar{R}$, we have first

THEOREM 7.—

$$|\bar{Q}(z, q_0)| \leqslant \frac{A(q_0)}{1-|z|}, \tag{2.31}$$

$$|\bar{P}(z, p_0)| \leqslant \exp\left(\frac{A(p_0)}{1-|z|}\right), \tag{2.32}$$

$$|\bar{R}(z, r_0)| \leqslant \frac{A(r_0)}{(1-|z|)^2}. \tag{2.33}$$

The first result is due to Lévy;* the second and third follow at once, since $\bar{P} = \exp \bar{Q}$, $R = \bar{Q}^2$. A proof of (2.31) is given in Appendix 1, different from Lévy's, and better suited to our further ends.

Applying now the principle (1.11) or (1.53), we have

THEOREM 8.—

$$M(\rho, Q) \leqslant \frac{A(q_0)}{1-\rho}, \tag{2.34}$$

$$M(\rho, P) \leqslant \exp\left(\frac{A(p_0)}{1-\rho}\right), \tag{2.35}$$

$$M(\rho, R) \leqslant \frac{A(r_0)}{(1-\rho)^2}. \tag{2.36}$$

The second result is Schottky's sharpened form of the Picard theorem; the first gives

$$|\log P(\rho e^{i\theta})| \leqslant \frac{A(p_0)}{1-\rho},$$

the still more sharpened form due to Lévy.†

So far there is substantially nothing novel in our results, and little, if anything, in our arguments; they are applications of the known

* P. Lévy, *Bull. de la Soc. Math. de France*, 40 (1912), 25–39.

† *Loc. cit.*

principle (1 . 11), which is the case $\lambda = \infty$ of Theorem 2. Other cases of the theorem, however, lead to new results.

There is little more of interest to be said about functions P, as such, and we shall be chiefly concerned with functions $Q(=\log P)$ and R. We now enter upon a short digression preliminary to a further study of these functions.

2 . 4. If we may assume a complete knowledge of the function $\bar{f}$, the general principles (1 . 53), Theorem 3, (1 . 52), and Theorem 4 give us results respectively of the following types :—

(A) an upper bound for $|f|$;

(B) an upper bound for $|a_n|$;

(C) an upper bound for $\Sigma|a_n|^2\rho^{2n} = M_2^2(\rho, f)$;

(D) results of the type $\Pi\rho_n(a) \geqslant \Pi\bar{\rho}_n(a)$.

Suppose now that the function

$$B(z) = \Sigma b_n z^n, \tag{2 . 41}$$

regular in γ, has the property

(2 . 42) *$B(z)$ takes, in γ, no value w for which $\Re w = 0$.*

It is evident that (2 . 42) is exactly equivalent to the condition

$$\Re B(z) > 0 \quad (|z| < 1), \tag{2 . 43}$$

or to

$$\Re B(z) < 0 \quad (|z| < 1), \tag{2 . 44}$$

according as $\Re b_0 > 0$ or $\Re b_0 < 0$. Now it is well known that for functions satisfying (2 . 43) or (2 . 44) there are results of types (A) and (B). We have, in fact,

THEOREM 9.—*If B satisfies* (2 . 42), *then for* $|z| = \rho < 1$,

$$(a) \qquad |B(z)| \leqslant |b_0| + \frac{2|\beta_0|\rho}{1-\rho} \leqslant |b_0|\frac{1+\rho}{1-\rho} \quad (b_0 = \beta_0 + i\gamma_0),$$

$$(b) \qquad |b_n| \leqslant 2|\beta_0| \quad (n > 0),$$

$$(c) \qquad \Sigma|b_n|^2\rho^{2n} \leqslant |b_0|^2 + \frac{4\beta_0^2\rho^2}{1-\rho^2} \leqslant |b_0|^2\frac{1+3\rho^2}{1-\rho^2}.$$

I have included a result of type (C); the results of type (D), however, belong more naturally to section 3, and I omit them here.

The result (*a*) is a best possible one ; it is substantially a well known theorem due to Carathéodory. To prove it here we need only observe (following Rogosinski*) that B is subordinate to

$$\bar{B}(z) = b_0 + \frac{2\beta_0 z}{1-z},$$

the domain W being either $\mathfrak{R}w > 0$ or $\mathfrak{R}w < 0$. Alternatively it is an immediate consequence of (*b*), which shows, indeed, that the inequality (*a*) is satisfied, not only by $|B(z)|$, but also by its majorant

$$\mathfrak{B}(\rho) = \Sigma |b_n| \rho^n.$$

(*c*) is also an immediate consequence of (*b*). The result (*b*) itself is due to M. Riesz. An inequality $|b_n| \leqslant A|\beta_0|$ had previously been proved [as a means to a corresponding version of (*a*)], by Borel, following an idea of Hadamard.†

Consider now the functions Q. Whereas functions B have a set of "missing values" filling the whole straight line $\mathfrak{R}w = 0$, the assigned missing values of Q are only the discrete set $\pm 2n\pi i$; the B's are a very special sub-class of the Q's. None the less (2 . 34) shows that an inequality similar to (*a*) holds also for Q. I put forward the conjecture that this parallelism holds also for the results of types (B) and (C),‡ the only change being a constant multiplier $A(q_0)$ on the right-hand side of the

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* *Loc. cit.*, 262. Cp. also Lindelöf, *loc. cit.*, § 7.

† Riesz's modification of the Borel-Hadamard argument, giving the full result (*b*), is as follows. Let $\beta_0 > 0$, say,

$$B(z) = P(\rho, \theta) + iQ(\rho, \theta) \quad (P > 0).$$

We employ the well known formulae

$$\beta_0 = \frac{1}{2\pi}\int_0^{2\pi} P\, d\theta, \qquad b_n \rho^n = \frac{1}{\pi}\int_0^{2\pi} P e^{-ni\theta}\, d\theta \quad (n > 0).$$

If now $\arg b_n = \alpha$, we have

$$|b_n|\rho^n = e^{-i\alpha} b_n \rho^n = \frac{1}{\pi}\int_0^{2\pi} P e^{-ni\theta - i\alpha}\, d\theta = \frac{1}{\pi}\int_0^{2\pi} P\cos(n\theta+\alpha)\, d\theta$$

$$= 2\beta_0 - \frac{1}{\pi}\int_0^{2\pi} P\{1-\cos(n\theta+\alpha)\}\, d\theta \leqslant 2\beta_0,$$

since the integrand is never negative. Making $\rho \to 1$ in this we obtain $|b_n| \leqslant 2\beta_0$.

‡ I have not as yet investigated in detail the results of type (D) for functions Q and R.

inequalities. That is, I conjecture that

$$|q_n| < A(q_0), \tag{2.45}$$

and

$$\Sigma |q_n|^2 \rho^{2n} < \frac{A(q_0)}{1-\rho}. \tag{2.46}$$

It will be found that we can prove (2.46). Our methods are essentially incapable of proving the full result (2.45), but they do give

$$|q_n| < A(q_0)(\log n)^2 \quad (n > 1),$$

which is a considerable step in the desired direction.*

There is a corresponding parallelism between functions R and functions $C(z)$, regular in γ, with missing values filling the half line $v = 0$, $u \leqslant 0$; and here we can actually prove our case.

A function C is of the form B^2. Hence if $\mathfrak{B}(\rho)$ and $\mathfrak{C}(\rho)$ are the majorants of $B(z)$ and $C(z)$, we have

$$\mathfrak{C}(\rho) \leqslant \mathfrak{B}^2(\rho),$$

the inequality, moreover, extending to the separate powers of ρ in the two expansions. We therefore deduce from the results for $\mathfrak{B}$,

$$|C(z)| \leqslant \mathfrak{C}(\rho) \leqslant |c_0| \left(\frac{1+\rho}{1-\rho}\right)^2, \tag{2.47}$$

$$|c_n| \leqslant 4|c_0| n \quad (n > 0). \tag{2.48}$$

We have seen already (Theorem 8) that (2.47) holds also for functions R, save for a constant multiplier on the right, and we shall see presently that the same is true of (2.48).†

2.5. We require the following results about $\bar{Q}(z) = \bar{Q}(z, q_0)$.

THEOREM 10.—

$$M_1(\rho, \bar{Q}) = \frac{1}{2\pi} \int_0^{2\pi} |\bar{Q}(\rho e^{i\theta})| d\theta < A(q_0) \left(\log \frac{2}{1-\rho}\right)^2. \tag{2.51}$$

* Existing arguments lead only to $|q_n| \leqslant A(q_0) n$.

† The type (C) result, for $\Sigma |c_n|^2 \rho^{2n}$, is obtained by substitution from (2.48). But this time we can *prove* the analogue of (2.48) for R, and the parallelism in type (C) is a trivial deduction from this. [So, indeed, is that in type (A).]

THEOREM 11.—

$$M_2^2(\rho, \bar{Q}) = \Sigma |\bar{q}_n|^2 \rho^{2n} < \frac{A(q_0)}{1-\rho}. \tag{2.52}$$

The proofs are long and difficult, and quite unrelated to the rest of the paper. I have postponed them to Appendix 1.

From Theorems 3 and 10 we now have

THEOREM 12.— $$|q_n| < A(q_0)(\log n)^2 \quad (n > 1).$$

From Theorems 1 and 11 we have, again,

THEOREM 13.—

$$M_2^2(\rho, Q) = \Sigma |q_n|^2 \rho^{2n} < \frac{A_0(q)}{1-\rho}.$$

From Theorem 13 it follows, by a result due to Hardy, that

$$\mathfrak{A}(\rho) < \frac{A(q_0)}{1-\rho} \quad (\rho < 1), \tag{2.53}$$

where $\mathfrak{A}(\rho)$ is the majorant* $\Sigma |q_n| \rho^n$ of $Q(z)$. Thus the inequality for $|Q(z)|$ extends to $\mathfrak{A}(\rho)$, in accordance with the behaviour of the functions B, and Theorem 13 implies among other things a generalization, to this extent, of the result (2.34).

Finally, since $R = Q^2$, we have (with $r_0 = q_0^2$)

$$M_1(\rho, R) = M_2^2(\rho, Q) \leqslant M_2^2(\rho, \bar{Q}) < \frac{A(r_0)}{1-\rho}.$$

From this there follows, by Theorem 3,

THEOREM 14.— $$|r_n| < A(r_0)\, n \quad (n > 0).$$

3. *Functions S and $\bar{S}$.*

3.1. We define a function $\bar{S}$ to be any function $\bar{s}_1 z + \bar{s}_2 z^2 + \ldots$, regular and "schlicht"† in γ, for which $\bar{s}_0 = \bar{S}(0) = 0$. It is well known that a "schlicht" $w = f(z)$ conformally represents γ on a simply-connected

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* In fact if $1-\rho_1 = \frac{1}{2}(1-\rho)$, we have

$$\Sigma |q_n| \rho^n \leqslant \{\Sigma |q_n|^2 \rho_1^{2n} \Sigma (\rho/\rho_1)^{2n}\}^{\frac{1}{2}} < \left\{\frac{A(q_0)}{1-\rho_1} \frac{1}{\rho_1 - \rho}\right\}^{\frac{1}{2}} < \frac{A(q_0)}{1-\rho}.$$

† A function f is said to be "schlicht" in γ if $f'(z) \neq 0$ for z in γ, and $f(z_1) \neq f(z_2)$ for distinct z_1, z_2 in γ.

domain W of the w-plane which does not overlap itself (is "schlicht"). We define a function S to be any function that is subordinate to some $\bar{S}$.

We denote by Γ the set of "missing values" of $\bar{S}$ in γ.

LEMMA 4.—*The set Γ is a connected (closed) continuum extending to ∞,* and contains at least one point w for which*

$$|w| = d = \operatorname*{Min}_{|z|<1} \left| \frac{\bar{S}(z)}{z} \right| \leqslant |\bar{s}_1|.$$

The number d is the distance of Γ from $w = 0$, and is positive.†

We shall see in a moment that Γ contains at least one point. This being so, it is the complement of a domain W, not the whole w-plane, and is therefore closed and not null. It contains no bounded component Γ_1 isolated from $\Gamma - \Gamma_1$, in particular no isolated point, since W is simply-connected. Thus Γ is perfect and connected.

The function $$\frac{\bar{S}(z)}{z} = \bar{s}_1 + \bar{s}_2 z + \ldots$$

is regular and never zero in γ. Hence

$$d(\rho) = \operatorname*{Min}_{|z|=\rho} \left| \frac{\bar{S}(z)}{z} \right|$$

is a decreasing function of ρ, tending to

$$d = \operatorname*{Min}_{|z|<1} \left| \frac{\bar{S}(z)}{z} \right|$$

as $\rho \to 1$. Since $d(0) = |\bar{s}_1|$ we have $d \leqslant |\bar{s}_1|$.

By the conformal representation properties of $\bar{S}$ the curve $c(\rho)$, or $w = \bar{S}(\rho e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, is a simple closed contour containing $w = 0$, and "expands" as ρ increases. Also $w = \bar{S}(z)$ conformally represents $|z| < \rho$ on the interior of $c(\rho)$; and $c(\rho)$ contains any closed set contained in W if ρ is sufficiently near to unity. Now

$$\rho d(\rho) = \operatorname*{Min}_{|z|=\rho} |\bar{S}(z)| = \{\textit{distance of } w = 0 \textit{ from } c(\rho)\},$$

---

* That is, Γ is perfect, and contains no bounded component Γ_1 isolated from $\Gamma - \Gamma_1$. For convenience of statement we regard, *e.g.*, two curves extending to ∞ as connected "at ∞".

† The substance of the Lemma is known, but I am unable to refer to any connected account of its separate contents.

and is an increasing function of ρ evidently tending to d as $\rho \to 1$. Hence, if $|w_0| < d$, we have $\rho d(\rho) > |w_0|$ $(\rho_0 < \rho < 1)$, $c(\rho)$ contains w_0 for sufficiently large ρ, and w_0 is a point of W, therefore not a point of Γ. On the other hand, W cannot contain the whole of $|w| \leqslant d$, otherwise some $c(\rho)$ would also contain it, and we should have $\rho d(\rho) > d$, which is impossible. It follows that Γ (being closed) contains at least one point of $|w| = d$, and no point of smaller absolute value. Thus d is the distance of $w = 0$ from Γ, evidently positive since $w = 0$ is a value of $\bar{S}$, and so not a point of Γ. This completes the proof of the Lemma.

In what follows Γ will always denote a connected continuum extending to ∞ and not containing $w = 0$; $d(\Gamma)$ the distance of $w = 0$ from Γ; $\Gamma(\bar{S})$ the set of missing values of $\bar{S}$. We write also

$$d(\bar{S}) = \operatorname*{Min}_{|z|<1} \left| \frac{\bar{S}(z)}{z} \right|,$$

so that, by the Lemma, $d(\bar{S}) = d\{\Gamma(\bar{S})\}$.

3.2. THEOREM 15.—*The necessary and sufficient condition for a function, regular in γ and vanishing at $z = 0$, to be an S, is that its set of missing values should contain a Γ. When Γ is given, S is subordinate to an $\bar{S}$, uniquely determined by Γ, whose set of missing values contains Γ.*

The condition is evidently necessary, since the superordinate $\bar{S}$ satisfies it, by Lemma 4. If, on the other hand, the condition is satisfied for a given Γ, the complementary set of Γ is a sum of *simply-connected* domains, each having as complete boundary part of Γ. One of these domains, W say, contains $w = 0$. Let $w = \Sigma(z)$ give the conformal representation of γ on W with $\Sigma(0) = 0$, $\Sigma'(0)$ being, further, real and positive (say), so that Σ is uniquely determined. Γ is clearly a part of the set of missing values of Σ. Now any contour in γ beginning and ending with $z = 0$ transforms by $w = S(z)$ into a contour lying in W, and beginning and ending at $w = 0$ (otherwise the transform has a point in common with the boundary of W, and so with Γ). That is, S is subordinate to Σ. This completes the proof.

We may now abandon our original definition of functions S, and take instead: *A function f is said to be an S if $f(0) = 0$, and the set of missing values of f in γ contains a (given) set Γ.*

We may for convenience write $\Gamma(S)$ for the Γ of the definition, and $\bar{S}(\Gamma)$ for the (unique) function $\bar{S}$ of Theorem 15.

We see then that we can obtain inequalities of types (A) to (D) for the general function S of the class just defined, provided we can obtain appropriate inequalities for the general function $\bar{S}$, that is, for the general function "schlicht" in γ (and vanishing at $z = 0$).

3 . 3. THEOREM 16.— $|\bar{s}_2| \leqslant 2|\bar{s}_1|$.

THEOREM 17.— $M(\rho, \bar{S}) \leqslant \dfrac{|\bar{s}_1|\rho}{(1-\rho)^2}$.

THEOREM 18.— $d = \underset{|z|<1}{\text{Min}} \left| \dfrac{\bar{S}(z)}{z} \right| \geqslant \frac{1}{4}|\bar{s}_1|$.

There are known results. It was shown by Koebe* that

$$M(\rho, \bar{S}) \leqslant |\bar{s}_1| A(\rho)$$

and $d > A|\bar{s}_1|$, results fundamental in the theory of conformal representation. From the first follows, taking, *e.g.* $\rho = \frac{1}{2}$ in $|\bar{s}_2| \leqslant \rho^{-2} M(\rho, \bar{S})$, that $|\bar{s}_2| \leqslant c|\bar{s}_1|$, where c is an absolute constant. It was shown by Pick† that this last theorem involves

$$(3.31) \qquad M(\rho, \bar{S}) \leqslant |\bar{s}_1| \int_0^\rho \frac{(1+r)^{c-1}}{(1-r)^{c+1}}\, dr$$

so that, in particular (as is easily verified), Theorem 17 follows from Theorem 16. Theorem 16, due to Bieberbach,‡ gives the best possible value of the constant c, and every best possible inequality in the theory of "schlicht" functions, in particular (Bieberbach's) Theorem 18 ultimately depends upon it. That Theorems 16, 17, and 18 are best possible results is shown at once by the example

$$(3.32) \qquad \bar{S}_0(z) = \frac{z}{(1-z)^2},$$

a "schlicht" function for which $\bar{s}_1 = 1$ and Γ is the set of points $v = 0$, $u \leqslant -\frac{1}{4}$.

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* For proofs see Landau's book, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, § 27.

† G. Pick, *Leipziger Berichte*, 68 (1916), 58–64. Pick used the "ζ-argument", see Appendix 2 below).

‡ L. Bieberbach, *Berliner Sitzungsberichte* (1916), 940–955.

3.4. Theorem 17 solves the problem of type (A) for functions S. We have, in fact,

THEOREM 19.— $$M(\rho, S) \leqslant \frac{4d\rho}{(1-\rho)^2},$$

where $d = d(\Gamma)$.

By Theorem 15 S is subordinate to an $\bar{S}$ whose $\Gamma(\bar{S})$ contains Γ. Hence $d(\Gamma) \geqslant d\{\Gamma(\bar{S})\} = d(\bar{S}) \geqslant \frac{1}{4}|\bar{s}_1|$, by Theorem 18. Theorem 19 now follows from Theorem 17.

3.5. Turning now to type (B) we prove first

THEOREM 20.— $$M_1(\rho, \bar{S}) \leqslant \frac{|\bar{s}_1|\rho}{1-\rho},$$

$$|\bar{s}_n| < e|\bar{s}_1|n \quad (n > 0).*$$

Clearly $$\phi(z) = \left\{\frac{\bar{S}(z^2)}{\bar{s}_1}\right\}^{\frac{1}{2}} = z + b_2 z^2 + b_3 z^3 + \ldots$$

is regular and "schlicht" in γ, and in virtue of Theorem 17

$$|\phi(\rho e^{i\theta})| \leqslant \frac{\rho}{1-\rho^2}. \tag{3.51}$$

Now

$$\pi \sum_1^\infty n|b_n|^2\rho^{2n} = \int_0^\rho r dr \left(2\pi \sum_1^\infty |nb_n|^2 r^{2n-2}\right) = \int_0^\rho r dr \int_0^{2\pi} |\phi'(re^{i\theta})|^2 d\theta$$

$$= \textit{area of the transform of } |z| \leqslant \rho \textit{ by } w = \phi(z).\dagger$$

But, *since* ϕ *is "schlicht"*, this area does not exceed π times the square of $\operatorname{Max}|\phi(\rho e^{i\theta})|$ (for varying θ), and so, by (3.51), does not exceed $\pi\rho^2(1-\rho^2)^{-2}$. Hence

$$\sum_1^\infty n|b_n|^2\rho^{2n-1} \leqslant \frac{\rho}{(1-\rho^2)^2}. \tag{3.52}$$

---

* The best inequality previously known for $|\bar{s}_n|$ is $|\bar{s}_n| < \frac{1}{4}e^3|\bar{s}_1|n^2$. See L. Bieberbach, *Math. Zeitschrift*, 2 (1918), 160–170 (162 f.n.).

† For an arbitrary (not "schlicht") function ϕ this formula is valid provided portions of the w-plane covered multiply are counted multiply in the area: in the present case, of course, no portion is covered multiply. The formula is employed in various connexions by Bieberbach.

Integrating this from 0 to ρ we obtain

$$\sum_1^\infty |b_n|^2 \rho^{2n} \leqslant \frac{\rho^2}{1-\rho^2}.$$

But

$$\int_0^{2\pi} |\bar{S}(\rho^2 e^{i\psi})|\, d\psi = \int_0^{2\pi} |\bar{S}(\rho^2 e^{2i\theta})\, d\theta|$$

$$= |\bar{s}_1| \int_0^{2\pi} |\phi(\rho e^{i\theta})|^2 d\theta = 2\pi |\bar{s}_1| \sum_1^\infty |b_n|^2 \rho^{2n}.$$

Hence (3 . 52) becomes

$$2\pi M_1(\rho^2, \bar{S}) \leqslant \frac{2\pi |\bar{s}_1| \rho^2}{1-\rho^2}.$$

We may write ρ for ρ^2 in this, obtaining the first part of the theorem. The second part follows from Theorem 3 (since $\bar{S}$ is an S).

From Theorem 20 we deduce

THEOREM 21.— $$M_1(\rho, S) \leqslant \frac{4d\rho}{1-\rho},$$

$$|s_n| < 4edn \quad (n > 0),$$

where $d = d(\Gamma)$.

For $|s_n| \leqslant e|\bar{s}_1|n$, by Theorems 3 and 20, and $|\bar{s}_1| \leqslant 4d$, by Theorem 18.

The questions, whether $|\bar{s}_n| \leqslant |\bar{s}_1|n$, as is conjectured by Bieberbach, and, if so, whether $|s_n| \leqslant |\bar{s}_1|n \leqslant 4dn$, remain open; the example $\bar{S}_0(z)$ shows that more cannot be true.

We can, as it happens, dispose of the case $n = 2$.

THEOREM 22.— $$|s_2| \leqslant 2|\bar{s}_1| \leqslant 8d.$$

For, by Theorem 1,

$$s_1 z + s_2 z^2 + \ldots = S(z) = \bar{S}\{\omega(z)\} = \bar{s}_1 \omega + \bar{s}_2 \omega^2 + \ldots.$$

If $\omega(z) = \omega_1 z + \omega_2 z^2 + \ldots$ we have, comparing coefficients of z^2 and using Theorem 16,

$$|s_2| = |\bar{s}_1 \omega_2 + \bar{s}_2 \omega_1^2| \leqslant |\bar{s}_1||\omega_2| + 2|\bar{s}_1||\omega_1|^2$$

(3 . 53) $$\leqslant |\bar{s}_1|(|\omega_2| + 2|\omega_1|),$$

since $|\omega_1| \leqslant 1$. But since $|\omega_1 + \omega_2 z + \ldots| \leqslant 1$ in γ, we have, by a known

theorem (due to Fejér),* $|\sigma_n(\phi)| \leqslant 1$, where $\sigma_n(\phi)$ is the first Cesàro's mean, to n terms, of $\omega_1+\omega_2 e^{i\phi}+\ldots$. For $n=2$ this gives

$$\tfrac{1}{2}|2\omega_1+\omega_2 e^{i\phi}| \leqslant 1.$$

If now we take $\phi = \arg\omega_1 - \arg\omega_2$, we obtain

$$2|\omega_1|+|\omega_2| \leqslant 2,$$

and the theorem follows from (3 . 53).

3 . 6. Our best result of type (C) for functions S is merely the obvious consequence of Theorem 21, and need not be considered further. We turn now to type (D), which is particularly simple for functions S. We define generally for a function f

$$\varpi(a) = \varpi(a, f) = \Pi\left(\frac{1}{\rho_n(a)}\right),$$

where (as in Theorem 4) $z_1(a), z_2(a), \ldots$ are the non-zero roots, in order of increasing moduli $\rho_1(a), \rho_2(a), \ldots$, of the equation $f-a=0$, the product being taken over all ρ_n, and a product containing no factors being interpreted to be unity. $\varpi(a) < \infty$ means, of course, that the product is convergent.

THEOREM 23.—*For a function S,*

$$\varpi(a) = \varpi(a, S) \leqslant 1+2\frac{d}{|a|}+2\sqrt{\left(\frac{d}{|a|}+\frac{d^2}{|a|^2}\right)} \quad (a \neq 0), \tag{3 . 61}$$

$$\varpi(0) \leqslant \frac{|\bar{s}_1|}{|s_\nu|} \leqslant \frac{4d}{|s_\nu|}, \tag{3 . 62}$$

where s_ν is the first coefficient in $\Sigma s_n z^n$ that does not vanish. (3 . 62) *gives in particular* $|s_\nu| \leqslant |\bar{s}_1| \leqslant 4d$.

For a "schlicht" function $\bar{S}$ there is at most one $\bar{z}_n(a)$, $\bar{z}(a)$ say, when $a \neq 0$. By Theorem 4 we have

$$\prod_{n \leqslant N}\left(\frac{\rho}{\rho_n(a)}\right) \leqslant \frac{\rho}{\bar{\rho}(a)}, \tag{3 . 63}$$

where N is determined by $\rho_N(a) < \rho \leqslant \rho_{N+1}(a)$. But, by a well known argument, the left side of (3 . 63) is not increased if we alter the value

* See p. 17 of Landau's book cited above.

of N, and it follows that (3 . 63) is valid for all values of N and ρ. Making $\rho \to 1$ we have therefore, for any N,

$$\prod_{n \leqslant N} \left(\frac{1}{\rho_n(a)} \right) \leqslant \frac{1}{\bar{\rho}(a)}.$$

Hence

$$\varpi(a) \leqslant \frac{1}{\bar{\rho}(a)}. \tag{3 . 64}$$

Now by Theorem 19

$$|a| = |\bar{S}\{\bar{z}(a)\}| \leqslant \frac{|\bar{s}_1|\bar{\rho}(a)}{\{1-\bar{\rho}(a)\}^2} \leqslant \frac{4d\bar{\rho}(a)}{\{1-\bar{\rho}(a)\}^2},$$

so that

$$\frac{1}{\bar{\rho}(a)} \leqslant 1 + 2\frac{d}{|a|} + 2\sqrt{\left(\frac{d}{|a|} + \frac{d^2}{|a|^2}\right)},$$

and the first part of the theorem follows from (3 . 64).

If $a = 0$, we have $\mu = 1$, $\Pi = 1$, in the right-hand side of (1 . 72), and so

$$|s_\nu| \rho^{\nu-1} \prod_{n \leqslant N} \left(\frac{\rho}{\rho_n(0)} \right) \leqslant |\bar{s}_1| \leqslant 4d,$$

first for a special N, and then, as above, for all N and ρ. Making first $\rho \to 1$ and then $N \to \infty$ we obtain the second part.

For special functions $\bar{S}$ $\bar{\rho}(a)$ is known, and (3 . 64) gives more precise results. Consider, for example, functions $B(z)$. Here

$$\bar{B}(z) - b_0 = \frac{2\beta_0 z}{1-z},$$

which is an $\bar{S}$, $\bar{S}_1(z)$ say;

$$\bar{z}(a, \bar{B}) = \bar{z}(a - b_0, \bar{S}_1) = \frac{a-b_0}{a+b_0'} \quad (a \neq b_0),$$

and we have

$$\varpi(a, B) \leqslant \left| \frac{a+b_0'}{a-b_0} \right| \quad (a \neq b_0), \tag{3 . 65}$$

where b_0' is the conjugate of b_0. Similar results hold, *e.g.*, for functions C, and for functions f satisfying $|f| \leqslant 1$ in γ.

3 . 7. The inequalities (2 . 47) show that a set of missing values filling a half-line restricts the "order" of the function to that of $(1-\rho)^{-2}$ at

most; (2.36) shows that the same restriction is effected by a discrete set $(-4\pi^2 n^2)$ of points on a half-line.*

Finally Theorem 19 shows that the same restriction of order is effected by any Γ of missing values, for example, any Jordan curve extending to ∞.† A comparison of these results inevitably suggests that if the curve, in its turn, is replaced by a discrete string of points, with gaps not too large, the order of the function will still not exceed $(1-\rho)^{-2}$. I am unable to prove this theorem: since it combines the depth of the very special modular theorems with the generality of the "schlicht" functions, it can hardly be easy. Provided, however, that we abandon the ideal of a *best possible* power of $(1-\rho)^{-1}$ for the order, we can not only prove the suggested result, but extend it materially. It will appear, in particular, that a very sparse set of missing values is enough to reduce the order of the function below a constant power of $(1-\rho)^{-1}$.

THEOREM 24.—*Suppose we are given an integer $k \geqslant 0$, a constant $c > 1$, and an infinite sequence $w_1, w_2, \ldots$, where $|w_n| = r_n$, $r_1 > 0$, $r_n < r_{n+1} < cr_n$, and $r_n \to \infty$ as $n \to \infty$. Suppose now that, in γ, $f(z) = a_0 + a_1 z + \ldots$ is regular and takes no value w_n more than k times. Then*

$$M(\rho, f) < A(k)\, m(1-\rho)^{-h},$$

where $h = A(k)\, c(r_1+1)$, $m = \operatorname{Max}(1, |a_0|, |a_1|, \ldots, |a_k|)$.

The proof depends on

LEMMA 5.—*Suppose that, in γ, $\phi(z) = u_0 + u_1 z + \ldots$ is regular and takes neither of the values 0 and 1 more than k times. Suppose, further, that*

$$|u_n| \leqslant 1 \quad (n \leqslant k). \tag{3.71}$$

Then

$$M(\rho, \phi) < A(k, \rho), \tag{3.72}$$

$$|u_{k+1}| < A(k). \tag{3.73}$$

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* This is, we may observe in passing, in striking contrast to the effect of even a *domain* of missing values, provided the domain is *bounded*. Thus the function $\exp\left(\frac{1+z}{1-z}\right)$ has a domain $|w| < 1$ of missing values, but its order is as high as $\exp\left(\frac{A}{1-\rho}\right)$.

† A precisely similar group of results holds for the order of the n-th coefficient.

Bieberbach* has proved that, if (3 . 71) is replaced by

$$|u_n| \leqslant K_n \quad (n \leqslant k),$$

then $M(\rho, \phi) < A(K_0, K_1, \ldots, K_k, k, \rho)$. The first part of the Lemma is the particular case $K_0 = K_1 = \ldots = 1$. The second part follows at once by taking $\rho = \frac{1}{2}$ in $|u_{k+1}\rho^{k+1}| \leqslant M(\rho)$.

3 . 8. Returning now to the theorem, let $|z_0| < 1$ and

$$F(z) = f(\zeta) = f\left(\frac{z_0 - z}{1 - z_0' z}\right) = b_0 + b_1 z + \ldots,$$

where z_0' is the conjugate of z_0. Since the transformation $\zeta = \zeta(z)$ leaves the unit circle invariant, the function $F(z)$ is regular in γ, and takes no value w_n more than k times. Let

$$\mu = \mu(z_0) = r_1 + |b_0| + |b_1| + \ldots + |b_k| \geqslant r_1. \tag{3 . 81}$$

It is evident from the hypotheses about the w_n that *there exists a* $\nu > 1$ *such that*

$$2\mu \leqslant |w_\nu| \leqslant 2c\mu.$$

Thus

$$|w_\nu - w_1| \leqslant 2|w_\nu| \leqslant 4c\mu \tag{3 . 82}$$

and

$$\mu \leqslant 2\mu - r_1 \leqslant |w_\nu| - |w_1| \leqslant |w_\nu - w_1|. \tag{3 . 83}$$

Let now $$\phi(z) = \frac{F(z) - w_1}{w_\nu - w_1} = u_0 + u_1 z + \ldots.$$

Clearly ϕ takes neither of the values 0 and 1 more than k times; also, for $0 < n \leqslant k$,

$$|u_n| = \left|\frac{b_n}{w_\nu - w_1}\right| \leqslant \frac{\mu}{|w_\nu - w_1|} \leqslant 1,$$

and

$$|u_0| = \frac{|b_0 - w_1|}{|w_\nu - w_1|} \leqslant \frac{|b_0| + r_1}{\mu} \leqslant 1.$$

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* L. Bieberbach, *Math. Annalen*, 85 (1922), 141–148. A new and shorter proof is given by E. Landau, *Math. Annalen*, 86 (1922), 158–160.

Thus Lemma 5 is applicable to ϕ, and we have, from (3.73),

$$|u_{k+1}| < A(k),$$

$$(3.84) \quad |b_{k+1}| = |u_{k+1}(w_\nu - w_1)| \leqslant 4c\mu A(k) < cA(k)(r_1 + |b_0| + \ldots + |b_k|)$$

$$< h(1 + |b_0| + |b_1| + \ldots + |b_k|),$$

where we denote by h a constant of the form $A(k)\,c(r_1+1)$ (not always the same at different occurrences).

We now write $$|z_0| = \rho,$$

$$\frac{d\zeta}{dz} = -\frac{1-\rho^2}{(1-z_0'z)^2} = \lambda = \lambda(z),$$

$$\lambda_n = \lambda_n(z) = \frac{d^n\lambda(z)}{dz^n}.$$

It is easily verified that

$$(3.85) \quad F^{(m)}(z) = \lambda^m f^{(m)}(\xi) + P_{1,m}(z) f^{(m-1)}(\xi) + \ldots + P_{m-1,m}(z) f'(\xi) \quad (m > 0),$$

where $P_{n,m}(z)$ is a homogeneous polynomial in $\lambda, \lambda_1, \ldots, \lambda_n$, of degree $m-n$ in the combined variables, the coefficients depending only on m and n. In (3.85) we now take $z = 0$, $\xi = z_0$, observing that

$$\lambda_n(0) = -(n+1)!\, z_0'^n(1-\rho^2), \qquad |\lambda_n(0)| < 2(n+1)!\,(1-\rho).$$

We obtain, first for $m = k+1$,

(3.86)

$$|\lambda(0)|^{k+1}|f^{(k+1)}(z_0)| \leqslant |F^{(k+1)}(0)| + \big(|P_{1,k+1}(0) f^{(k)}(z_0)| + \ldots + |P_{k,k+1}(0) f'(z_0)|\big)$$

$$\leqslant (k+1)!\,|b_{k+1}| + A(k)\big((1-\rho)^k |f^{(k)}(z_0)|$$

$$+ (1-\rho)^{k-1}|f^{(k-1)}(z_0)| + \ldots + (1-\rho)|f'(z_0)|\big);$$

and secondly, for $0 < m \leqslant k$,

$$(3.87) \quad m!\,|b_m| = |F^{(m)}(0)|$$

$$\leqslant |\lambda(0)|^m |f^{(m)}(z_0)| + |P_{1,m}(0) f^{(m-1)}(z_0)| + \ldots + |P_{m-1,m}(0) f'(z_0)|$$

$$< A(m)\big((1-\rho)^m |f^{(m)}(z_0)| + (1-\rho)^{m-1}|f^{(m-1)}(z_0)| + \ldots + (1-\rho)|f'(z_0)|\big).$$

From (3 . 84) and (3 . 87), observing that $b_0 = f(z_0)$, we obtain

$$|b_{k+1}| < h(1+|b_0|+\ldots+|b_k|)$$

$$< h\Big((1-\rho)^k|f^{(k)}(z_0)|+\ldots+(1-\rho)|f'(z_0)|+|f(z_0)|+1\Big).$$

Substituting this in (3 . 86) we have finally

$$(3\,.\,88)\quad (1-\rho)^{k+1}|f^{(k+1)}(z_0)| \leqslant (1-\rho^2)^{k+1}|f^{(k+1)}(z_0)| = |\lambda(0)|^{k+1}|f^{(k+1)}(z_0)|$$

$$< h_1\Big((1-\rho)^k|f^{(k)}(z_0)|+\ldots+(1-\rho)|f'(z_0)|+|f(z_0)|+1\Big) = h_1 T,$$

say. Now let $\psi(\rho) = (1-\rho)^{h_1+1}T$. Since*

$$\frac{d}{d\rho}|f^{(n)}(z_0)| \leqslant |f^{(n+1)}(z_0)|,$$

we have

$$\frac{d}{d\rho}\psi(\rho) = (1-\rho)^{h_1+1}\sum_{n=0}^{k}(1-\rho)^n\frac{d}{d\rho}|f^{(n)}(z_0)|$$

$$-\sum_{n=0}^{k}(h_1+1+n)(1-\rho)^{h_1+n}|f^{(n)}(z_0)|-(h_1+1)(1-\rho)^{h_1}$$

$$\leqslant (1-\rho)^{h_1}\sum_{n=1}^{k+1}(1-\rho)^n|f^{(n)}(z_0)|-(h_1+1)(1-\rho)^{h_1}\Big(\sum_{n=0}^{k}(1-\rho)^n|f^{(n)}(z_0)|+1\Big)$$

$$\leqslant (1-\rho)^{h_1}\Big\{(1-\rho)^{k+1}|f^{(k+1)}(z_0)|-h_1\Big(\sum_{n=0}^{k}(1-\rho)^n|f^{(n)}(z_0)|+1\Big)\Big\}$$

$$\leqslant 0,$$

by (3 . 88). Hence

$$\psi(\rho) = \int_0^\rho \psi'(\rho)\,d\rho+\psi(0) \leqslant \psi(0)$$

$$= \sum_{n=0}^{k}|f^{(n)}(0)|+1 < A(k)(1+|a_0|+\ldots+|a_k|) < A(k)m,$$

and so finally $(1-\rho)^{h_1+1}|f(z_0)| < \psi(\rho) < A(k)\,m,$

which is an inequality of the desired form.

It is not without interest to examine the special case of functions $Q(z)$, for which $k=0$, and the set of w's is the set $(\pm 2n\pi i)$. We start

* $\frac{d}{d\rho}|\phi(z_0)|$ is used to mean $\frac{\partial}{\partial\rho}|\phi(\rho e^{i\theta})|$, where $z_0 = \rho e^{i\theta}$.

from the theorem :*

$$(A) \qquad \textit{If} \quad |p_0| < \varpi, \quad \textit{then} \quad M(\rho, P) < A(\rho, \varpi),$$

or from its immediate consequence :

$$(B) \qquad \textit{If} \quad |p_0| < 1, \quad \textit{then} \quad |p_1| < A.$$

Consider now $Q(z)$: the general argument reduces to the following form :—Let $|z_0| = \rho$, $\zeta = (z-z_0)/(1-zz_0')$; let n be the integer for which $2(n-1)\pi \leqslant |Q(z_0)| < 2n\pi$, and let

$$\phi(z) = \frac{Q(\zeta)}{2n\pi i} = b_0 + b_1 z + \ldots.$$

Evidently $|b_0| < 1$. Since $\phi \neq 0, 1$, we have, by Theorem (B),

$$|b_1| < A,$$

$$(1-\rho^2)|Q'(z_0)| < 2n\pi A < A(|Q(z_0)|+1),$$

$$\frac{d}{d\rho} \log (|Q(z_0)|+1) < \frac{A}{1-\rho^2} < \frac{A}{1-\rho},$$

whence, integrating from 0 to ρ,

$$|Q(z_0)| < |Q(z_0)|+1 < (1-\rho)^{-A}\big(|Q(0)|+1\big) = (1+|q_0|)(1-\rho)^{-A},$$

a result of the type of (2.31), but without the best possible index. Two points are noteworthy. First that, starting from (A) with a quite undetermined function $A(\rho, \varpi)$, we deduce the result (A) (sharpened to apply to $\log P$ in place of P) with a *definite* form of $A(\rho, \varpi)$ not far from the best possible one. Secondly that, if we retrace our argument to first principles, it is seen that we use no modular theory properly so-called, but appeal only to the existence of the function representing a curvilinear triangle on a half-plane (and the inversion principle).

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* We use the concise notation of § 2.1. Theorem (A), though more general than the theorem

$$M(\rho, P) \leqslant M(\rho, \bar{P}) = A(\rho, p_0),$$

is deducible from it without very serious trouble.

3 . 9. I conclude by noting one or two results more or less connected with the work of § 3 . 8.

THEOREM 25.—*Let* $f(z) = z+a_2z^2+\ldots$ *be regular and "schlicht" in* γ, *and let* β *be one of its missing values. Then, for* $|z| = \rho < 1$, *we have*

$$\left|\frac{f'(z)}{f(z)-\beta}\right| < \frac{A}{1-\rho}, \quad \left|\log\left(1-\frac{f(z)}{\beta}\right)\right| A \leqslant \log\left(\frac{1}{1-\rho}\right).*$$

Let $|z_0| = \rho$. Evidently *there exists a point* β_1 *on the* Γ *of* f, *such that*

$$\left|\frac{f(z_0)-\beta}{\beta_1-\beta}\right| = 1.$$

Let $$\zeta = \frac{z_0-z}{1-z_0'z}, \quad \phi(z) = \frac{f(\zeta)-\beta}{\beta_1-\beta} = b_0+b_1z+\ldots.$$

Clearly ϕ is regular and never 0 or 1 in γ, and

$$|b_0| = \left|\frac{f(z_0)-\beta}{\beta_1-\beta}\right| = 1 \leqslant 1.$$

It follows, by Lemma 5†, that $|b_1| < A$, or

$$\left|\frac{-(1-\rho^2)f'(z_0)}{\beta_1-\beta}\right| < A = A\left|\frac{f(z_0)-\beta}{\beta_1-\beta}\right|,$$

$$\left|\frac{(1-\rho^2)f'(z_0)}{f(z_0)-\beta}\right| < A, \tag{3 . 91}$$

and the first part of the theorem follows at once. Since

$$\frac{d}{d\rho}\left|\log\left(1-\frac{f(z_0)}{\beta}\right)\right| \leqslant \left|\frac{d}{dz_0}\log\left(1-\frac{f(z_0)}{\beta}\right)\right| = \left|\frac{f'(z_0)}{f(z_0)-\beta}\right| < \frac{A}{1-\rho},$$

the second part follows at once by integration with respect to ρ.

An alternative proof of the first part is as follows :—

$$\psi(z) = -\frac{f(\zeta)-f(z_0)}{(1-\rho^2)f'(z_0)} = z+b_2z^2+\ldots$$

is regular and "schlicht" in γ, and has the missing value

$$\beta' = -\frac{\beta-f(z_0)}{(1-\rho^2)f'(z_0)}.$$

---

* The logarithm on the left is supposed to have its principal value, viz. 0, at $z = 0$; it is, of course, regular in γ.

† Or by Landau's well known theorem.

Hence $$\left|\frac{(1-\rho^2)f'(z_0)}{\beta-f(z_0)}\right| = \frac{1}{|\beta'|} \leqslant \frac{1}{d(\psi)} \leqslant 4,$$

by Theorem 18, and this is (3 . 91) (with $A = 4$).

From Theorem 25 we can deduce

THEOREM 26.—*Let* $f(z) = z+a_2z^2+\ldots$ *be regular and "schlicht" in* γ. *Then, for* $|z| = \rho < 1$,

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{A}{1-\rho}, \quad \left|\log\frac{f(z)}{z}\right| \leqslant A\log\frac{1}{1-\rho}.$$

By Lemma 4 there exists a β for which $|\beta| \leqslant |\bar{s}_1| = 1$. Thus

$$\left|\frac{zf'}{f}\right| = \left|z-\frac{\beta z}{f}\right|\left|\frac{f'}{f-\beta}\right|.$$

The second factor on the right is less than $A/(1-\rho)$, by Theorem 25. The first does not exceed

$$1+\left|\frac{z}{f}\right| \leqslant 1+\frac{1}{d(f)} \leqslant 1+4 = A,$$

by Lemma 4 and Theorem 18, and this proves the first part. Now

$$\chi(z) = \frac{f'(z)}{f(z)} - \frac{1}{z}$$

is regular in γ. Also

$$|z\chi| \leqslant \left|\frac{zf'}{f}\right| + 1 < \frac{A}{1-\rho}.$$

It follows that* $$|\chi| < \frac{A}{1-\rho}.$$

Hence $$\frac{d}{d\rho}\left|\log\frac{f}{z}\right| \leqslant \left|\frac{d}{dz}\log\frac{f}{z}\right| = |\chi(z)| < \frac{A}{1-\rho},$$

and the second part follows by integration.

In Theorem 26 the constant furnished by our argument is not explicitly determined; the best possible theorem is likely to be difficult.

---

* The deduction is trivial for $\rho \geqslant \frac{1}{2}$; and from

$$|\chi| < \frac{A}{1-\rho} < A \quad (\rho = \tfrac{1}{2}),$$

follows

$$|\chi| < A < \frac{A}{1-\rho} \quad (\rho < \tfrac{1}{2}).$$

4. *Appendix* 1.—*Proof of Theorems* 7, 10, *and* 11.

4.1. These are theorems concerning the modular function $k^2(\tau)$. Our notation is that of Tannery and Molk, except that we write $\vartheta(q)$* sometimes for $\vartheta(0\,|\,\tau)$.

LEMMA 6.—

(i) $$q\frac{d}{dq}(\log k^2) = \vartheta_4^4 = \vartheta_4^4(0\,|\,\tau).$$

(ii) *There exists a unique determination* $\kappa(\tau)$ *of* $\log k^2(\tau)$, *such that*

$$\kappa(\tau) = \pi i\tau + \phi(q) + 4\log 2, \tag{4.11}$$

where

$$\phi(q) = \int_0^q \frac{\vartheta_4^4 - 1}{q}\,dq, \tag{4.12}$$

and is regular in $|q|<1$. *Also*

$$\kappa'(\tau) = \pi i\vartheta_4^4. \tag{4.13}$$

The result (i) is known,† and (ii) follows without difficulty, since $k^2 \sim 16q$ as $q \to 0$.

4.2. We have

$$\bar{Q}(z) = \bar{Q}(z, q_0) = q_0 - \kappa(\tau_0) + \kappa(\tau), \quad z = \frac{\tau-\tau_0}{\tau-\tau_0'}, \tag{4.21}$$

since $\bar{Q}$ has the value q_0 at $z=0$.

We write $$\rho = e^{-\eta}, \quad \tau = u+iv, \quad \tau_0 = u_0+iv_0,$$

so that $|u_0| \leqslant 1$, and we denote for brevity constants $A(q_0)$ or $A(\tau_0)$ by B. The results of Theorems 7, 10, and 11 are trivial if η is greater than a B, and we may suppose any convenient inequality $\eta < B$ (in particular $\eta<1$), and any convenient inequality $1 > \rho > B$, to be satisfied.

Since

$$\tau = \frac{\tau_0 - \tau_0' z}{1-z} = \tau_0' + \frac{2iv_0}{1-z}, \tag{4.22}$$

---

* Of course q here means $e^{\pi i\tau}$.

† See, *e.g.*, Weber, *Ell. Funkt.* (2 ed.), 168, l. 4.

it follows from (4 . 11) that it is enough, in proving Theorem 7, to show that

$$|\phi(q)| < \frac{B}{\eta} \quad (|z| = \rho). \tag{4 . 23}$$

Now $$\vartheta_4 = 1 + 2\sum_1^\infty (-)^n q^{n^2},$$

$$\left|\frac{\vartheta_4^4 - 1}{q}\right| \leqslant \frac{(1+2\Sigma|q|^{n^2})^4 - 1}{|q|} = \frac{\vartheta_3^4(0|iv)-1}{|q|} = \psi(|q|) = \psi(e^{-\pi v}),$$

say. If $|q| \leqslant \frac{1}{2}$, we have $|\phi(q)| < B$. If $|q| > \frac{1}{2}$, $q = |q|e^{i\psi}$, we have

$$|\phi(q)| \leqslant |\phi(\tfrac{1}{2}e^{i\psi})| + \left|\int_{\frac{1}{2}e^{i\psi}}^{q} \frac{\vartheta_4^4 - 1}{q}\, dq\right| < B + \int_{\frac{1}{2}}^{|q|} \psi(r)\, dr$$

$$< B + B\int_{\frac{1}{2}}^{|q|} \{\vartheta_3^4(r) - 1\}\, dr < B + B\int_v^B \vartheta_3^4(0|iv)\, dv.*$$

Since in the last integral

$$|\vartheta_3(0|iv)| = \left|\sqrt{\left(\frac{i}{iv}\right)}\, \vartheta_3\left(0\middle| -\frac{1}{iv}\right)\right| < \frac{B}{\sqrt{v}},$$

we have

$$|\phi(q)| < B + Bv^{-1}$$

$$< B\eta^{-1},$$

since (as is easily verified by geometrical considerations) v_1, the minimum value of v for $|z| = \rho$, occurs when $u = u_0$, and then

$$-\rho = \frac{v - v_0}{v + v_0}, \qquad v = v_0\frac{1-\rho}{1+\rho} > B(1-\rho) > B\eta.$$

This completes the proof of Theorem 7.

4 . 3. *Proof of Theorem* 10.—We reduce Theorem 10 successively to simpler propositions. In the first place it is enough to prove that

$$\int_0^{2\pi} |\bar{Q}'(z)|\, d\theta < \frac{B}{1-\rho} \log\left(\frac{2}{1-\rho}\right).\dagger \tag{4 . 31}$$

---

* Since $d|q| = -\pi|q|\, dv < A|dv|$.

† It follows at once from (4 . 31) (by the method of Theorem 3) that

$$|\bar{q}_n| < B\log n \quad (n > 1).$$

For assuming the truth of (4 . 31),

$$\int_0^{2\pi} |\bar{Q}(z)|\,d\theta = \int_0^{2\pi} \left| \int_0^{\rho} \bar{Q}'(re^{i\theta})\, e^{i\theta}\, dr + q_0 \right| d\theta$$

$$< \int_0^{2\pi} d\theta \int_0^{\rho} |\bar{Q}'(re^{i\theta})|\,dr + B = \int_0^{\rho} dr \int_0^{2\pi} |\bar{Q}'(re^{i\theta})|\,d\theta + B$$

$$< \int_0^{\rho} dr\, \frac{B}{1-r} \log\left(\frac{2}{1-r}\right) + B$$

$$< B \log\left(\frac{2}{1-\rho}\right) \int_0^{\rho} \frac{dr}{1-r} + B < B \left\{ \log\left(\frac{2}{1-\rho}\right) \right\}^2,$$

the desired result.

Next we have, by Lemma 6 (if $\rho > \frac{1}{2}$, say),

$$|\bar{Q}'(z)\,d\theta| \leqslant B\,|\bar{Q}'(z)\,dz| = B\,|\kappa'(\tau)\,d\tau|.$$

It is therefore enough to prove

$$\int |\kappa'(\tau)\,d\tau| < \frac{B}{\eta} \log\left(\frac{1}{\eta} + 2\right), \tag{4 . 32}$$

where τ is given by (4 . 22), and $|z| = \rho$. Moreover, we may suppose $\eta < \eta_0$, where η_0 is any convenient B.

4 . 4. Corresponding to the circumference $|z| = \rho$ there is in the τ-plane a circumference K, with radius R lying between $B_1\eta^{-1}$ and $B_2\eta^{-1}$, and with centre lying in $|u| \leqslant 1$, $v > v_0$. Let K' be the part of the circumference for which $v \leqslant v_0$, K'' the remainder. Consider the strips U_m or $|u - 2m| \leqslant 1$. Let μ be the greatest $|m|$ for which K' has points in common with U_m. If η_0 is small enough, (i) K' cuts the line $v = v_0$ at an angle less than, say, $\frac{1}{4}\pi$, and (ii) for $|m| \leqslant \mu$ U_m intercepts two distinct arcs on K. Call the lower of these arcs K_m. The following results are easily verified. Let $v_m = \eta(1+m^2)$. Then, for τ on the arc K_m and for $|m| \leqslant \mu$,

v/v_m *lies between bounds* B_3 *and* B_4; (4 . 41)

$$B(1+m^2) > |\tau - \tau_0'|^2 > B(1+m^2); \tag{4 . 42}$$

$$\left| \frac{d\tau}{du} \right| < B; \tag{4 . 43}$$

$$\mu < B\eta^{-\frac{1}{2}}. \tag{4 . 44}$$

4.5. We have now

$$\int |\kappa'(\tau)\,d\tau| \leqslant B\int_{K'} |\kappa'(\tau)|\,du + B\int_{K''} |\kappa'(\tau)\,d\tau|.$$

On K'' $\qquad \Im\tau > B, \quad |q| < 1-B, \quad |\vartheta_4| < B,$

and so $\qquad |\kappa'(\tau)| < B,$

$$\int_{K''} < B\,(\textit{radius of } K) < \frac{B}{\eta}. \tag{4.51}$$

Also, by (4.42),

$$\int_{K'} \leqslant B \sum_{m=-\mu}^{\mu} \int_{K_m} |\kappa'(\tau)|\,du = B\sum_{-\mu}^{\mu} I_m, \tag{4.52}$$

say. Now on K_m

$$|u-2m| \leqslant 1, \quad v = v_m\lambda, \quad B_3 < \lambda < B_4.$$

Hence, writing $u_1 = u-2m$,

$$I_m = \int_{-1}^{1} \left|\kappa'\left(u_1+2m+iv_m\lambda(u_1)\right)\right| du_1 = \int_{-1}^{1} \left|\kappa'\left(u+iv_m\lambda(u)\right)\right| du \tag{4.53}$$

(since κ' has period 2), where

$$B_3 < \lambda(u) < B_4. \tag{4.54}$$

Our proof will be completed if we show that

$$\int_{-1}^{1} \left|\kappa'\left(u+iv_m\lambda(u)\right)\right| du < \frac{B}{v_m}\log\left(\frac{1}{v_m}+2\right) \tag{4.55}$$

for *every* continuous function $\lambda(u)$ satisfying (4.54). For then

$$I_m < \frac{B}{v_m}\log\left(\frac{1}{\eta}+2\right) < \frac{B}{\eta}\log\left(\frac{1}{\eta}+2\right)\frac{1}{1+m^2},$$

$$\sum_{m=-\mu}^{\mu} I_m < \frac{B}{\eta}\log\left(\frac{1}{\eta}+2\right)\sum_{m=-\infty}^{\infty}\frac{1}{1+m^2} = \frac{B}{\eta}\log\left(\frac{1}{\eta}+2\right).$$

and the desired conclusion follows from (4.51) and (4.52). Now in the proposition we are to prove m occurs only in v_m, which we may replace in (4.55) by a number h, known to be *less* than some B. The desired result is then trivial if $h > 1$, on account of $h\lambda(u) > B$. Also, since

$$|\kappa'| \leqslant |\vartheta_3|^4 + |\vartheta_4|^4$$

and the right-hand side has period 1, we see finally that it is enough to prove

$$J=\int_0^1(|\vartheta_3|^4+|\vartheta_4|^4)\,du<\frac{B}{h}\log\left(\frac{1}{h}+2\right)\quad(0<h<1), \tag{4.56}$$

where $$\tau=u+ih\lambda(u),\quad B<\lambda<B.$$

4.6. This requires a "Farey dissection." Let $N=[h^{-\frac{1}{2}}]+1$. With each q satisfying $1<q\leqslant N$ we associate the positive integers p less than and prime to q; with $q=1$ we associate $p=0$, $p=1$. Let $\xi_{p,q}$ denote the interval

$$\left|x-\frac{p}{q}\right|\leqslant\frac{1}{qN}.$$

Then it is known that the set of $\xi_{p,q}$, for $q\leqslant N$ and the associated p, covers the interval (0, 1). When u belongs to $\xi_{p,q}$, we apply to τ the substitution

$$\mathbf{T}=S_{p,q}\tau=\frac{-p_1+q_1\tau}{-p+q\tau}=\frac{q_1}{q}-\frac{1}{q^2\{\tau-(p/q)\}}, \tag{4.61}$$

where p_1, q_1 are the (positive or negative) integers, in absolute value not exceeding q, for which $p_1q-pq_1=1$.*

Then

$$\Im\mathbf{T}=-\Im\frac{1}{q^2\{\tau-(p/q)\}}=\frac{h\lambda}{q^2[\{u-(p/q)\}^2+h^2\lambda^2]}>\frac{hB}{q^2(q^{-2}N^{-2}+h^2B)}$$

$$>\frac{BhN^2}{1+Bhq^2.hN^2}>B,$$

since $hq^2\leqslant hN^2<B$. Hence

$$|\vartheta_3(0|\tau)|^4+|\vartheta_4(0|\tau)|^4=\left|\frac{\vartheta_i(0|\mathbf{T})}{\sqrt{(a+b\tau)}}\right|^4+\left|\frac{\vartheta_j(0|\mathbf{T})}{\sqrt{(a+b\tau)}}\right|^4\dagger$$

$$<\frac{B}{|-p+q\tau|^2}=\frac{B}{q^2}\frac{1}{\{u-(p/q)\}^2+h^2\lambda^2}<\frac{B}{q^2}\frac{1}{\{u-(p/q)\}^2+Bh^2}.$$

$$\int_{\xi_{p,q}}|\kappa'(\tau)|\,du<\frac{B}{q^2}\int_{-\infty}^{\infty}\frac{dx}{x^2+Bh^2}<\frac{B}{q^2h},$$

$$J\leqslant\sum_{q\leqslant N}\sum_{p}\int_{\xi_{p,q}}<\frac{B}{h}\sum_{q=1}^{N}\frac{1}{q^2}q<\frac{B}{h}\log(N+2)<\frac{B}{h}\log\left(\frac{1}{h}+2\right).$$

This completes the proof.

---

* For $p=0$, $q=1$ we have $p_1=1$ and q_1 arbitrary: we may choose $q_1=0$.

† i and j are each one of 1, 2, 3, 4. See Tannery and Molk, Table (XLII).

4.7. *Proof of Theorem* 11.—It is enough to prove

$$\int_0^{2\pi} |\bar{Q}'(z)|^2 d\theta < \frac{B}{\eta^3} \quad (\eta < 1). \tag{4.71}$$

For then

$$\mu(r) = \sum_{n=0}^{\infty} |n\bar{q}_n|^2 r^{2(n-1)} = \int_0^{2\pi} |\bar{Q}'(re^{i\theta})|^2 d\theta < \frac{B}{(1-r)^3},$$

$$M_2^2(\rho, \bar{Q}) = \sum_{n=0}^{\infty} |\bar{q}_n|^2 \rho^{2n} \leqslant |\bar{q}_0|^2 + 4\Sigma \frac{|n\bar{q}_n|^2}{2n(2n-1)} \rho^{2n}$$

$$= |\bar{q}_0|^2 + 4\int_0^{\rho} dr \int_0^{r} \mu(r_1)\, dr_1$$

$$< B + \frac{B}{1-\rho} < \frac{B}{1-\rho}.$$

Now (for $\eta < 1$)

$$\left|\frac{d\tau}{dz}\right| = \frac{B}{|1-z|^2} < B|\tau - \tau_0'|^2,$$

$$|\bar{Q}'(z)|^2 d\theta < A|\bar{Q}'(z)|^2 |dz| = B|\kappa'(\tau)|^2 \left|\frac{d\tau}{dz}\, d\tau\right| < B|\kappa'(\tau)|^2 |(\tau-\tau_0')^2\, d\tau|,$$

$$\int_0^{2\pi} |\bar{Q}'(z)|^2 d\theta < \int_{K'+K''} |\kappa'(\tau)|^2 |(\tau-\tau_0')^2\, d\tau|.$$

Since $$\int_{K''} < B\int |\tau-\tau_0'|^2 |d\tau| < B\int \left(\frac{B}{\eta}\right)^2 |d\tau| < \frac{B}{\eta^3},$$

it is enough to show that $$\int_{K'} < \frac{B}{\eta^3}.$$

Now

$$\int_{K'} = \sum_{-\mu}^{\mu} \int_{K_m} B|\vartheta_4|^8 |\tau-\tau_0'|^2 |d\tau| < B\sum_{-\mu}^{\mu} \int_{2m-1}^{2m+1} \{|\vartheta_3|^8 + |\vartheta_4|^8\} (1+m^2)^2\, du$$

$$< B\sum_{-\mu}^{\mu} (1+m^2)^2 \int_0^1 \psi\{u + i\lambda(u) v_m\}\, du,$$

where $$B < \lambda = \lambda(u) < B,$$

and $$\psi(\tau) = |\vartheta_3|^8 + |\vartheta_4|^8.$$

It is therefore sufficient to show that for all continuous $\lambda(u)$ satisfying $B < \lambda < B$, and $0 < h < 1$,

$$\int_0^1 \psi(\tau)\, du < \frac{B}{h^3} \quad [\tau = u + i\lambda(u)\, h], \tag{4.72}$$

since then

$$\int_{K'} < B \sum_{-\mu}^{\mu} (1+m^2)^2 v_m^{-3} < B\eta^{-3} \sum_{-\infty}^{\infty} \frac{1}{1+m^2} = \frac{B}{\eta^3}.$$

To establish (4.72) we use the Farey dissection and the argument of (4.6), which give

$$\int_0^1 \psi(\tau)\, du \leqslant \sum_{p,q} \int_{\xi_{p,q}} \frac{B}{|-p+q\tau|^4}\, du \leqslant B \sum_{p,q} \frac{1}{q^4} \int_{\xi_{p,q}} \frac{du}{\left[\{u-(p/q)\}^2+h^2\lambda^2\right]^2}$$

$$\leqslant \sum_{p,q} \frac{B}{q^4} \int_{-\infty}^{\infty} \frac{du}{\left[\{(u-(p/q)\}^2+Bh^2\right]^2} = \sum_{p,q} \frac{B}{q^4} \frac{1}{h^3} < \frac{B}{h^3},$$

the desired result.

5. *Appendix* 2.

5.1. I conclude with some general remarks, with the chief end of elucidating the relations between (Rogosinski's) Theorem 1 and the closely connected principles of Lindelöf.

The applications in the present paper may be said to deal with the problem of the influence of missing values (in a generalized sense) of a function on the order of the function and its coefficients (and zeros). We are (in the applications) dealing with functions $f(z)$ whose "Gesamtverlauf" is confined to a *given* Riemann surface W.* Since the transformation $\zeta = (z-z_0)/(1-zz_0')$ ($|z_0|<1$) leaves γ invariant, *the function* $f(\zeta)$ *also has this relation to* W. This fact provides us with a powerful argument, and we have often used it. Thus any inequality necessarily holding between, say, the first two coefficients of $f(z)$ translates itself into one between $-(1-\rho^2)f'(z_0)$ and $f(z_0)$; we can, roughly speaking, transfer, by the "ζ-argument," properties of the origin to any point z_0.

Lindelöf explicitly confines himself to functions $f(z)$ of the type just explained; this enables him to import consequences of the ζ-argument into the general principles with which he begins.† These general principles are, in fact, substantially equivalent to Theorem 1 combined with the *use* of the ζ-argument. There are further but minor differences. Lindelöf contemplates a general domain X in place of the more special γ, and he states his results and arguments in terms of Green's functions. When we specialize X his principle, expressed in the language

* Independent, for example, of a_0.

† I am regarding as of principal importance the cases in which Lindelöf's domain X is a circle.

of the complex variable, becomes Theorem 1, with its advantages of simplicity and directness. The group of properties peculiar to functions whose domain of existence is a circle is, moreover, an important one, and Theorem 1 is admirably adapted to its study. Whether it is possible to claim in addition that Theorem 1 can replace the general principle in all circumstances is perhaps less certain.

To prevent a possible confusion, I should perhaps observe finally that the class of functions f related in the manner described above to some (undetermined) surface W is not the same conception as a class of functions subordinate to some function $\bar{f}$, but a rather more complex one. It would have been possible (by naming it) to bring the former conception into the greater prominence, but I have preferred the course actually adopted.

Postscript. (*Added November 11th*, 1924.)—The following results are supplementary to different parts of the paper.

(i) [2 . 5]. It may be shown that the means $M_\lambda(\rho)$ behave differently for the functions $\bar{B}$, $\bar{Q}$ when $0 < \lambda < 1$. In the first place it is easily seen that

$$M_\lambda(\rho, \bar{B}) \leqslant A(l, b_0) \quad (0 \leqslant \lambda \leqslant l < 1). \tag{2 . 54}$$

On the other hand

$$M_\lambda(\rho, \bar{Q}) \neq O(1) \quad (\lambda > 0), \tag{2 . 55}$$

as $\rho \to 1$. A direct proof of (2 . 55) is difficult, but we may argue indirectly as follows. If the mean is bounded (for any particular λ) it follows, by a theorem of F. Riesz, that $\bar{Q}$ tends almost always to a finite limit as z tends radially to the circumference of γ; further, that this limit cannot have the same value in any set of positive measure, and therefore differs almost always from all the values of any given denumerable set, in particular differs almost always from all numbers $\pm 2n\pi i$. Then $\bar{P} = e^{\bar{Q}}$ tends almost always to a finite limit other than 0 or 1. Now the interior of γ can be divided into fundamental regions for the function $\bar{P}$. These are curvilinear triangles with corners on $|z| = 1$, and the sides are of three types, corresponding to (real) values of w between $-\infty$ and 0, 0 and 1, 1 and $+\infty$ respectively. A radius vector crosses in general an infinity of sides of at least two types, it follows that the limit of $\bar{P}$ can in general only be 0, 1, or ∞, and we arrive at a contradiction.

It is not without interest to observe that we *can* prove

$$\int_0^{2\pi} |\Re \bar{Q}(\rho e^{i\theta})|^\lambda d\theta < A(l, q_0) \quad (0 \leqslant \lambda \leqslant l < 1), \tag{2.56}$$

the argument being very similar to that of Appendix 1.

(ii) [Theorem 20]. The argument of Theorem 20 extends to prove:

$$M_\lambda^\lambda(\rho, S) \leqslant M_\lambda^\lambda(\rho, \bar{S}) \leqslant 2\lambda\pi |\bar{s}_1|^\lambda \int_0^\rho r^{\lambda-1}(1-r)^{-2\lambda} dr \quad (\lambda > 0). \tag{3.54}$$

In particular

$$M_\lambda^\lambda(\rho, S) \leqslant M_\lambda^\lambda(\rho, \bar{S}) < A(l) |\bar{s}_1|^\lambda \quad (0 \leqslant \lambda \leqslant l < \tfrac{1}{2}). \tag{3.55}$$

We may suppose $\bar{s}_1 = 1$. Let p and q be positive integers, and let

$$\psi(z) = \{\bar{S}(z^{2q})\}^{1/2q}.$$

ψ is regular and "schlicht" in γ, and

$$\phi = \psi^p = z^p + b_{p+1} z^{p+1} + \ldots$$

is regular, and takes no value more than p times, in γ. Hence

$$\pi \Sigma n |b_n|^2 \rho^{2n} = \textit{area of the transform of } |z| \leqslant \rho \textit{ by } w = \phi(z)$$

$$\leqslant p\pi \{\operatorname{Max} |\phi(\rho e^{i\theta})|\}^2 \leqslant p\pi\rho^{2p}(1-\rho^{2q})^{-(2p/q)},$$

$$\int_0^{2\pi} |\bar{S}(z^{2q})|^{p/q} d\theta = \int_0^{2\pi} |\phi(z)|^2 d\theta = 2\pi\Sigma |b_n|^2 \rho^{2n}$$

$$\leqslant 4p\pi \int_0^\rho \frac{dr}{r} r^{2p}(1-r^{2q})^{-(2p/q)},$$

$$\int_0^{2\pi} |\bar{S}(Re^{i\phi})|^{p/q} d\phi = \int_0^{2\pi} |\bar{S}(Re^{2qi\theta})|^{p/q} d\theta \leqslant 4p\pi \int_0^R \frac{dR}{2qR} R^{p/q}(1-R)^{-(2p/q)}.$$

In this we write ρ for R and make $p/q \to \lambda$, obtaining (3.54).

It follows from (3.55) and the theorem of F. Riesz mentioned above that a function S tends almost always to a limit as z tends radially to the circumference of γ.

(iii) [3.5].

THEOREM.—*The perimeter of the transform of* $|z| \leqslant \rho$ *by* $w = \bar{S}(z)$ *does not exceed*

$$A|\bar{s}_1|\frac{\rho}{(1-\rho)^2}.^*$$

The perimeter being $\int_0^{2\pi} |\bar{S}'(\rho e^{i\theta})|\rho d\theta$, our result follows from Theorem 20 and the following lemma :—

If $f(z)$ *is regular in* γ, *and*

$$M_1(\rho, f) \leqslant K(1-\rho)^{-\alpha} \quad (\alpha \geqslant 0),$$

then

$$M_1(\rho, f') \leqslant A(\alpha)K(1-\rho)^{-\alpha-1}.\dagger$$

Let $1-\rho_1 = \frac{1}{2}(1-\rho)$, $x = \rho e^{i\theta}$, $z = \rho_1 e^{i\phi}$. We have

$$\int_0^{2\pi} |f'(x)|d\theta = \int_0^{2\pi} d\theta \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z)}{(z-x)^2} izd\phi \right| \leqslant A\int_0^{2\pi} d\phi |f(z)| \int_0^{2\pi} \frac{d\theta}{|z-x|^2}$$

$$\leqslant A\int_0^{2\pi} d\phi |f(z)| \frac{A}{\rho_1-\rho} \leqslant \frac{A}{1-\rho} M_1(\rho_1, f),$$

and

$$M_1(\rho_1, f) \leqslant K(1-\rho_1)^{-\alpha} \leqslant KA(\alpha)(1-\rho)^{-\alpha}.$$

(iv) [Theorems 20, 21]. It is possible to prove

THEOREM 20A.—

$$M_1(\rho, \bar{S}) < \vartheta|\bar{s}_1|\frac{\rho}{1-\rho} + A|\bar{s}_1|, \tag{3.56}$$

$$|\bar{s}_n| < \vartheta_1 e|\bar{s}_1|n \quad (n > 0), \tag{3.57}$$

where ϑ, ϑ_1 *are absolute constants less than* 1, *with corresponding modifications in Theorem* 21.

We prove first another theorem of some independent interest.

Let $w = f(z) = z + a_2z^2 + \ldots$ *be regular and "schlicht" in* γ, *and let* $M = r(1-r)^{-2}$, *and* $r \geqslant \frac{1}{2}(3-\sqrt{5}) = r_0$. *Then there is a segment of the circle* $|w| = M$, *cut off by a chord distant* $\frac{1}{2}M$ *from* $w = 0$ *and containing no point of the transform of* $|z| \leqslant r$ *by* $w = f(z)$.

---

* The corresponding result for functions S is false.

† If $\alpha = 0$ it is possible to prove a little more, viz. that

$$M_1(\rho, f') = o\{(1-\rho)^{-1}\}.$$

Since $M \geqslant r_0(1-r_0)^{-2} = 1$, there exists a $\beta = \beta(r)$ of the set $\Gamma(f)$ for which $|\beta| = M$. Then

$$F(z) = F(z, r) = \frac{\beta f(z)}{\beta - f(z)}$$

is regular and "schlicht" in $|z|<1$, and $F'(0) = 1$. Hence for $|z| \leqslant r$

$$\left|\frac{\beta f}{\beta - f}\right| \leqslant \frac{r}{(1-r)^2} = M,$$

$$\left|\frac{f/\beta}{1-f/\beta}\right| \leqslant 1, \quad \Re \frac{f}{\beta} \leqslant \tfrac{1}{2},$$

which proves the result.

If now $r^2 \geqslant 3-\sqrt{8}$ and $\phi(z) = \{f(z^2)\}^{\frac{1}{2}}$, it follows without difficulty that the transform of $|z| \leqslant r$ by $w = \phi(z)$ covers at most a proportion ϑ of the whole area of the circle $|w| \leqslant r(1-r^2)^{-1}$. Introducing this modification into the argument of Theorem 20, we obtain (3 . 56).

It follows now, by Theorem 3, that

$$t_n = |\bar{s}_n|/|\bar{s}_1| < \vartheta en + A$$
$$< \tfrac{1}{2}(1+\vartheta)en \quad (n \geqslant A_1).$$

Since also the left side is *less* than en for each $n < A_1$, we obtain (3 . 57) with

$$\vartheta_1 = \text{Max}\,\{\tfrac{1}{2}(1+\vartheta), \underset{n<A_1}{\text{Max}}\, t_n e^{-1} n^{-1}\} < 1.$$

(v) Prof. E. Landau, to whom I communicated the special argument for $Q(z)$ on p. 506, has pointed out to me that we may also argue as follows. It is enough to prove our result for the special point $z_0 = \rho$. Consider now

$$\psi(z) = Q\{\rho + (1-\rho)z\}/(2n\pi),$$

where n has its former meaning. ψ is a P whose $|p_0|$ and $|p_1|$ are $|Q(\rho)|/(2n\pi) < 1$ and $(1-\rho)|Q'(\rho)|/(2n\pi)$. It follows that

$$(1-\pi)|Q'(\rho)| < A\{|Q(\rho)|+1\},$$

and the modifications required in the text are evident.

Prof. Landau's remark is of general application. His method does not give so good an inequality as the ξ-argument, but it may replace the latter, wherever a best possible inequality is not in question, and the gain in simplicity of detail would be considerable in a complicated piece of analysis such as occurs in § 3 . 8.

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BALANCE SHEET, 31st October, 1924.

Liabilities.	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				341	0	0
Life Compositions Fund				2872	9	0
Reserve Fund in 1923	63	0	1			
Add Invested Surplus Fund	700	0	0			
	763	0	1			
Less Depreciation of Investments	169	11	4			
Reserve Fund				593	8	9
Sundry Creditors				19	6	0
Revenue Account balance in hand				137	0	4
				£4662	11	6

Assets.	£	s.	d.
London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 82 ...	315	14	0
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 56½	395	13	7
Funding Loan £2500 4 per Cent. at 90½ ...	2262	10	0
Conversion Loan £1700 3½ per Cent. at 79¼	1347	5	0
The above-mentioned stocks are valued at the present market price.			
Deposit at Bank	250	0	0
Cash at Bank	91	8	11
	£4662	11	6

REVENUE ACCOUNT for the year 1st November, 1923, to 31st October, 1924.

	Year 1922-23.					
	£	s.	d.	£	s.	d.
Balance from last year	332	9	7	*187*	*15*	*8*
Income tax returned	51	9	11	*50*	*17*	*0*
Dividends on—						
Lord Rayleigh's Fund ...	35	14	3	*34*	*16*	*10*
The Society's Investments	170	1	3	*156*	*9*	*9*
Entrance Fees	25	4	0	*46*	*4*	*0*
Annual Subscriptions	434	1	0	*418*	*3*	*0*
Sale of *Proceedings*, &c.	296	4	1	*352*	*5*	*0*
Advertisements revenue	7	0	0	*7*	*0*	*0*
Grant from Royal Society Brunner Mond Donation Fund for 1924	100	0	0		—	
	£1452	4	1	*1253*	*11*	*3*

	Year 1922-23.					
	£	s.	d.	£	s.	d.
Printing and Circulating *Proceedings*, &c.	1140	1	6	*751*	*8*	*6*
Purchases and Binding for Library ...	138	9	4	*129*	*9*	*9*
Rent	30	0	0	*30*	*0*	*0*
Postages and Sundries	6	12	11	*10*	*3*	*5*
Balance in hand on 31st October, 1924...	137	0	4	*332*	*9*	*7*
	£1452	4	1	*1253*	*11*	*3*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.
Balance in hand 31st October, 1923 ...		1	2
Dividends received	2	6	6
Income tax returned		13	10
	£3	1	6

	£	s.	d.
Cash at Bank 31st October, 1924	3	1	6
	£3	1	6

STATEMENT OF TRUST FUNDS, 31st October, 1924.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

13th November, 1924.

Signed on behalf of the Council:—

L. N. G. FILON, *Vice-President.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM,

26th November, 1924. *Lt.-Col. late R.E., Auditor.*

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